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Some results on real valued continuous functions on an interval

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Abstract. In this paper, results mainly on the structures of some special class of real valued functions on an interval are discussed. Another significant result is also established. These results have been derived and presented mainly in the perspective of study of relations between real valued functions and their derivatives. The results are very fundamental in nature and may be useful in the next course of generalizations or improvements in this direction.

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1. Introduction

The notion of derivative of functions is a supreme fundamental concept in the differential and integral calculus and it is foundational for other powerful branches of mathematics like ordinary and partial differential equations, Numerical Analysis, dynamical systems, differential geometry etc. That is why the area of structural study of real valued functions is an interesting as well as important branch of Real Analysis. Results in this direction involve various differentiability structures on corresponding domains, and for different class of real valued functions. Differentiability and continuity properties of real valued functions of real variables have been studied vastly. For the primary reading, one can refer [2], and [3]. Further reading can be done with [4], and [1].
A new approaches to differential calculus based on mathematical structures are always being tried across the world in different directions and methods. This will lead to presenting continuous or differentiable functions and solving differential or integral equations. The concept of derivative of a real-valued function depends on the choice of the coordinate system used. For functions on finite Euclidean spaces, the derivative is an element of a countably based continuous domain which can be given an effective structure that characterizes various other properties like fixed points, additivity or any other special properties of functions.

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Results of the kind to be discussed in this article are considered to be very fundamental in nature and may be useful in pure and applied sciences. All the results are principally from the point of view of study on the characteristics of some special class of real valued functions, and the additional properties they enjoyed in. Let us present the results, which may be given in more generalized settings in the coming years.

2. Theorems and proofs

Based on the theoretic framework for differential calculus, one can embark on the task of constructing consequences of mean value theorems for subspace of Euclidean spaces, which would extend the set-theoretic model for computational geometry in solving models carved out of the physical problems. In this context, it is quite natural to have a class of real valued continuous functions defined on an interval I relating its extreme values. It is equally interesting to link this structure with some restricted class of continuous functions on I. We will first derive a result as a consequence of Intermediate Value Theorem.

Theorem 1. *Let f be a continuous real valued function on $(0,1)$ such that*

$$\max_{x\in(0,1)} f(x) > 0, \quad \min_{x\in(0,1)} f(x) < 0.$$

Then there exists $c \in (0,1)$ such that

$$cf(c) = 2\int_0^c tf(t)dt. \tag{1}$$

Proof. If $f = 0$, then there is nothing to prove. So assume that f is not a zero function. By hypothesis, there exists $a, b \in (0,1)$ such that $a \neq b$ and

$$f(a) = \max_{x\in(0,1)} f(x) > 0, \quad f(b) = \min_{x\in(0,1)} f(x) < 0.$$

Define

$$g(x) = xf(x) - 2\int_0^x tf(t)dt.$$

Clearly $g(x)$ is continuous on $(0,1)$. Now

$$g(a) \geq af(a) - 2\int_0^a tf(a)dt = af(a) - a^2 f(a).$$

But then

$$g(a) \geq af(a)(1-a) > 0.$$

Also

$$g(b) \leq bf(b) - 2\int_0^b tf(b)dt,$$

which is equivalent

$$g(b) \leq bf(b)(1-b) < 0,$$

since $f(b) < 0$.

By Intermediate Value Theorem there exists $c \in (a, b)$ such that $g(c) = 0$. Hence (1) is proved.

Real valued functions on the set of real numbers with any classical structure like continuity, bijection etc normally enjoy the properties like '0 going to 0', even-odd property or the additivity etc. But a typical function of the kind given below in the next theorem, being a 'translated' composition of itself exhibits these properties and that is why the result seems to be some what interesting to present over here.

Theorem 2. *If* $f : \Re \to \Re$ *defined by*

$$f(x + nf(y)) = f(x) + (n-1)y + f(y) \tag{2}$$

for $n > 1$, *then*

(a) $f(0) = 0$

(b) f *is an odd function*

(c) $f(x + y) = f(x) + f(y), \;\; \forall x, y \in \Re$.

Proof. Let us prove (a) first. Assume that $f(0) = a$. Now taking $x = 0, \; y = 0$, in (2)we have

$$f(na) = 2a.$$

Also if we take $x = 0, \; y = na$ in (2) we have

$$f(2na)) = 3a + (n-1)na. \tag{3}$$

Taking $x = na, y = 0$, again in (2) we have

$$f(2na) = 3a. \tag{4}$$

From (3) and (4) it follows that $a = 0$.

Let us prove the part (b) now. Taking $x = 0$, in (2), we get

$$f(nf(y)) = (n-1)y + f(y) \tag{5}$$

Now replacing x by $nf(x)$ in the definition and then using (5), we get

$$f(nf(x) + nf(y)) = f(nf(x)) + (n-1)y + f(y) = (n-1)(x+y) + f(x) + f(y). \tag{6}$$

Let $f(x) + f(-x) = g$. Now again taking x as $nf(x)$ and $y = -x$ in (2) and then using (6) we have

$$f(nf(x) + nf(-x)) = f(x) + f(-x),$$

which is equivalent to

$$f(ng) = g.$$

But then
$$f(nf(ng)) = f(ng). \tag{7}$$
Also from (5),
$$f(nf(ng)) = (n-1)ng + f(ng). \tag{8}$$
From (7) and (8) it follows that $g = 0$ since $n > 1$.
Lastly, we prove that f is an additive function.
Suppose
$$f(x+y) - f(x) - f(y) = h(x, y).$$
Now
$$f(nh) = f(-n(f(x)+f(y))+nf(x+y)) = f(-n(f(x)+f(y)))+(n-1)(x+y)+f(x+y),$$
by definition given in (2). Since f is an odd function, and using (6) we get
$$f(nh) = -f(n(f(x) + f(y))) + (n-1)(x+y) + f(x+y)$$
$$= -[(n-1)x + f(x) + (n-1)y + f(y)] + (n-1)(x+y) + f(x+y) = h(x, y),$$
and hence
$$f(nh) = h. \tag{9}$$
Also taking $y = nh$ in (5) we will have
$$f(nf(nh)) = (n-1)nh + f(nh).$$
By using (9) in the above expression we get
$$f(nh) = (n-1)nh + h. \tag{10}$$
The equations (9) and (10) make us to conclude $h = 0$. Thus the proof is complete.

In our next result, we show that a special type of functions associated with the continuous functions on an interval can take the shape of exponential functions of some continuous functions. As exponential functions are of great useful while studying differentiability structure and they in turn help us to get the corresponding flavor of uniformly continuous functions.

Theorem 3. *If f and g are real valued continuous functions on $[0, 1]$ such that $|g(x)| < |f(x)|$ for all $x \in [0, 1]$, then $f + g$ can be expressed as fe^{ψ} where ψ is a real valued continuous function on $[0, 1]$.*

Proof. Since $|g(x)| < |f(x)|$ for all $x \in [0, 1]$,then $f + g$ is always a non-zero function on $[0, 1]$. If $\phi = g/f$ then $|\phi(x)| < 1$, for all $x \in [0, 1]$. Therefore the range of $1 + \phi$ lies in the positive x-axis. Hence there exist a continuous function on D given by $\psi = \log(1 + \phi)$ such that $f + g = f(1 + \phi) = fe^{\psi}$.

Theorem 4. *Let $g : [0,1] \to \Re$ be a non-constant continuous function vanishing nowhere in its domain and for $x \in [0,1]$ define*

$$f(x) = \max\{g(y); 0 \le y \le x\}.$$

Then f is uniformly continuous function in its domain.

Proof. Take $g(x) = g(b)$ for all $x > b$. It can be easily seen that

$$|f(x) - f(y)| \le \max|g(s) - g(t)|$$

for all $s, t \in [x, y]$. But then

$$|f(x) - f(y)| \le \max|g(s) - g(t)|$$

whenever $|s - t| \le |x - y|$. So given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|s - t| < \delta \implies |g(s) - g(t)| < \epsilon$$

since g is a continuous real valued function on a compact set $[0,1]$ and thereby uniformly continuous. So using the same δ we have

$$|f(x) - f(y)| \le \max|g(s) - g(t)| < \epsilon$$

whenever $|s - t| \le |x - y| < \delta$. Therefore, f is uniformly continuous in its domain.

Our next result is originated from the characterization of continuous functions between metric spaces. The necessary and sufficient condition for a function $f : \Re \to \Re$ to be continuous is image of convergent sequences in the domain is convergent in the range and limits correspond each other can characterize the continuity only if the image of the convergent sequence converges? The below theorem is some what motivated by this question.

Theorem 5. *Let $f : \Re \to \Re$, be a connected map such that $\{x_n\} \to a \implies \{f(x_n)\} \to f(a)$ for any convergent sequences $\{x_n\}, \{f(x_n)\}$ in $\Re$. Then f must be continuous.*

Proof. By hypothesis, if $\{x_n\}$ is a sequence of terms in $\Re$ such that $x_n \to a$ then $f(x_n) \to f(a)$, provided $f(x_n)$ also converges. Note that if $x_n \to a$ but $f(x_n)$ does not converge then clearly $\{f(x_n)\}$ is an unbounded sequence.

Now we suppose that f fails to be continuous at $x = a$. Let $\delta > 0$ such that whenever $|x - a| < \delta$ we have

$$|f(x) - f(a)| < \alpha \tag{11}$$

for some fixed $\alpha > 0$ or

$$|f(x) - f(a)| > (|f(a)| + \alpha). \tag{12}$$

If such a δ does not exist, then we can find a sequence $\{x_n\} \to a$ such that

$$|f(x_n) - f(a)| \ge \alpha > 0, \text{or}|f(x_n)| \le (|f(a)| + \alpha)$$

for all n. But then the sequence $\{f(x_n)\}$ is bounded and it will have a subsequence converging to a limit other than $f(a)$. But this is not possible. Therefore our claim of existence of (11) and (12) is correct.
Let us define a function

$$g(x, f(x)) = |f(x)|$$

for $|x - a| \leq \delta$, where δ is as given previously. Clearly g is a continuous function. Then g has image points in the set $(-\infty, |f(a)| + \alpha) \bigcup (|f(a)| + \alpha, \infty)$. The point $g(a, f(a)) = |f(a)| \in (-\infty, |f(a)| + \alpha)$ and the unboundedness of f near a imply that there are points of the image of g in $(|f(a)| + \alpha, \infty)$. Therefore $g\,(\Re \times f(\Re))$ must be disconnected. Since g is continuous, $\Re \times f(\Re)$ must also be disconnected. This is a contradiction to the fact $\Re \times f(\Re)$ is connected being a product of two connected sets. Hence our assumption is wrong and f has no discontinuity. Hence the proof.

Finally we will present an inequality involving real numbers and their behaviour on the unit circle in L^p settings. It is quite interesting to see such results with a 'non-normed' structure.

Theorem 6. *If* $0 \leq a \leq 1$, $0 < c \leq b \leq 1$ *and* $p > 0$ *then*

$$\frac{a+b}{\left\{\int_0^{2\pi} |e^{i\theta} + b|^p d\theta\right\}^{1/p}} \geq \frac{a+c}{\left\{\int_0^{2\pi} |e^{i\theta} + c|^p d\theta\right\}^{1/p}}. \tag{13}$$

Proof. To prove the Inequality (13), it suffices to show that

$$\int_0^{2\pi} \left(\frac{|e^{i\theta} + b|}{a+b}\right)^p d\theta \leq \int_0^{2\pi} \left(\frac{|e^{i\theta} + c|}{a+c}\right)^p d\theta,$$

for which we will show

$$\left(\frac{|e^{i\theta} + b|}{a+b}\right)^p \leq \left(\frac{|e^{i\theta} + c|}{a+c}\right)^p, \tag{14}$$

for any $\theta \in [0, 2\pi]$ and $0 \leq a \leq 1$, $c \leq b \leq 1$.

Let us consider the function $f : (0,1] \mapsto R$ defined by $f(x) = \frac{|e^{i\theta}+x|}{a+x}$ defined on $(0,1]$, and show that f is non-increasing. Observe that

$$f'(x) \leq 0 \text{ if and only if } x(a - \cos\theta) + a\cos\theta - 1 \leq 0.$$

A simple exercise makes us conclude for any real values of θ, we have $x(a-\cos\theta)+a\cos\theta-1 \leq 0$, implying that f and f^p are non-increasing. Hence the Inequality (14) follows and thus the proof is complete.

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