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The (CLR_g)-property for coincidence point theorems and Fredholm integral equations in modular metric spaces

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Abstract. In paper, we prove some common fixed point theorems for the pair of self-mappings with the g-quasi-condition in modular metric spaces. Also, we modify and prove some common fixed point theorems by using the (CLR_g)-property along with the weakly compatible mapping. Finally, we give some applications on integral equations to illustrate our main results.

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1. Introduction

In 1998, Jungck and Rhoades [1] introduced the notion of weakly compatible mappings as follows:

Let X be a nonempty set. Two mappings $f, g : X \to X$ are said to be *weakly compatible* if $fx = gx$ implies $fgx = gfx$ for any $x \in X$.

In 2011, Sintunavarat and Kumam [4] introduced a new relax condition is called the (CLR_g)-property as follows:

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Suppose that (X, d) is a metric space and $f, g : X \to X$ be two mappings. The mappings f and g are said to satisfy the *common limit in the range of* g (shortly, (CLR_g)-property) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$$

for some $x \in X$. The importance of (CLR_g)-property ensures that one does not require the closeness of range subspaces

On the other hand, in 2010, Chistyakov [2] introduced the notion of a modular metric space which is a new generalization of a metric space. In the same way, Mongkolkeha et al. [3] proved the existence of fixed point theorems for contraction mappings as following:

Let ω be a metric modular on X and X_ω be a modular metric space induced by ω. If X_ω is a complete modular metric space and $T : X_\omega \to X_\omega$ be a mapping such there exists $k \in [0, 1)$ with

$$\omega_\lambda(Tx, Ty) \leq k\omega_\lambda(x, y)$$

for all $x, y \in X_\omega$ and $\lambda > 0$, then T has a unique fixed point in X_ω.

Currently Aydi et al. [5] established some coincidence and common fixed point results for three self-mappings on a partially ordered cone metric space satisfying a contractive condition and proved an existence theorem of a common solution of integral equations. In the same way, Shatanawi et al.[6] studied some new real generalizations on coincidence points for weakly decreasing mappings satisfying a weakly contractive condition in an ordered metric space. Many author studies in modular metric spaces [11, 12, 13, 14, 15, 16, 17].

In this paper, we study and prove the existence of some coincidence point theorems for generalized contraction mappings in modular metric spaces and give some applications on integral equations for our main results.

2. Preliminaries

In this section, we give some definitions and their properties for our main results.

Definition 1. *[7] Let (X, d) be a metric space. Two mappings $f : X \to X$ and $g : X \to X$ are said to satisfy the $(E.A)$-property if there exist a sequences $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

Next, we introduce the notion of a modular metric space as follows:

Definition 2. *Let X be a linear space over $\mathbb{R}$ with $\theta \in X$ as its zero element. A functional $\rho : X \to [0, +\infty]$ is called a* modular *on X if, for all $x, y, z \in X$, the following conditions hold:*

(M1) $\rho(x) = 0$ if and only if $x = \theta$;

(M2) $\rho(x) = \rho(-x)$;

(M3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ *whenever* $\alpha, \beta \geq 0$ *and* $\alpha + \beta = 1$.

The linear subspace $X_\rho := \left\{x \in X : \lim_{\lambda \to \infty} \rho(\lambda x) = 0\right\}$ *is called a modular space.*

Definition 3. *[2] Let X be a nonempty set.*

(1) A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ *is called a metric modular on X if, for all $x, y, z \in X$, the following conditions hold:*

(MM1) $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0$ *if and only if* $x = y$;

(MM2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$;

(MM3) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ *for all* $\lambda, \mu > 0$.

(2) If, instead of the condition (MM1), we have the following condition:

(MM1′) $\omega_\lambda(x, x) = 0$ *for all* $\lambda > 0$,

then ω is called a (metric) pseudo-modular on X.

Remark 1. *A modular ω on a set X, the function* $0 < \lambda \mapsto \omega_\lambda(x, y) \in [0, \infty]$ *for all* $x, y \in X$, *is a non-increasing on* $(0, \infty)$. *In fact, if* $0 < \mu < \lambda$, *then the conditions (MM3), (MM1′) and (MM2) imply*

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y). \tag{1}$$

It follows that, at each point $\lambda > 0$, the right limit $\omega_{\lambda+0}(x, y) := \lim_{\epsilon \to +0} \omega_{\lambda+\epsilon}(x, y)$ and the left limit $\omega_{\lambda-0}(x, y) := \lim_{\varepsilon \to +0} \omega_{\lambda-\varepsilon}(x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$\omega_{\lambda+0}(x, y) \leq \omega_\lambda(x, y) \leq \omega_{\lambda-0}(x, y). \tag{2}$$

for all $x, y \in X$. We know that, if $x_0 \in X$, the set $X_\omega = \{x \in X : \lim_{\lambda \to \infty} \omega_\lambda(x, x_0) = 0\}$ is a metric space, which is called a *modular space*, whose metric is given by

$$d^0_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda \lambda(x, y) \leq \lambda\}$$

for all $x, y \in X_\omega$. Also, it follows that, if X is a real linear space, $\rho : X \to [0, \infty]$ and

$$\omega_\lambda(x, y) = \rho\Big(\frac{x - y}{\lambda}\Big)$$

for all $\lambda > 0$ and $x, y \in X$, then ρ is a modular on X if and only if ω is a metric modular on X (see [2]).

Example 1. *[8] The following indexed objects ω are simple examples of a modular on a set X. Let $\lambda > 0$ and $x, y \in X$. Then we have*

(1) $\omega_\lambda(x, y) = \infty$ *if* $\lambda \leq d(x, y)$, *and* $\omega_\lambda(x, y) = 0$ *if* $\lambda > d(x, y)$;

(2) $\omega_\lambda(x, y) = \infty$ *if* $\lambda < d(x, y)$, *and* $\omega_\lambda(x, y) = 0$ *if* $\lambda \geq d(x, y)$.

Definition 4. *[3] Let X_ω be a modular metric space.*

(1) The sequence $\{x_n\}$ in X_ω is said to be ω-convergent to a point $x \in X_\omega$ if $\omega_\lambda(x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$;

(2) The sequence $\{x_n\}$ in X_ω is called an ω-Cauchy sequence if $\omega_\lambda(x_m, x_n) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$;

(3) A subset C of X_ω is said to be ω-closed if the limit of a convergent sequence $\{x_n\}$ of C always belongs to C;

(4) A subset C of X_ω is said to be ω-complete if any ω-Cauchy sequence $\{x_n\}$ in C is ω-convergent to a point is in C;

(5) A subset C of X_ω is said to be ω-bounded if, for all $\lambda > 0$, $\delta_\omega(C) = \sup\{\omega_\lambda(x, y) : x, y \in C\} < \infty$.

Definition 5. *Let X_ω be a modular metric space and $f, g : X \to X$ be two mappings. The mappings f and g are said to satisfy the common limit in the range of g (shortly, (CLR_g)-property) if*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$$

for some $x \in X_\omega$.

Definition 6. *[9] Let X_ω be a modular metric space. We say that ω satisfies the Δ_2-condition if, for any sequence $\{x_n\} \subset X_\omega$ and $x \in X_w$, there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x, such that $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$.*

Note that, in this paper, we suppose that ω is a modular on X and satisfies the Δ_2-condition on X.

3. Fixed point results for the contractive condition

Lemma 1. *Let f and g be weakly compatible self-mappings of a set X_ω. If f and g have a unique coincidence point, that is, $t = fx = gx$, then t is the common fixed point of f and g.*

Theorem 1. *Let X_ω be a modular metric space and $f, g : X_\omega \to X_\omega$ be weakly compatible mappings such that $f(X_\omega) \subset g(X_\omega)$ and $g(X_\omega)$ is a ω-complete subspace of X_ω. Suppose there exists number $a \in [0, \frac{1}{4})$ for all $x, y \in X_\omega$ and $\lambda > 0$ such that*

(a) there exists $x_0, x_1 \in X_\omega$ such that $\omega_\lambda(fx_0, gx_1) < \infty$;

(b) $\omega_\lambda(fx, fy) \le a[\omega_\lambda(fx, gy) + \omega_{2\lambda}(fy, gx) + \omega_\lambda(fx, gx) + \omega_\lambda(fy, gy)]$.

Then f and g have a coincidence point.

Proof. Let x_0 be an arbitrary point in X_ω. Since $f(X_\omega) \subset g(X_\omega)$, there exists a sequence $\{x_n\}$ in X_ω such that

$$gx_n = fx_{n-1}$$

for all $n \geq 1$. Now, setting $x = x_n$ and $y = x_{n+1}$ in (b), we have

$$\begin{aligned}\omega_\lambda(fx_n, fx_{n+1}) &\leq a[\omega_\lambda(fx_n, fx_n) + \omega_{2\lambda}(fx_{n+1}, fx_{n-1}) + \omega_\lambda(gx_{n+1}, gx_n) + \omega_\lambda(fx_{n+1}, fx_n)] \\ &= a[\omega_{2\lambda}(fx_{n+1}, fx_{n-1}) + \omega_\lambda(gx_{n+1}, gx_n) + \omega_\lambda(fx_{n+1}, fx_n)]\end{aligned}$$

for all $\lambda > 0$. On the other hand, we have

$$\begin{aligned}\omega_{2\lambda}(fx_{n+1}, fx_{n-1}) &\leq \omega_\lambda(fx_{n+1}, fx_n) + \omega_\lambda(fx_n, fx_{n-1}) \\ &= \omega_\lambda(fx_{n+1}, fx_n) + \omega_\lambda(gx_{n+1}, gx_n)\end{aligned}$$

and so

$$\omega_\lambda(fx_n, fx_{n+1}) \leq a[\omega_\lambda(fx_{n+1}, fx_n) + \omega_\lambda(gx_{n+1}, gx_n) + \omega_\lambda(gx_{n+1}, gx_n) + \omega_\lambda(fx_{n+1}, fx_n)].$$

This implies that

$$\omega_\lambda(fx_n, fx_{n+1}) \leq \frac{2a}{1-2a}\omega_\lambda(gx_n, gx_{n+1})$$

for all $n \in \mathbb{N}$, where $\alpha = \dfrac{2a}{1-2a} < 1$. By induction, we have

$$\omega_\lambda(fx_n, fx_{n+1}) \leq \alpha^n \omega_\lambda(gx_0, gx_1) \tag{3}$$

for all $n \in \mathbb{N}$. By (a), it follows that $\{fx_n\}$ is a ω-Cauchy sequence. Since $g(X_\omega)$ is ω-complete, there exists $u, v \in X_\omega$ such that $u = g(v)$ and $fx_n \to u$ as $n \to \infty$. Since ω satisfy the Δ_2-condition on X, we have $\lim_{n\to\infty} \omega_\lambda(fx_n, u) = 0$ for all $\lambda > 0$ and hence

$$\lim_{n\to\infty} \omega_\lambda(fx_n, u) = \lim_{n\to\infty} \omega_\lambda(gx_n, u) = 0 \tag{4}$$

for all $\lambda > 0$. Letting $x = x_n$ and $y = v$ in (b), we have

$$\begin{aligned}\omega_\lambda(fx_n, fv) &\leq a[\omega_\lambda(fx_n, gv) + \omega_{2\lambda}(fv, gx_n) + \omega_\lambda(fx_n, gx_n) + \omega_\lambda(fv, gv)] \\ &\leq a[\omega_\lambda(fx_n, gv) + \omega_{2\lambda}(fv, fx_n) + \omega_\lambda(fx_n, gx_n) + \omega_\lambda(fv, gv)]\end{aligned}$$

and, by Remark 1, since the function $\lambda \mapsto \omega_\lambda(x, y)$ is non-increasing, we have

$$\omega_\lambda(fx_n, fv) \leq a[\omega_\lambda(fx_n, gv) + \omega_\lambda(fv, fx_n) + \omega_\lambda(fx_n, gx_n) + \omega_\lambda(fv, gv)].$$

By (b), letting $n \to \infty$ in the above inequality, we have

$$\omega_\lambda(fv, gv) \leq [\omega_\lambda(fv, gv) + \omega_\lambda(fv, fv) + \omega_\lambda(fv, gv) + \omega_\lambda(fv, gv)].$$

Thus $(1 - 4k)\omega_\lambda(fv, gv) \leq 0$ for all $\lambda > 0$ and so

$$gv = fv = u,$$

which proves that g and f have a coincidence point.

Now, we generalize Theorem 1 by using $(CLRg)$-property for weakly compatible mappings as follows:

Theorem 2. *Let X_ω be a modular metric space and $f, g : X_\omega \to X_\omega$ be weakly compatible mappings such that $f(X_\omega) \subset g(X_\omega)$. Suppose there exists a number $a \in [0, \frac{1}{4})$ for all $x, y \in X_\omega$ and $\lambda > 0$ such that*

(a) there exists $x_0, x_1 \in X_\omega$ such that $\omega_\lambda(fx_0, gx_1) < \infty$;

(b) $\omega_\lambda(fx, fy) \leq a[\omega_\lambda(fx, gy) + \omega_{2\lambda}(fy, gx) + \omega_\lambda(fx, gx) + \omega_\lambda(fy, gy)]$.

If f and g satisfy the (CLR_g)-property, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLR_g)-property, there exists a sequence $\{x_n\}$ in X_ω such that $\lim_{n\to\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = gx$ for some $x \in X_\omega$. From (b), we have

$$\omega_\lambda(fx_n, fx) \leq a[\omega_\lambda(fx_n, gx) + \omega_{2\lambda}(fx, gx_n) + \omega_\lambda(fx_n, gx_n) + \omega_\lambda(fx, gx)]$$

for all $n \geq 1$. Letting $n \to \infty$, we have $gx = fx$. Let $t = fx = gx$. Since f and g are weakly compatible mappings, $fgx = gfx$ implies that $ft = fgx = gfx = gt$.

Now, we claim that $ft = t$. From (b), we have

$$\begin{aligned}
\omega_\lambda(ft, t) &= \omega_\lambda(ft, fx) \\
&\leq a[\omega_\lambda(ft, gx) + \omega_{2\lambda}(fx, gt) + \omega_\lambda(ft, gt) + \omega_\lambda(fx, gx)] \\
&= a[\omega_\lambda(ft, gx) + \omega_{2\lambda}(fx, gt)] \\
&= a[\omega_\lambda(ft, t) + \omega_{2\lambda}(t, ft)]
\end{aligned}$$

and, by Remark 1, since the function $\lambda \mapsto \omega_\lambda(x, y)$ is non-increasing, we have

$$\omega_\lambda(ft, t) \leq a[\omega_\lambda(ft, t) + \omega_\lambda(t, ft)].$$

This implies that $(1 - 2a)\omega_\lambda(ft, t) \leq 0$ for all $\lambda > 0$, that is, $\omega_\lambda(ft, t) = 0$ and so $ft = t = gt$. Thus t is a common fixed point of f and g.

For the uniqueness of the common fixed point, we suppose that u is another common fixed point in X_ω such that $fu = gu$. From (b), we have

$$\begin{aligned}
\omega_\lambda(gu, gt) &= \omega_\lambda(fu, ft) \\
&\leq a[\omega_\lambda(fu, gt) + \omega_{2\lambda}(ft, gu) + \omega_\lambda(fu, gu) + \omega_\lambda(ft, gt)] \\
&= a[\omega_\lambda(fu, gt) + \omega_{2\lambda}(ft, gu)] \\
&= a[\omega_\lambda(gu, gt) + \omega_{2\lambda}(gt, gu)]
\end{aligned}$$

and, by Remark 1, since the function $\lambda \mapsto \omega_\lambda(x, y)$ is non-increasing, we have

$$\omega_\lambda(gu, gt) \leq a[\omega_\lambda(gu, gt) + \omega_\lambda(gt, gu)].$$

This implies $gu = gt$. Thus, by Lemma 1, we have f and g have a unique common fixed point.

Theorem 3. *Let X_ω be a modular metric space and $f, g : X_\omega \to X_\omega$ be weakly compatible mappings such that $f(X_\omega) \subset g(X_\omega)$. Suppose that there exist $a_1, a_2, a_3, a_4, a_5 \in [0, \frac{1}{4})$ and $\sum_{i=1}^{5} a_i < 1$ such that, for all $x, y \in X_\omega$ and $\lambda > 0$,*

(a) there exists $x_0, x_1 \in X_\omega$ such that $\omega_\lambda(fx_0, gx_1) < \infty$;

(b) $\omega_\lambda(fx, fy) \leq a_1\omega_\lambda(fx, gx)+a_2\omega_\lambda(fy, gy)+a_3\omega_\lambda(fy, gx)+a_4\omega_\lambda(fx, gy)+a_5\omega_\lambda(gy, gx)$.

If f and g satisfy (CLR_g)-property, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLR_g)-property, there exists a sequence $\{x_n\}$ in X_ω such that $\lim_{n\to\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = gx$ for some $x \in X_\omega$. From (b), we have

$$\begin{aligned}\omega_\lambda(fx_n, fx) &\leq a_1\omega_\lambda(fx_n, gx_n) + a_2\omega_\lambda(fx, gx) + a_3\omega_\lambda(fx, gx_n)\\ &\quad + a_4\omega_\lambda(fx_n, gx) + a_5\omega_\lambda(gx, gx_n)\end{aligned}$$

for all $n \geq 1$. By taking the limit $n \to \infty$, we have

$$\begin{aligned}\omega_\lambda(gx, fx) &\leq a_1\omega_\lambda(gx, gx) + a_2\omega_\lambda(fx, gx) + a_3\omega_\lambda(fx, gx)\\ &\quad + a_4\omega_\lambda(gx, gx) + a_5\omega_\lambda(gx, gx)\\ &= (a_2 + a_3)\omega_\lambda(fx, gx).\end{aligned}$$

This implies that $(1 - a_2 - a_3)\omega_\lambda(fx, gx) \leq 0$ for all $\lambda > 0$, which is a contradiction. Thus $fx = gx$. Now, let $t = fx = gx$. Since f and g are weakly compatible mappings, we have $fgx = gfx$, which implies that $ft = fgx = gfx = gt$.

Now, we show that $gt = t$. Suppose $\omega_\lambda(gt, t) > 0$. Then, from (b), we have

$$\begin{aligned}\omega_\lambda(gt, t) &= \omega_\lambda(ft, fx)\\ &\leq a_1\omega_\lambda(ft, gt) + a_2\omega_\lambda(fx, gx) + a_3\omega_\lambda(fx, gt)\\ &\quad + a_4\omega_\lambda(ft, gx) + a_5\omega_\lambda(gx, gt)\\ &\leq a_3\omega_\lambda(t, gt) + a_4\omega_\lambda(gt, t) + a_5\omega_\lambda(t, gt)\\ &= (a_3 + a_4 + a_5)\omega_\lambda(gt, t).\end{aligned}$$

This implies that $(1 - a_3 - a_4 - a_5)\omega_\lambda(gt, t) \leq 0$ for all $\lambda > 0$, which is a contradiction. Thus t is a common fixed point of f and g.

For the uniqueness of the common fixed point, we suppose that u is another common fixed point in X_ω such that $fu = gu$. From (b), we have

$$\begin{aligned}\omega_\lambda(u, t) &= \omega_\lambda(gu, gt)\\ &= \omega_\lambda(fu, ft)\\ &\leq a_1\omega_\lambda(fu, gu) + a_2\omega_\lambda(ft, gt) + a_3\omega_\lambda(ft, gu)\\ &\quad + a_4\omega_\lambda(fu, gt) + a_5\omega_\lambda(gt, gu)\\ &\leq a_3\omega_\lambda(ft, gu) + a_4\omega_\lambda(gu, ft) + a_5\omega_\lambda(ft, gu)\\ &= (a_3 + a_4 + a_5)\omega_\lambda(u, t).\end{aligned}$$

This implies that $(1 - a_3 - a_4 - a_5)\omega_\lambda(u,t) \leq 0$ for all $\lambda > 0$, which is a contradiction. Thus $\omega_\lambda(u,t) = 0$ and so $u = t$. Hence f and g have a unique common fixed point.

By setting $g = I_{X_\omega}$, we deduce the following result of fixed point for one self-mapping from Theorem 3.

Corollary 1. *Let X_ω be an ω-complete modular metric space and $f : X_\omega \to X_\omega$ such that, for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x_0, fx_0) < \infty$ and*

$$\omega_\lambda(fx, fy) \leq a_1\omega_\lambda(fx, x) + a_2\omega_\lambda(fy, y) + a_3\omega_\lambda(fy, x) + a_4\omega_\lambda(fx, y) + a_5\omega_\lambda(x, y)$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, \frac{1}{4})$ with $\sum_{i=1}^{5} a_i < 1$. Then f has a unique fixed point z. Further, for any $x_0 \in X_\omega$, the Picard sequence $\{fx_n\}$ with an initial point x_0 is ω-convergent to the fixed point z.

Corollary 2. *Let X_ω be an ω-complete modular metric space and $f : X_\omega \to X_\omega$ such that, for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x_0, fx_0) < \infty$ and*

$$\omega_\lambda(fx, fy) \leq a_1\omega_\lambda(fx, x) + a_2\omega_\lambda(fy, y) + a_3\omega_\lambda(x, y)$$

where $a_1, a_2, a_3 \in [0, \frac{1}{4})$ with $0 \leq a_1 + a_2 + a_3 < 1$. Then f has a unique fixed point.

Corollary 3. *Let X_ω be an ω-complete modular metric space and $f : X_\omega \to X_\omega$ such that, for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x_0, fx_0) < \infty$ and*

$$\omega_\lambda(fx, fy) \leq a\omega_\lambda(x, y)$$

where $0 \leq a < 1$. Then f has a unique fixed point.

Now, we give some examples of the (CLR_g)-property as follows:

Example 2. *Let $X_\omega = [0, \infty)$ be a modular metric space. Define two mappings $f, g : X_\omega \to X_\omega$ by $fx = x + 4$ and $gx = 5x$ for all $x \in X_\omega$, respectively. Now, we consider the sequence $\{x_n\}$ defined by $x_n = \{1 + \frac{1}{n}\}$ for each $n \geq 1$. Since*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 5 = g(1) \in X_\omega,$$

f and g satisfy the (CLR_g)-property.

Example 3. *The conclusion of Example 2 remains true if the self-mappings f and g is defined on X_ω by $f(x) = \frac{x}{4}$ and $g(x) = \frac{x}{2}$ for all $x \in X_\omega$, respectively. Let a sequence $\{x_n\}$ be defined by $x_n = \{\frac{1}{n}\}$ in X_ω. Since*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g(0) \in X_\omega,$$

f and g satisfy the (CLR_g)-property.

4. Fixed point results for the strict contractive condition

Definition 7. *Let (X,ω) be a modular metric space and (f,g) be a pair of self-mappings on X_ω. For any $x,y \in X_\omega$, consider the following sets:*

$$\mathcal{M}_0^{f,g}(x,y) = \{\omega_\lambda(gx,gy), \omega_\lambda(gx,fx), \omega_\lambda(gy,fy), \omega_\lambda(gx,fy), \omega_\lambda(gy,fx)\},$$

$$\mathcal{M}_1^{f,g}(x,y) = \Big\{\omega_\lambda(gx,gy), \omega_\lambda(gx,fx), \omega_\lambda(gy,fy), \frac{\omega_\lambda(gx,fy)+\omega_\lambda(gy,fx)}{2}\Big\},$$

$$\mathcal{M}_2^{f,g}(x,y) = \Big\{\omega_\lambda(gx,gy), \frac{\omega_\lambda(gx,fx)+\omega_\lambda(gy,fy)}{2}, \frac{\omega_\lambda(gx,fy)+\omega_\lambda(gy,fx)}{2}\Big\}.$$

and define the following conditions:

(C1) *for any $x,y \in X_\omega$, there exists $\alpha_0(x,y) \in \mathcal{M}_0^{f,g}(x,y)$ such that*

$$\omega_\lambda(fx,fy) < \alpha_0(x,y),$$

(C2) *for any $x,y \in X_\omega$, there exists $\alpha_1(x,y) \in \mathcal{M}_1^{f,g}(x,y)$ such that*

$$\omega_\lambda(fx,fy) < \alpha_1(x,y),$$

(C3) *for any $x,y \in X_\omega$, there exists $\alpha_2(x,y) \in \mathcal{M}_2^{f,g}(x,y)$ such that*

$$\omega_\lambda(fx,fy) < \alpha_2(x,y).$$

These conditions are called the strict contractive conditions.

Definition 8. *Let (X,ω) be a modular metric space. Let f,g be self-mappings on X_ω. Then f is called a g-quasi-contraction if, for some constant $a \in (0,1)$, there exists $\alpha(x,y) \in \mathcal{M}_0^{f,g}(x,y)$ such that*

$$\omega_\lambda(fx,fy) \leq a\alpha(x,y)$$

for all $x,y \in X_\omega$.

Theorem 4. *Let X_ω be a modular metric space and $f,g : X_\omega \to X_\omega$ are weakly compatible mappings such that $f(X_\omega) \subset g(X_\omega)$ satisfies the condition (C3) for all $x,y \in X_\omega$ and $\lambda > 0$. If f and g satisfy the (CLR_g)-property, then f and g have a unique common fixed point.*

Proof. Since f and g satisfy the (CLR_g)-property, there exists a sequence $\{x_n\}$ in X_ω such that $\lim_{n\to\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = gx$ for some $x \in X_\omega$. From (C3), we have

$$\omega_\lambda(fx_n,fx) < \alpha_2(x_n,x),$$

where $\alpha_2(x_n,x) \in \mathcal{M}_2^{f,g}(x_n,x)$. Therefore, we have

$$\mathcal{M}_2^{f,g}(x_n,x) = \Big\{\omega_\lambda(gx_n,gx), \frac{\omega_\lambda(gx_n,fx_n)+\omega_\lambda(gx,fx)}{2}, \frac{\omega_\lambda(gx_n,fx)+\omega_\lambda(gx,fx_n)}{2}\Big\}.$$

Now, we show that $fx = gx$. Suppose that $fx \neq gx$. From (C3), we have the following three cases:

Case 1. $\omega_\lambda(fx_n, fx) < \alpha_2(x_n, x)$. Taking limit as $n \to \infty$, we have $\omega_\lambda(gx, fx) < \omega_\lambda(gx, gx) = 0$, which is a contradiction.

Case 2. $\omega_\lambda(fx_n, fx) < \dfrac{\omega_\lambda(gx_n, fx_n) + \omega_\lambda(gx, fx)}{2}$. Taking limit as $n \to \infty$, we have

$$\omega_\lambda(gx, fx) < \frac{\omega_\lambda(gx, gx) + \omega_\lambda(gx, fx)}{2} = \frac{1}{2}\omega_\lambda(gx, fx),$$

which is a contradiction.

Case 3. $\omega_\lambda(fx_n, fx) < \dfrac{\omega_\lambda(gx_n, fx) + \omega_\lambda(gx, fx_n)}{2}$. Taking limit as $n \to \infty$, we have

$$\omega_\lambda(gx, fx) < \frac{\omega_\lambda(gx, fx) + \omega_\lambda(gx, gx)}{2} = \frac{1}{2}\omega_\lambda(gx, fx),$$

which is a contradiction. Hence $gx = fx$ in all the cases. Let $t = fx = gx$. Since f and g are weakly compatible mappings, $fgx = gfx$, which implies that $ft = fgx = gfx = gt$.

Now, we show that $ft = t$. Suppose that $ft \neq t$. From (C3), we have

$$\omega_\lambda(ft, t) = \omega_\lambda(ft, fx) < \alpha_2(t, x),$$

where $\alpha_2(t, x) \in \mathcal{M}_2^{f,g}(t, x)$. Therefore, we have

$$\begin{aligned}\mathcal{M}_2^{f,g}(t, x) &= \left\{\omega_\lambda(gt, gx), \frac{\omega_\lambda(gt, ft) + \omega_\lambda(gx, fx)}{2}, \frac{\omega_\lambda(gt, fx) + \omega_\lambda(gx, ft)}{2}\right\} \\ &= \{\omega_\lambda(ft, t), 0, \omega_\lambda(ft, t)\}.\end{aligned}$$

So, we have only two possible cases:

Case 4. $\omega_\lambda(ft, t) < \omega_\lambda(ft, t)$, which is a contradiction.

Case 5. $\omega_\lambda(ft, t) < 0$, which is a contradiction.

Hence $ft = t = gt$. Therefor, t is a common fixed point of f and g.

For the uniqueness of the common fixed point, we suppose that u is another common fixed point in X_ω such that $fu = gu$. From (C3), we have

$$\omega_\lambda(t, u) = \omega_\lambda(gt, gu) = \omega_\lambda(ft, fu) < \alpha_2(t, u),$$

where $\alpha_2(t, u) \in \mathcal{M}_2^{f,g}(t, u)$. Therefore, we have

$$\begin{aligned}\mathcal{M}_2^{f,g}(t, u) &= \left\{\omega_\lambda(gt, gu), \frac{\omega_\lambda(gt, ft) + \omega_\lambda(gu, fu)}{2}, \frac{\omega_\lambda(gt, fu) + \omega_\lambda(gu, ft)}{2}\right\} \\ &= \{\omega_\lambda(gt, gu), 0, \omega_\lambda(gt, gu)\}.\end{aligned}$$

So, we have only two possible cases.

Case 6. $\omega_\lambda(gu, gt) < \omega_\lambda(gu, gt)$, which is a contradiction.

Case 7. $\omega_\lambda(gu, gt) < 0$, which is a contradiction.

Hence $gu = gt$. implies $u = t$ and so f and g have a unique common fixed point.

Theorem 5. *Let X_ω be a modular metric space and $f, g : X_\omega \to X_\omega$ be weakly compatible mappings such that f is the g-quasi-contraction for all $x, y \in X_\omega$ and $\lambda > 0$. If f and g satisfy (CLR_g)-property, then f and g have a unique common fixed point.*

Proof. Since f and g satisfy the (CLR_g)-property, there exists a sequence $\{x_n\}$ in X_ω such that $\lim_{n\to\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = gx$ for some $x \in X_\omega$. Since f is the g-quasi-contraction, we have

$$\omega_\lambda(fx_n, fx) \leq a\alpha_0(x_n, x),$$

where $\alpha_0(x_n, x) \in \mathcal{M}_0^{f,g}(x_n, x)$. Therefore, we have

$$\mathcal{M}_0^{f,g}(x_n, x) = \{\omega_\lambda(gx_n, gx), \omega_\lambda(gx, fx), \omega_\lambda(gx_n, fx_n), \omega_\lambda(gx_n, fx), \omega_\lambda(gx, fx_n)\}.$$

Now, we have the following five cases:

Case 1. $\omega_\lambda(fx_n, fx) \leq a\omega_\lambda(gx_n, gx)$. Taking the limit as $n \to \infty$, we have $gx = fx$.

Case 2. $\omega_\lambda(fx_n, fx) \leq a\omega_\lambda(gx_n, fx_n)$. Taking the limit as $n \to \infty$, we have $gx = fx$.

Case 3. $\omega_\lambda(fx_n, fx) \leq a\omega_\lambda(gx, fx)$. Taking the limit as $n \to \infty$, we have $gx = fx$.

Case 4. $\omega_\lambda(fx_n, fx) \leq a\omega_\lambda(gx, fx_n)$. Taking the limit as $n \to \infty$, we have $gx = fx$.

Case 5. $\omega_\lambda(fx_n, fx) \leq a\omega_\lambda(gx, fx_n)$. Taking the limit as $n \to \infty$, we have $gx = fx$.

Hence, in all the possible cases, $gx = fx$. Now, let $t = fx = gx$. Since f and g are weakly compatible mappings, it follows that $fgx = gfx$, which implies that $ft = fgx = gfx = gt$.

Now, we claim that $ft = t$. Since f is the g-quasi-contraction, we have

$$\omega_\lambda(ft, t) = \omega_\lambda(ft, fx) \leq a\alpha_0(t, x),$$

where $\alpha_0(t, x) \in \mathcal{M}_0^{f,g}(t, x)$. Therefore, we have

$$\begin{aligned}\mathcal{M}_0^{f,g}(t, x) &= \{\omega_\lambda(gt, gx), \omega_\lambda(gt, ft), \omega_\lambda(gx, fx), \omega_\lambda(gt, fx), \omega_\lambda(gx, ft)\} \\ &= \{\omega_\lambda(ft, t), 0, 0, \omega_\lambda(ft, t), \omega_\lambda(t, ft)\}.\end{aligned}$$

Now, we have two cases.

Case 6. $\omega_\lambda(ft, t) \leq a\omega_\lambda(ft, t)$. This implies $ft = t$.

Case 7. $\omega_\lambda(ft, t) \leq 0$. This implies $ft = t$.

Hence $ft = t = gt$ and so t is a common fixed point of f and g.

For the uniqueness of the common fixed point t, we suppose that u is another common fixed point in X_ω such that $fu = gu$. Since f is the g-quasi-contraction, we have

$$\omega_\lambda(gt, gu) = \omega_\lambda(ft, fu) \leq a\alpha_0(t, u),$$

where $\alpha_0(t, u) \in \mathcal{M}_0^{f,g}(t, u)$. Therefore, we have

$$\begin{aligned}\mathcal{M}_0^{f,g}(t, u) &= \{\omega_\lambda(gt, gu), \omega_\lambda(gt, ft), \omega_\lambda(gu, fu), \omega_\lambda(gt, fu), \omega_\lambda(gu, ft)\} \\ &= \{\omega_\lambda(ft, fu), 0, 0, \omega_\lambda(ft, fu), \omega_\lambda(fu, ft)\}\end{aligned}$$

So, we have only two possible cases.

Case 8. $\omega_\lambda(ft, fu) \leq a\omega_\lambda(ft, fu)$. This implies $ft = fu$.

Case 9. $\omega_\lambda(ft, fu) \leq 0$. This implies $ft = fu$.

Therefore, f and g have a unique common fixed point.

Example 4. *Let $X_\omega = (0, 1]$ with $\omega_\lambda(x, y) = \frac{1}{\lambda}|x-y|$ for all $\lambda > 0$. Consider the functions f and g defined by*

$$fx = \begin{cases} \frac{4}{5}, & \text{if} \quad x \in (0, \frac{4}{5}], \\ \\ \frac{1}{5}, & \text{if} \quad x \in (\frac{4}{5}, 1]. \end{cases}$$

$$gx = \begin{cases} 1 - \frac{x}{4}, & \text{if} \quad x \in (0, \frac{4}{5}], \\ \\ \frac{9}{10}, & \text{if} \quad x \in (\frac{4}{5}, 1]. \end{cases}$$

Choosing a sequences $\{x_n\} = \{\frac{4}{5} - \frac{1}{n}\}$, we can see that f and g enjoy the $(CLRg)$-property

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \frac{4}{5} = g(\frac{4}{5}).$$

Also,

$$f(\frac{4}{5}) = g(\frac{4}{5}) \quad \text{implies} \quad fg(\frac{4}{5}) = gf(\frac{4}{5}),$$

which shows that f and g are weakly compatible.

Case 1. *For each $x, y \in (0, \frac{4}{5}]$, we have*

$$\begin{aligned} \omega_\lambda(fx, fy) &= \frac{1}{\lambda}|fx - fy| \\ &= \frac{1}{\lambda}|\frac{4}{5} - \frac{4}{5}| \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_0^{f,g}(x, y) &= \{\omega_\lambda(gx, gy), \omega_\lambda(gx, fx), \omega_\lambda(gy, fy), \omega_\lambda(gx, fy), \omega_\lambda(gy, fx)\} \\ &= \{\frac{1}{\lambda}|gx - gy|, \frac{1}{\lambda}|gx - fx|, \frac{1}{\lambda}|gy - fy|, \frac{1}{\lambda}|gx - fy|, \frac{1}{\lambda}|gy - fx|\} \\ &= \{\frac{1}{\lambda}|1 - \frac{x}{4} - (1 - \frac{y}{4})|, \frac{1}{\lambda}|1 - \frac{x}{4} - \frac{4}{5}|, \frac{1}{\lambda}|1 - \frac{y}{4} - \frac{4}{5}|, \\ &\quad \frac{1}{\lambda}|1 - \frac{x}{4} - \frac{4}{5}|, \frac{1}{\lambda}|1 - \frac{y}{4} - \frac{4}{5}|\}. \end{aligned}$$

Thus, we obtain $\omega_\lambda(fx, fy) \leq a\alpha_0(x, y)$, where $a \in (0, 1)$.

Case 2. *For* $x \in (0, \frac{4}{5}]$ *and* $y \in (\frac{4}{5}, 1]$ *we have*

$$\begin{aligned} \omega_\lambda(fx, fy) &= \frac{1}{\lambda}|fx - fy| \\ &= \frac{1}{\lambda}|\frac{4}{5} - \frac{1}{5}| \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_0^{f,g}(x, y) &= \{\omega_\lambda(gx, gy), \omega_\lambda(gx, fx), \omega_\lambda(gy, fy), \omega_\lambda(gx, fy), \omega_\lambda(gy, fx)\} \\ &= \{\frac{1}{\lambda}|gx - gy|, \frac{1}{\lambda}|gx - fx|, \frac{1}{\lambda}|gy - fy|, \frac{1}{\lambda}|gx - fy|, \frac{1}{\lambda}|gy - fx|\} \\ &= \{\frac{1}{\lambda}|1 - \frac{x}{4} - \frac{9}{10}|, \frac{1}{\lambda}|1 - \frac{x}{4} - \frac{4}{5}|, \frac{1}{\lambda}|\frac{9}{10} - \frac{1}{5}|, \\ &\quad \frac{1}{\lambda}|1 - \frac{x}{4} - \frac{1}{5}|, \frac{1}{\lambda}|\frac{9}{10} - \frac{4}{5}|\}. \end{aligned}$$

Thus, we obtain $\omega_\lambda(fx, fy) \le a\alpha_0(x, y)$, *where* $a \in (0, 1)$.

Case 3. *For* $x \in (\frac{4}{5}, 1]$ *and* $y \in (0, \frac{4}{5}]$, *we have*

$$\begin{aligned} \omega_\lambda(fx, fy) &= \frac{1}{\lambda}|fx - fy| \\ &= \frac{1}{\lambda}|\frac{1}{5} - \frac{4}{5}| \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_0^{f,g}(x, y) &= \{\omega_\lambda(gx, gy), \omega_\lambda(gx, fx), \omega_\lambda(gy, fy), \omega_\lambda(gx, fy), \omega_\lambda(gy, fx)\} \\ &= \{\frac{1}{\lambda}|gx - gy|, \frac{1}{\lambda}|gx - fx|, \frac{1}{\lambda}|gy - fy|, \frac{1}{\lambda}|gx - fy|, \frac{1}{\lambda}|gy - fx|\} \\ &= \{\frac{1}{\lambda}|\frac{9}{10} - (1 - \frac{y}{4})|, \frac{1}{\lambda}|\frac{9}{10} - \frac{1}{5}|, \frac{1}{\lambda}|1 - \frac{y}{4} - \frac{4}{5}|, \\ &\quad \frac{1}{\lambda}|\frac{9}{10} - \frac{4}{5}|, \frac{1}{\lambda}|1 - \frac{y}{4} - \frac{1}{5}|\}. \end{aligned}$$

Thus, we obtain $\omega_\lambda(fx, fy) \le a\alpha_0(x, y)$, *where* $a \in (0, 1)$.

Case 4. *For each* $x, y \in (\frac{4}{5}, 1]$, *we have*

$$\begin{aligned} \omega_\lambda(fx, fy) &= \frac{1}{\lambda}|fx - fy| \\ &= \frac{1}{\lambda}|\frac{1}{5} - \frac{1}{5}| \end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_0^{f,g}(x,y) &= \{\omega_\lambda(gx,gy),\omega_\lambda(gx,fx),\omega_\lambda(gy,fy),\omega_\lambda(gx,fy),\omega_\lambda(gy,fx)\}\\ &= \{\frac{1}{\lambda}|gx-gy|,\frac{1}{\lambda}|gx-fx|,\frac{1}{\lambda}|gy-fy|,\frac{1}{\lambda}|gx-fy|,\frac{1}{\lambda}|gy-fx|\}\\ &= \{\frac{1}{\lambda}|\frac{9}{10}-\frac{1}{5}|,\frac{1}{\lambda}|\frac{9}{10}-\frac{1}{5}|,\frac{1}{\lambda}|\frac{9}{10}-\frac{1}{5}|,\\ &\frac{1}{\lambda}|\frac{9}{10}-\frac{1}{5}|,\frac{1}{\lambda}|1-\frac{9}{10}-\frac{1}{5}|\}.\end{aligned}$$

Thus, we obtain $\omega_\lambda(fx,fy)\le a\alpha_0(x,y)$, *where* $a\in(0,1)$.

Therefore, f *and* g *satisfy all conditions of Theorem 5 are satisfied and* $x=\frac{4}{5}$ *is the unique common fixed point of* f *and* g.

5. Some applications to Fredholm integral equations

The purpose of this section is to show the existence and uniqueness of a solution of Fredholm integral equations in modular metric spaces with a function space $(C(I,\mathbb{R}),\omega_\lambda)$ and a contraction by using our main results.

Consider the integral equation:

$$fx(t)-\mu\int_0^r K(t,s)hx(s)ds=g(t), \tag{5}$$

where $x:I\to\mathbb{R}$ is an unknown function, $g:I\to\mathbb{R}$ and $h,f:\mathbb{R}\to\mathbb{R}$ are two functions, μ is a parameter. The kernel K of the integral equation is defined by $I\times\mathbb{R}\to\mathbb{R}$, where $I=[0,r]$.

Theorem 6. *Let* K,f,g,h *be continuous. Suppose that* $C\in\mathbb{R}$ *is such that, for all* $t,s\in I$,

$$|K(t,s)|\le C$$

and, for each $x\in(C(I,\mathbb{R}),\omega_\lambda)$, *there exists* $y\in(C(I,\mathbb{R}),\omega_\lambda)$ *such that*

$$(fy)(t)=g(t)+\mu\int_0^r K(t,s)hx(s)ds$$

for all $r\in C(I,\mathbb{R})$. *If* f *is injective, there exists* $L\in\mathbb{R}$ *such that, for all* $x,y\in\mathbb{R}$,

$$|hx-hy|\le L|fx-fy|$$

and $\{fx:x\in(C(I,\mathbb{R}),\omega_\lambda)\}$ *is complete, then, for any* $\mu\in\big(-\frac{1}{CrL},\frac{1}{CrL}\big)$, *there exists* $w\in(C(I,\mathbb{R}),\omega_\lambda)$ *such that, for any* $x_0\in(C(I,\mathbb{R}),\omega_\lambda)$,

$$fw(t)=\lim_{x\to\infty}fx_n(t)=\lim_{x\to\infty}\Big[g(t)+\mu\int_0^r K(t,s)hx_{n-1}(s)ds\Big] \tag{6}$$

and w *is the unique solution of the equation* (5).

Proof. Let $X_\omega = Y_\omega = (C(I,\mathbb{R}),\omega_\lambda)$ and define $d(x,y) = \max_{t\in I}|x(t)-y(t)|$ for all $x,y \in X_\omega$. Let $T,S \in X_\omega \to X_\omega$ be the mappings defined as follows:

$$(Tx)(t) = g(t) + \mu\int_0^r K(t,s)(hx)(s)ds, \quad Sx = fx.$$

Then, by the assumptions, $S(X_\omega) = \{Sx : x \in X_\omega\}$ is complete. Let $x^* \in T(X_\omega)$ for any $x \in X_\omega$ and $x^*(t) = Tx(t)$. By the assumptions, there exists $y \in X_\omega$ such that $Tx(t) = fy(t)$ and hence $T(X_\omega) \subseteq S(X_\omega)$. Since

$$\begin{aligned}
\omega_\lambda(Tx,Ty) &= |\mu|\Big|\int_0^r [K(t,s)(hx)(s)ds] - \int_0^r [K(t,s)(hy)(s)ds]\Big| \\
&\le |\mu|\int_0^r c|(hx)(s)-(hy)(s)|ds \\
&\le L|\mu|C\int_0^r |(fx)(s)-(fy)(s)|ds \\
&\le L|\mu|C\int_0^r |(Sx)(s)-(Sy)(s)|ds \\
&\le \Big(\sup_{t\in I}|(Sx)(t)-(Sy)(t)|\Big)L|\mu|C\int_0^r ds \\
&\le L|\mu|Crd(Sx,Sy).
\end{aligned}$$

Therefore, for any $\mu \in \Big(-\frac{1}{CrL},\frac{1}{CrL}\Big)$, there exists a unique $w \in X_\omega$ such that

$$fw(t) = \lim_{x\to\infty} Sx_n(t) = \lim_{x\to\infty} Tx_{n-1}(t) = T(w)(t), \quad x_0 \in X_\omega$$

for all $t \in I$, which is the unique solution of the equation (5). So, S and T have a coincidence point in X_ω. Moreover, if either T or S is injective, then S and T have a unique coincidence point in X_ω.

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