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Identifiability and Minimality in Rational Models

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Abstract. This paper uses key algebraic relationships between matrix Padé approximation and certain multivariate time series models. These relationships help us to obtain relevant results for solving the problems of identifiability and exchangeability in several models. We develop a new generalization of the corner method and apply it to the multivariate case. One advantage of the procedure is the presentation of the results in easily interpretable tables. We define new canonical representations. The paper also contains additional theoretical results improving on formulations of the corresponding algorithm that will assist us. The technique is illustrated in Vectorial Autoregressive Moving Average models by using a theoretical example.

2000 Mathematics Subject Classifications: 41A20 (Approximation by rational functions), 37M10 (Time series analysis)

Key Words and Phrases: matrix Padé approximation; multivariate time series; rational models; specification stage; exchangeability; identifiability.

1. Introduction

The aim of this work focuses on the specification stage of multivariate time series models discussed in [5, 9, 11, 16, ...]. These books contain what we believe is the most outstanding published compilation on specification methodologies. More recent references are, among others, [8, 15]. Some of the properties involving minimum orders and the unique irreducible representation for univariate time series models cannot be transferred to the multivariate case. In particular, two specific problems arise when considering rational matrix models: i) *identifiability*, especially when a unique representation for a pair of minimum orders (m.o.) does not exist; and ii) *exchangeable models*, particularly when several pairs of m.o. do exist. Identifiable models have been discussed from different points of view in [3, 10, ...]. The approach to exchangeable models using Scalar Component Models (SCM) introduced in [17]

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is relevant but does not consider the identifiability problem. [2] highlights the difficulty and complexity of studying m.o.

We are unaware of any process in the literature that allows for the determination of minimum order pairs, and of pairs with identifiable corresponding representations. Our research is motivated by these problems. We have shown how Matrix Padé Approximation (MPA) results ([12, 14]) can be obtained in a more practical and easily interpretable manner, thus leading to the discovery of all the identifiable representations with m.o. The characterization of rational matrix functions proposed in [12] involves $m \times n$ matrices. This general theoretical context can be applied in several models, such as Vectorial Autoregressive Moving Average (VARMA) models involving square matrices, Systems of Transfer Function Equations (STFE) with rectangular matrices, and others. In essence, we have generalized the corner method ([1]) to the multivariate case. Our approach provides a way to solve some special problems in time series analysis and mathematical modeling.

The paper is structured as follows. Section 2 contains theoretical relationships between certain multivariate time series models and rational matrix functions. We mention several possible applications of the main theoretical contributions in [12, 14] and analyze the problems of identifiability, minimality and exchangeability. Section 3 illustrates the use of the algorithm in VARMA models with a theoretical example. We conclude with some comments on the most relevant aspects of our research and discuss some possible considerations for future study.

This paper improves on [13] by proposing new theoretical results (theorems and properties) and reformulating some of the material in [13] so as to broaden the scope to include the characterization of rational models which require the use of non-square matrices. Moreover, definitions for new canonical representations are included, all of this illustrated with an example.

2. Rational Models in Multivariate Time Series

We are interested in studying if a process X_t, a k-vector of random variables, conforms to certain rational matrix models, for instance VARMA or STFE models. The results discussed in this section depend on algebraic properties that characterize rational functions and which do not depend on the location of the zeros in the polynomials representing them.

The large number of representations available for a rational matrix function, as compared to a scalar one, has resulted in several new concepts worthy of detailed study. We have chosen the following definitions for uniqueness and minimum degrees.

Definition 1. *We say that any $m \times n$ rational matrix function F(z) in the complex domain has a unique left representation for $(h, g) \in \mathbb{N}_0^2$ if there exists a single pair of matrix polynomials $N(z)$ and $D(z)$, called the numerator and the denominator respectively, of degrees bounded by h and g respectively, such that $F(z) \equiv D^{-1}(z)N(z)$ and D(0)=I hold. We consider $N(z)$ is an $m \times n$ matrix polynomial and $D(z)$ is an $m \times m$ matrix polynomial.*

Definition 2. *(q, p) is said to be a pair of minimum degrees (m.d.) for a rational matrix function $F(z)$ if $F(z) \equiv R^{-1}(z)S(z)$ (where $S(z)$ and $R(z)$ are matrix polynomials with degrees q and p respectively, $R(0) = I$) and in the case $F(z) \equiv D^{-1}(z)N(z)$ (where $N(z)$ and $D(z)$ are matrix* *polynomials with degrees h and g respectively, D(0)=I), $h < q$ implies $g > p$ and $g < p$ implies $h > q$.*

The results and properties presented in this Section follow from results and properties set forth in [14], which were written within the scope of MPA. They have been rewritten and adapted so as to be clearer and more practical in the context of multivariate time series. This revision is not obvious. The theoretical properties of matrix rational functions suggest different possibilities: For instance, if we consider the VARMA models as a particular case, we know that if X_t follows a $VARMA(p,q)$ model, $A(L)X_t = B(L)\varepsilon_t$, (non necessarily stationary, non necessarily invertible)[†] where L is the backshift operator (i.e. $X_{t-n} = L^n X_t$), ε_t a vector white noise process such that $E(\varepsilon_t) = 0$, $E(\varepsilon_t\varepsilon_t') = \Sigma$, $E(\varepsilon_t\varepsilon_{t+f}') = 0$ if $f \neq 0$, $A(z) = \sum_{i=0}^{p} A_i z^i$,

$B(z) = \sum_{i=0}^{q} B_i z^i, A_p \neq 0$,
$B_q \neq 0, A_0 = B_0 = I, A_i$ and B_j $(i = 1,2,...,p; j = 1,2,...,q)$ are $k \times k$ matrices; then:

There exists a rational matrix function $M(z)$, such that $B(z)M(z) \equiv A(z)$ for any z that is not a pole of $M(z)$. Equivalently, there also exists a rational matrix function $W(z)$, such that $A(z)W(z) \equiv B(z)$ for any z that is not a pole of $W(z)$.

Note that the following statements are equivalent: a) (p,q) are m.d. for $M(z)$; b) (q,p) are m.d. for $W(z)$; c) (p,q) are m.o. for the VARMA representation of the process X_t.

In particular, we have the following results:

Theorem 1. *If X_t is a stationary process, the following statements are equivalent:*

a There exist two matrix polynomials $A(z)$ and $B(z)$ such that X_t can be represented as a VARMA(p,q) model $A(L)X_t = B(L)\varepsilon_t$ where (p,q) are m.o.

b It holds that $X_t = W(L)\varepsilon_t$, where $W(L) \equiv A^{-1}(L)B(L) \equiv \sum_{j=0}^{\infty} W_j L^j$, $W_0 = I$, W_j being a $k \times k$ matrix for $j = 0,1...$

c There exist p matrices $A_1 \dots A_p$ such that $A_p \neq 0$, $A_p C_{q-p+1+i} + ... + A_1 C_{q+i} = -C_{q+1+i}$ and $A_p C_{q-p} + ... + A_1 C_{q-1} \neq -C_q$ for any $i \geq 0$ (Yule-Walker Equations). Here C_h represents the autocovariance matrix $C_h = cov(X_t, X_{t-h})(h \in \mathrm{Z})$.

d Given $h = max\{-q + p - 1, 0\}$ and $C_h = cov(X_t, X_{t-h})$, there exist $C_{(-h)}(z)$, a rational matrix function with m.d. (q+h,p), and $H_{q+h}(z)$, a matrix polynomial of degree q+h, fulfilling $A(z)C_{(-h)}(z) \equiv H_{q+h}(z)$ whenever z is not a pole of $C_{(-h)}(z) \equiv \sum_{i=0}^{\infty} C_{i-h} z^i$.

e Given a $g \geq \max\{-q + p - 1, 0\}$ and $C_h = cov(X_t, X_{t-h})$, there exist $C_{(-g)}(z)$, a rational matrix function with m.d. $(q + g, p)$, and $H_{q+g}(z)$, a matrix polynomial of degree $q + g$, satisfying $A(z)C_{(-g)}(z) \equiv H_{q+g}(z)$ whenever z is not a pole of $C_{(-g)}(z) \equiv \sum_{i=0}^{\infty} C_{i-g} z^i$.

[†]The X_t process is stationary if the roots of the determinantal equation $|A(z)| = 0$ are outside the unit circle and it is invertible if the roots of the determinantal equation $|B(z)| = 0$ are outside the unit circle.

The Proof is a consequence of

i) the recurrence relationships that characterize a rational matrix function, that is $\exists A(z)$ and $B(z)$ such that $W(z) \equiv A^{-1}(z)B(z)$ iff $A_p W_{q-p+1+i} + ... + A_1 W_{q+i} = -W_{q+i+1}$ $\forall i \geq 0$ and $A_p W_{j-p} + ... + A_1 W_{j-1} = B_j,\ j = 0, 1, ..., q$;

ii) results in the field of MPA: Proposition 2 and its Corollaries in [14];

iii) the fact that series in L can be treated as formal power series in z [2].

Theorem 2. *If X_t is an invertible process, the following statements are equivalent:*

a There exist two matrix polynomials $A(z)$ and $B(z)$ such that X_t can be represented as a VARMA(p,q) model $A(L)X_t = B(L)\varepsilon_t$, where (p,q) are m.o.

b $M(L)X_t = \varepsilon_t$, where $M(L) \equiv B^{-1}(L)A(L) \equiv \sum_{j=0}^{\infty} M_j L^j$, $M_0 = I$, (M_j being a $k \times k$ matrix for $j = 0, 1...$)

The Proof follows from the fact that $M(L)$ can be treated as a formal power series in z, as well as from the recurrence relationships that characterize a rational matrix function, that is, $\exists A(z)$ and $B(z)$ such that $M(z) \equiv B^{-1}(z)A(z)$ iff $B_q M_{p-q+1+i} + ... + B_1 M_{p+i} = -M_{p+1+i}$ $\forall i \geq 0$ and $B_q M_{j-q} + ... + B_1 M_{j-1} = A_j$, $j = 0, 1, ..., p$.

In some situations we may also consider the case $k = k_1 + k_2$, with k_1 and k_2 representing the number of endogenous and exogenous variables, respectively, in X_t. If y_t is the k_1-endogenous vector, z_t is the k_2-exogenous vector and the process is invertible, then we have a STFE given by $y_t = V(L)z_t + u_t$ with $V(L) = \sum_{i=0}^{\infty} V_i L^i$ and V_i a general matrix for any $i \in \mathbb{N}_0$ [19]. It would be interesting to determine the m.o. of lag matrix polynomials, which represent $V(L)$ in rational form, and investigate the identifiability of the corresponding representation.

In both contexts of VARMA and STFE models, as well as in other model contexts involving multivariate series, a central question is to determine whether or not the model (or the associated matrix formal power series in z: $W(z)$, $M(z)$, $C_{(-g)}(z)$, $V(z)$, etc) can be characterized by rational, identifiable and m.o. representations. Note that identifiable representations do not always exist for given pairs of m.o.

In order to answer this question, let us denote by

$$\mathrm{F(z)} = \sum_{\mathrm{k=0}}^{\infty} \mathrm{c_k\, z^k}\ , \mathrm{c_k} \in \mathbb{C}^{\mathrm{mxn}}, \mathrm{z} \in \mathbb{C}, \tag{1}$$

the associated matrix formal power series and define two tables, *Table 1* and *Table 2* [12, 14]. Next, we will remember some of their properties and later, we will see the new and practical utility of these properties in VARMA models identification.

Table 1. The value $T1(i,j) \equiv rank((c_{i-j+h+k-1})_{h,k=1}^{j})$ is placed in each cell (i,j) of *Table 1*, i.e. at the intersection of the i^{th} column with the j^{th} row. By convention, we set $c_{-i} = 0\ \forall i \in \mathbb{N}$, $T1(i,0) = 0\ \ \forall i \in \mathbb{N}_0$.

Definition 3. *The set* $R1 = \left\{(i,j) \in \mathbb{N}_0^2 / T1(i,j) = T1(i+k, j+k)\ \forall k \in \mathbb{N}\right\}$ *is called "a staired block" of Table 1.*

Example 1. *If* $F(z) = \sum_{i=0}^{\infty} c_{2i} z^{2i}$, *with* $c_{2i} = \frac{1}{(2i)!}\begin{pmatrix} 1 & 3 \\ -1/4 & -1 \end{pmatrix}^{2i}$, *then Figure 1 shows Table 1 for* $F(z)$.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	0	2	0	2	0
2	4	4	4	4	4	4
3	6	4	6	4	6	4
4	8	8	8	8	8	8
5	10	8	10	8	10	8

Figure 1: *Table 1* of Example 1.

Example 2. *Let*

$$\begin{aligned} F(z) &= \begin{pmatrix} 5.31 & 1 & 4.31 \\ 1 & 1.33 & -0.33 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0.66 & 0 & 0.66 \end{pmatrix} z + \begin{pmatrix} 2.25 & 0.75 & 1.5 \\ 0 & 0 & 0 \end{pmatrix} z^2 \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ 1.125 & 0.375 & 0.75 \end{pmatrix} z^3. \end{aligned}$$

Figure 2 shows Table 1 for $F(z)$ *with the border of* $R1$ *outlined.* $R1 = \{(i,j)/i \geq 3 \wedge j \geq 0\} \cup \{(2,j)/j \geq 2\}$. *We observe that (3,0) and (2,2) are the corners of* $R1$.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	1	1	1	0	0
2	4	4	3	2	1	0
3	6	6	5	3	2	1
4	8	8	7	5	3	2
5	10	10	9	7	5	3

Figure 2: *Table 1* of Example 2.

Property 1. $F(z)$ *is a rational matrix function iff* $R1 \neq \emptyset$.

Proof. Follows from Theorem 10 [12, p. 177].

Therefore, $F(z)$ is rational in Example 2, and not rational for any (q,p) with $0 \leq q, p < 5$ in Example 1.

The following Properties guarantee that, in certain cases, a pair of degrees associated with a corner of $R1$ will (or will not) be a pair of m.d. (Properties 2-7), and that the left representation of $F(z)$ will (or will not) be unique for a given pair of degrees (Properties 8-9).

Property 2. *$(i,0) \in R1$ and $(i-1,0) \notin R1$ iff $(i,0)$ is a pair of m.d.*

Proof. Follows from Property 1 in [14, p. 29].

Therefore, in Example 2, $(3,0)$ is a pair of m.d.

Property 3. *If $T1(i-1,j) < T1(i,j)$, then $(i-u, j-v)$ is not a pair of m.d. whenever $1 \leq u \leq i$ and $0 \leq v \leq j$.*

Proof. Follows from Property 2 in [14, p. 29].

Property 4. *If $T1(i,j) = jm$ and $(i,j) \notin R1$, then $(i-u, j-v)$ is not a pair of m.d. whenever $1 \leq u \leq i$ and $0 \leq v \leq j$.*

Proof. Follows from Property 3 in [14, p. 30].

Therefore, in Example 2, if $a \leq 5$ and $b \leq 5$, then $(0,a)$ and $(1,b)$ are not pairs of m.d. (since $T1(1,5) = 10$ and $(1,5) \notin R1$).

Property 5. *If $(i,j) \in R1$, $(i-1,j) \notin R1$ and $T1(i,j) = jm$, then $(i-u, j-v)$ is not a pair of m.d. whenever $1 \leq u \leq i$ and $0 \leq v \leq j$.*

Proof. Follows from Property 4 in [14, p. 30].

Property 6. *Suppose $m = n$. Then $(0,j) \in R1$ and $(0,j-1) \notin R1$ iff $(0,j)$ is a pair of m.d.*

Proof. Follows from Property 5 in [14, p. 30].

Property 7. *Suppose $(i,j) \in R1$. If $(i-u,j) \notin R1$ and $(i,j-v) \notin R1$, for any $u \in [1,i]$ and $v \in [1,j]$, and if all these cells are not pairs of m.d., then (i,j) is a pair of m.d.*

Proof. Follows directly from the definition of m.d.

Property 8. *Suppose $(i,j) \in R1$. Then $T1(i,j) = jm$ iff the left representation of $F(z)$ for (i,j) is unique. Under these conditions, we can obtain the matrix coefficients $d_1, d_2, ..., d_j$ in the denominator $D(z)$ in connection to Definition 1 by solving the system*

$$d_j c_{i-j+h} + d_{j-1} c_{i-j+h+1} + \ldots + d_1 c_{i+h-1} = -c_{i+h} \; h = 1,2,...,j. \tag{2}$$

Proof. Follows from Theorem 2 and Corollary 1 in [14, p. 28].

Therefore, in Example 2, the left representation of $F(z)$ is unique for $(3,0)$, and it is not unique for any $(a,b) \in R1$ with $b \neq 0$.

Property 9. *If $m = n$ and $(0,j) \in R1$, then $T1(0,j) = jm$ and the left representation of $F(z)$ for $(0,j)$ is unique.*

Proof. Follows from Property 5 in [14, p. 30].

In Example 2, *Table 1* does not state whether $(2,1)$ is a pair of m.d. or not.

Although *Table 1* is preferable from a computational perspective, *Table 2* furnishes on alternative outlet when certain pairs of m.d. in *Table 1* cannot be identified. We will now define *Table 2*.

***Table 2**$_{sr}$*. Given any $(s,r) \in R1$, the value

$$T2_{sr}(i,j) = \begin{cases} 0 & if\ rank(M4_{sr}(i,j)) = rank(M5_{sr}(i,j)) \\ 1 & otherwise \end{cases}$$

is placed in the (i,j) cell of *Table 2$_{sr}$*, whenever $0 \le i \le s$ and $0 \le j \le r$. Here $M4_{sr}(i,j) = (c_{i-j+h+k-1})_{h,k=1}^{j,s+r-i}$, and $M5_{sr}(i,j) = (c_{i-j+h+k-1})_{h,k=1}^{j+1,s+r-i}$. Note that the cell (s,r) is the lower-right corner of *Table 2$_{sr}$* .

Definition 4. $R2_{sr} = \{(i,j) \in \mathbb{N}_0^2 / T2_{sr}(i,j) = 0\}$.

The *Table 2$_{35}$* of Example 2 appears in Figure 3. Observe that $R2_{35} = \{(3,j)/j \le 5\} \cup \{(2,k)/1 \le k \le 5\}$, and $(3,0)$ and $(2,1)$ are the *corners* of $R2_{35}$.

	0	1	2	3
0	1	1	1	0
1	1	1	0	0
2	1	1	0	0
3	1	1	0	0
4	1	1	0	0
5	1	1	0	0

Figure 3: *Table 2* of Example 2.

Property 10. *(i,j) is a pair of m.d. for F(z) iff, given any $(s,r) \in R1$ such that $0 \le i \le s$ and $0 \le j \le r$, the cell (i,j) is a corner of $R2_{sr}$.*

Proof. Follows from Property 9 in [14, p. 31].

As an application, we infer that $F(z)$ in Example 2 has two pairs of m.d. The first pair equals $(3,0)$, and is already identified in *Table 1*, while the second one is $(2,1)$.

Property 11. *If a cell (h,g) in Table 2 takes a zero value, then any cell in the lower right rectangle whose upper left corner is (h,g) also takes on a zero value.*

Proof. Follows from Property 11 in [14, p. 31].

Property 12. *The left representation of $F(z)$ for the degrees (h,g) is unique iff, given $(s,r) \in R1$ such that $(h,g) \in R2_s$, the $rank(M4_{sr}(h,g))$ is equal to mg. Under these conditions, we can obtain the coefficients of the denominator $D(z)$ in connection to Definition 1 by solving the system*

$$d_j c_{i-j+h} + d_{j-1} c_{i-j+h+1} + \ldots + d_1 c_{i+h-1} = -c_{i+h} \ h = 1,2,...,s+r-i. \quad (3)$$

It is preferable to choose $(s,r) \in R1$ with $(h,g) \in R2_{sr}$ and $s+r$ minimum.

Proof. Proof: Follows from Theorem 3 in [14, p. 28].

For instance, if in Example 2 we consider $(s,r) = (3,1)$, then the left representation of $F(z)$ for the m.d. $(2,1)$ is unique since $rank(M4_{3,1}(2,1)) = 2$. The denominator $D(z)$ of such a representation in connection to Definition 1 can be calculated by solving the system (3).

Example 3. *Let $X_t = W(L)\varepsilon_t$ be the following bivariate VARMA(0,2) model:*

$$X_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \varepsilon_t + \begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix} \varepsilon_{t-1} + \begin{pmatrix} 0 & 0 \\ 0.5 & 0.25 \end{pmatrix} \varepsilon_{t-2} \text{ where } \Sigma = E(\varepsilon_t \varepsilon_t') = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let us then consider the three series $W(z)$, $M(z)$ and $C_{(-g)}(z)$, for a given g (although, as it is shown in Theorems 1 and 2, one series is sufficient).

Figure 4 contains the *Table 1* for $W(z)$. By Property 4, the cells $(0, j)$ are not m.d. for $W(z)$ (whenever $j = 0, 1, ..., 5$) and the process X_t does not follow any $VAR(j) \equiv VARMA(j, 0)$ model. Property 2 also reveals that $(2,0)$ is a pair of m.d., so X_t follows a $VMA(2) \equiv VARMA(0,2)$ representation.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	1	1	0	0	0
2	4	3	2	1	0	0
3	6	5	4	2	1	0
4	8	7	6	4	2	1
5	10	9	8	6	4	2

Figure 4: $R1(W)$ of Example 3.

In Figure 5 we have the *Table 1* for $M(z)$. From Property 2 we deduce that, for any $j = 0, 1, ..., 5$ the cells $(j, 0)$ are not m.d. Further, Property 6 guarantees that $(0,2)$ is a pair of m.d. Finally, as a consequence of Property 8, we see that the left representation of $M(z)$ for $(0,2)$ is unique.

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	1	1	1	1	1
2	4	3	2	2	2	2
3	6	4	3	2	2	2
4	8	6	4	3	2	2
5	10	8	6	4	3	2

Figure 5: $R1(M)$ of Example 3.

We can obtain additional information from *Table 2*. In Figure 6 we give an arrangement for the set $R2_{25}(W)$, or equivalently, for the set $R2^*_{25}(M) = \{(i,j)/(j,i) \in R2_{52}(M)\}$. We see that $(1,1)$ is another pair of m.d. From Property 12, the left representations of $W(z)$ and $M(z)$ for $(1,1)$ are unique, since $rank(M4_{21}(1,1)) = 2$ for $W(z)$ and $rank(M4_{12}(1,1)) = 2$ for $M(z)$.

	0	1	2
0	1	1	0
1	1	0	0
2	1	0	0
3	1	0	0
4	1	0	0
5	1	0	0

Figure 6: $R2_{25}(W)$ of Example 3.

Let us now consider the statement e) of Theorem 1. Without loss of generality and in order to build the *Table 1* with $r \geq 2$ rows for $C_{(-g)}(z)$, we can assume that $g = r - 2$. (Note that $g \geq -a + b - 1$, where (a, b) represents any cell in these tables with r rows). The difference between the numerator degrees of $W(z)$ and $C_{(-g)}(z)$ is g. At this point, a definition for *staired blocks* is necessary:

$$\overline{R1}(C_{(-g)}) = \{(i, j) \in \mathbb{N}_0^2 / (i + g, j) \in R1(C_{(-g)})\} \text{ and}$$
$$\overline{R2}_{ab}(C_{(-g)}) = \{(i, j) \in \mathbb{N}_0^2 / (i + g, j) \in R2_{a+g,b}(C_{(-g)})\} \text{ for any } (a, b) \in \overline{R1}(C_{(-g)}).$$

The interpretation of the tables can be unified by ignoring the first g columns in *Table 1* and *Table 2* for $C_{(-g)}(z)$. Essentially, we place $T1(i + g, j)$ and $T2_{ab}(i + g, j)$ in cell (i, j) of *Table 1* and *Table 2*, respectively, and then highlight the borders of $\overline{R1}(C_{(-g)})$ and $\overline{R2}_{ab}(C_{(-g)})$.

Figure 7 shows $\overline{R1}(C_{(-4)})$. Observe that:

$$R1(W) = \{(i, j)/i \geq 2 \wedge j \geq 0\} \neq \overline{R1}(C_{(-4)}) = \{(i, j)/(i \geq 2 \wedge j \geq 0) \text{ or } (i = 1 \wedge j \geq 1)\}.$$

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	2	1	0	0	0
2	4	4	2	1	0	0
3	6	6	4	2	1	0
4	8	8	6	4	2	1
5	10	10	8	6	4	2

Figure 7: $\overline{R1}(C_{(-4)})$ of Example 3.

As a consequence of Property 4 we see that the $VMA(2)$ and $VARMA(1,1)$ models are the only ones with m.o. within the confines of *Table 1*; therefore, *Table 2* can be ignored. Property 8 shows that $VMA(2)$ and $VARMA(1,1)$ are identifiable. Since $A_0 = B_0 = I$, we need to estimate two matrix parameters for each model. Observation of the Tables reveals that the existence of m.o. cannot be ascertained for (p, q) with $q \geq 6$ or $p \geq 6$. In these cases the corresponding models are less parsimonious because there are at least six matrix parameters.

The following result is also important:

Result 1. *If* $(a, b) \in \overline{R1}(C_{(-g)}) \cap R1(W)$ *where* $g \geq -a + b - 1, g \geq 0$ *and* $(b, a) \in R1(M)$*, then* $(i, j) \in \overline{R2}_{ab}(C_{(-g)}) \Leftrightarrow (i, j) \in R2_{ab}(W) \Leftrightarrow (j, i) \in R2_{ba}(M)$.

Proof. Follows from Proposition 2 and its Corollaries in [14, p. 34-36].

3. Algorithm for Specifying Minimal Rational Models

This algorithm starts with a data sample and then proceeds to characterize a rational model, identify its m.o. and study parameter identifiability:

STEP 1. Choose the dimensions for *Table 1*: NR rows and NC columns.

STEP 2. Estimate the matrix coefficients:

Option i) the VMA coefficients $(W_0, W_1, \ldots, W_{NR+NC+1})$

Option ii) the VAR coefficients $(M_0, M_1, \ldots, M_{NR+NC+1})$

Option iii) the autocovariance matrices $(C_0, C_1, \ldots, C_{NR+NC+1})$, or

Option iv) the Transfer Function coefficients $(V_0, V_1, \ldots, V_{NR+NC+1})$, etc.

STEP 3. Construct *Table 1* for the above $C_{(-NR+2)}(z)$ or $W(z)$ or $M(z)$ or $V(z)$, etc. If $R1 \neq \emptyset$, the process has a rational representation. In such a case, evaluate the m.o. by using the properties of $R1$.

STEP 4. If *Table 1* does not suffice to identify all of m.o.'s, then construct *Table 2*.

STEP 5. Study the identifiability for each representation with m.o. (p,q), using Property 8 or 12, depending on the case.

STEP 6. From the conditions of Property 8 or 12, obtain initial estimators for the autoregressive coefficients by solving a) (2) or (3) with (q,p) and sr, if we estimated the VMA coefficients or the autocovariance matrices in STEP 2 or b) (2) or (3) with (p,q) and rs, if we estimated the VAR coefficients matrices in STEP 2.

Steps 1, 2 and 3 can be considered as an improvement over the VARMA model procedure given in [17], especially when determining an overall order, because the ambiguous parameter h in [17] is not necessary here, and then the dimensions of the matrices involved are not larger than necessary.

Simplified Table 1. In [12] a *simplified Table 1* was constructed to save on computational work.

Redundant parameters. If for a certain pair of m.o. the representation is not unique, then different "canonical"representations might be defined by fixing certain sets of free parameters. The *echelon* form furnishes such an example [see e.g. 9]. The specification problem in VARMA models was studied through SCM in [17], where Tiao and Tsay eliminated one type of redundant parameters by carefully studying the SCM. A non-identifiable $VARMA(p,q)$ representation is always a significant problem when estimating $A_1, ..., A_p, B_1, \ldots, B_q$.

Properties 8 and 12 offer a method of determining whether or not a process follows m.o. identifiable representations.

If in STEP 2 of the above algorithm we estimate $(W_0, W_1, ..., W_{NR+NC+1})$, or $(C_0, C_1, \ldots, C_{NR+NC+1})$ and if $(q,p) \in R2_{sr}$ and $rank(M4_{sr}(q,p)) = b < pk$, then the associated rational representation is not unique. So in what follows, we will define a new canonical representation with eliminated redundant parameters. To do so, observe that each ith row of $LS_{sr}(q,p)$ can be written as

$$(d_p^{(i)} d_{p-1}^{(i)} \ldots d_1^{(i)}) M4_{sr}(q,p) = (c_{q+1}^{(i)} \ldots c_{s+r}^{(i)}) \;\; i = 1, ..., k.$$

Obviously, there are several different ways to choose $f = pk - b$ rows of $M4_{sr}(q,p)$, all of them depending on the other b linear independent rows. Let us choose, for example, the i$_1^{th}$, the i$_2^{th}$,... and the i$_f^{th}$ rows. Then, we can define a "canonical" representation as follows:

The f columns $i_1, i_2, ..., i_f$ of the $k \times pk$ matrix $(d_p d_{p-1} ... d_1)$ are k-vectors with annihilated coordinates, while the remaining b columns of this matrix have unique estimation.

Many other alternative and similar representations could be easily defined (by choosing another set of f rows in $M4_{sr}(q,p)$) and all redundant parameters can be eliminated.

Remark: If in STEP 2 of the algorithm we estimate $(M_0, M_1, \ldots, M_{NR+NC+1})$, then we must substitute p with q and s with r.

4. Conclusions

In this contribution, we investigated a practical method to apply in the analysis of multivariate time series. The effects of such an application allowed us to *characterize* rational matrix models, study the possible pairs of *minimum orders*, recognize the *exchangeable* models that might exist and detect *identifiable* and non-identifiable representations. Note that our method does not require any knowledge of the matrix coefficients of the polynomials that appear in the model. Through the example, we have shown that the adopted approach offers the following advantages:

- The procedure is straightforward in the sense that the results are presented directly in tables that are easily interpretable.

- Presumably, one of the benefits of this method is computational efficiency.

- The procedure provides a (possible) solution of the minimality problem for the VARMA models. This problem was pointed out in [2, p. 310], as well as in other related studies, referring to the mathematical complexity of the question.

- The proposed definition of m.o. permits us to advance in a more global study of the identifiability problem.

- Though it would be interesting to compare the statistical procedures in a more general way using simulation exercises, the necessary software is not available. Nevertheless, procedures to determine the rank based on QR or similar algorithms using orthogonality properties [6] may still be explored.

- Further, we have also given theoretical relationships between rational matrix functions and rational representations of a multivariate time series within a wide range of applications. The results coming from our investigations could contribute to the development of new statistical procedures in the future, specifically by considering: a) $W(z)$ (with moving average

coefficients);
b) $M(z)$ (with autoregressive coefficients); c) non-stationary processes; d) redundant parameters and the definition of new canonical representations; e) systems of Transfer Function Equations.

We believe that these future studies will considerably enrich the field of multivariate time series analysis.

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