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# On N(k)-Mixed Quasi Einstein Manifolds

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**Abstract.** In this paper N(k)-Mixed Quasi Einstein Manifolds( $N(k)-(MQE)_n$ )are introduced and the existence of these manifolds is proved. We give hyper surfaces of Euclidean spaces as examples of $N(k)-(MQE)_n$ and semi symmetric, ricci symmetric and ricci recurrent $N(k)-(MQE)_n$ manifolds are studied.

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**Key Words and Phrases**: N(k)-mixed quasi Einstein, mixed quasi constant curvature, ricci recurrent, semi symmetric, ricci symmetric

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## 1. Introduction

M.C.Chaki and R.K.Maity [1] introduced the concept quasi Einstein manifolds. A non-flat Riemannian manifold$(M^n, g)(n > 2)$ is said to be a quasi Einstein manifold if its ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where $a$ and $b$ are smooth functions of which $b \neq 0$ and $A$ is a non zero 1-form such that $g(X,U) = A(X)$, for all vector fields $X$ and $U$ is a unit vector field. U.C.De and Gopal Chandra Ghosh [4, 5] generalized the quasi Einstein manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be a generalized quasi Einstein manifold if its ricci tensor $S$ of type $(0,2)$ is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

where $a$, $b$ and $c$ are certain smooth functions, $A$ and $B$ are non zero 1-forms, and $U$ and $V$ are unit vector fields corresponding to 1-forms $A$ and $B$ respectively such that $g(X,U) = A(X)$, $g(X,V) = B(X)$ and $g(U,V) = 0$. The vector fields $U$ and $V$ are called generators of

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*Email address:* `hgnraj@yahoo.com` (H. Nagaraja) the quasi Einstein manifold. The k-nullity distribution $N(k)$ [8]of a Riemannian manifold $M$ is defined by

$$N(k) : p \to N_p(k) = \{Z \in T_pM \backslash R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for all $X, Y \in TM$ and $k$ is a smooth function.
M.M.Tripathy and Jeong - Jik Kim [6] introduced the notion of $N(k)$-quasi Einstein manifold which is defined as follows: If the generator $U$ belongs to the $k$-nullity distribution $N(k)$, then a quasi Einstein manifold $(M^n, g)$ is called $N(k)$-quasi Einstein manifold. Motivated by the above definitions we give the following definition.

**Definition 1.** *Let $(M^n, g)$ be a non flat Riemannian manifold. If the ricci tensor $S$ of $(M^n, g)$ is non zero and satisfies*

$$S(X,Y) = ag(X,Y) + bA(X)B(Y) + cB(X)A(Y), \tag{1}$$

*where $a, b$ and $c$ are smooth functions and $A$ and $B$ are non zero 1-forms such that $g(X,U) = A(X)$ and $g(X,V) = B(X)$ for all vector fields $X$, and $U$ and $V$ being the orthogonal unit vector fields called generators of the manifold belong to $N(k)$, then we say that $(M^n, g)$ is a $N(k)$-mixed quasi Einstein manifold and is denoted by $N(k) - (MQE)_n$.*

In this paper we introduce another notion of a manifold of mixed quasi constant curvature similar to manifold of quasi constant curvature defined in [4]. A Riemannian manifold $(M^n, g)$ is called a manifold of mixed quasi constant curvature if it is conformally flat and the curvature tensor $'R$ of type (0,4) satisfies the condition

$$\begin{aligned} 'R(X,Y,Z,W) = {} & p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + q\left[g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W) + g(X,W)A(Z)B(Y)\right. \\ & \left. - g(X,Z)A(W)B(Y)\right] + s[g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X) \\ & + g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)] \end{aligned} \tag{2}$$

Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold.
Taking $X = W = e_i$ and summing over $i, 1 \le i \le n$ in (2), we obtain

$$\begin{aligned} S(Y,Z) = {} & (n-1)pg(Y,Z) + (n-1)q\,[A(Y)B(Z) + A(Z)B(Y)] \\ & + s\left[2g(Y,Z) - A(Z)B(Y) - A(Y)B(Z)\right] \end{aligned}$$

which implies

$$S(Y,Z) = ag(Y,Z) + bA(Y)B(Z) + cA(Z)B(Y) \tag{3}$$

where $b = c = (n-1)q - s, a = (n-1)p + 2s$. i.e. the space $(M^n, g)$is mixed quasi Einstein. Thus we have

**Theorem 1.** *A manifold of mixed quasi constant curvature is a mixed quasi Einstein manifold.*

Conversely suppose $(M^n, g)$ is conformally flat mixed quasi Einstein manifold. Then

$$\begin{aligned} R(X,Y)Z = & \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} \\ & -\frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{4}$$

Here $Q$ is Ricci operator defined by $S(X,Y) = g(QX,Y)$.
From the above equation, we get

$$\begin{aligned} {}'R(X,Y,Z,W) = & g(R(X,Y)Z,W) \\ = & \frac{1}{n-2}\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\} \\ & -\frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \end{aligned} \tag{5}$$

Taking $X = Y = e_i$ and taking summation over $i, 1 \le i \le n$ in (1), we obtain $r = na$. Substituting this in (5) and using (1), we get

$$\begin{aligned} R(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ q[g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W) + g(X,W)A(Z)B(Y) \\ &- g(X,Z)A(W)B(Y)] + s[g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X) \\ &+ g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)] \end{aligned}$$

where $p = \dfrac{a}{n-1}$, $q = \dfrac{b}{n-2}$, $s = \dfrac{c}{n-2}$.
i.e. $(M^n, g)$ is a manifold of mixed quasi constant curvature.

## 2. Existence Theorem of a $N(k)$-mixed Quasi Einstein Manifolds

**Theorem 2.** *If in a conformally flat Riemannian manifold $(M^n, g)$, the ricci tensor $S$ satisfies the relation*

$$S(X,Z)g(Y,W) - S(Y,Z)g(X,W) = \beta(g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \tag{6}$$

*where $\beta$ is a non zero scalar, then $(M^n, g)$ is a $N(k)$-mixed quasi Einstein manifold.*

*Proof.* Let $U$ be a vector field defined by $g(X,U) = A(X)$, $\forall X \in TM$.
Taking $X = W = U$ in (6), we obtain

$$S(Y,Z) = ag(Y,Z) + bA(Y)B(Z) + cA(Z)B(Y) \tag{7}$$

where $a = \dfrac{-\alpha\beta}{u}$, $\alpha = S(U,U)$, $u = g(U,U)$, $b = \frac{1}{u}$, $c = \frac{\beta}{u}$,
and $S(U,Z) = S(Z,U) = g(QZ,U) = A(QZ) = B(Z)$. Therefore $(M^n, g)$ is mixed quasi Einstein.

If $(M^n, g)$ is conformally flat, then taking $Z = U$ in (4), we obtain

$$R(X,Y)U = \frac{1}{n-2}\{A(Y)QX - A(X)QY + S(Y,U)X - S(X,U)Y\} \tag{8}$$

$$-\frac{r}{(n-1)(n-2)}\{A(Y)X - A(X)Y\} \tag{9}$$

Taking $\beta = 1$ in (6), we get

$$S(X,Z)g(Y,W) - S(Y,Z)g(X,W) - g(Y,Z)S(X,W) + g(X,Z)S(Y,W) = 0$$

Taking $Z = U$ in the above equation, we obtain

$$S(X,U)g(Y,W) - S(Y,U)g(X,W) - A(Y)S(X,W) + A(X)S(Y,W) = 0,$$

which can be rewritten as $g(S(X,U)Y - S(Y,U)X - A(Y)QX + A(X)QY, W) = 0$, $\forall W$.
Therefore we have $S(X,U)Y - S(Y,U)X - A(Y)QX + A(X)QY = 0$.
Substituting this in (8), we get $R(X,Y)U = k(A(Y)X - A(X)Y)$, where $k = \frac{-r}{(n-1)(n-2)}$.

Therefore we have $U \in N_p(k)$, where $k = \frac{-r}{(n-1)(n-2)}$.
Suppose $V$ is a unit vector field orthogonal to $U$. Then, we have $V \in N_p(k)$.
Hence $(M^n, g)$ is a $N(k)$-mixed quasi Einstein manifold.

As it is well known that a 3-dimensional Riemannian manifold is conformally flat.

Thus we have

**Corollary 1.** *A 3- dimensional manifold is* $N\left(\frac{-r}{(n-1)(n-2)}\right)$*-mixed quasi Einstein manifold provided* (6) *holds.*

## 3. Example of a $N(k) - (MQE)_n$ manifold

Let $(M^n, \tilde{g})$ be a hypersurface of the Euclidean space $E^{n+1}$. Let $A$ be a (1,1) tensor corresponding to the normal valued second fundamental tensor $H$.

$$\tilde{g}(A_\xi(X), Y) = g(H(X,Y), \xi) \tag{10}$$

where $\xi$ is a unit normal vector field and X and Y are tangent vector fields.
Further

$$H_\xi(X,Y) = \tilde{g}(A_\xi(X), Y) \tag{11}$$

The hypersurface $(M^n, \tilde{g})$ is quasi umbilical if

$$H_\xi(X,Y) = \alpha\tilde{g}(X,Y) + \beta C(X)D(Y) \tag{12}$$

In view of (10), we have

$$H(X,Y) = \alpha g(X,Y)\xi + \beta C(X)D(Y)\xi. \tag{13}$$

The Gauss equation of $M^n$ in $E^{n+1}$ can be written as

$$\tilde{g}(\tilde{R}(X,Y)Z,W) = \tilde{g}(H(X,W),H(Y,Z)) - \tilde{g}(H(W,Y),H(Z,X)) \tag{14}$$

From (12) and (14), we have

$$\begin{aligned} '\tilde{R}(X,Y,Z,W) \;&=\; \alpha^2 g(X,W)g(Y,Z) + \alpha\beta g(X,W)C(Y)D(Z) \\ &\quad + \alpha\beta g(Y,Z)C(X)D(W) + \beta^2 C(X)C(Y)D(W)D(Z) \\ &\quad - \alpha^2 g(W,Y)g(Z,X) - \alpha\beta g(W,Y)C(Z)D(X) \\ &\quad - \alpha\beta g(Z,X)C(W)D(Y) - \beta^2 C(W)D(Y)C(Z)D(X) \end{aligned}$$

Contracting the above equation with $X = W = e_i$ and taking summation over $i, 1 \leq i \leq n$, we obtain

$$\tilde{S}(Y,Z) = ag(Y,Z) + bC(Y)D(Z) + cC(Z)D(Y)$$

where $a = (n-1)\alpha^2, b = (n-1)\alpha\beta + \beta^2, c = -\beta(2\alpha + \beta)$.
Hence $(M^n, \tilde{g})$ is a mixed quasi Einstein manifold.
Suppose $U$ and $V$ are unit orthogonal vectorfields corresponding to the 1-forms $C$ and $D$ respectively. Then putting $Z = U$ in (13), we get

$$H(X,U) = \alpha C(X)\xi. \tag{15}$$

Putting $Z = U$ in (14) and using (15), we get

$$\tilde{R}(X,Y)U = k(C(Y)X - C(X)Y)$$

where $k = \alpha^2$. Similarly we can show that

$$\tilde{R}(X,Y)V = k(D(Y)X - D(X)Y)$$

where $k = \alpha^2$. Thus we have

**Theorem 3.** *A quasi umbilical hypersurface of a Euclidean space $E^{n+1}$is a $N(k)$-mixed quasi Einstein manifold.*

## 4. Ricci Curvature, Eigen Vectors and Associated Scalars of a $N(k) - (MQE)_n$

From (1) we have $S(U,U) = a = S(V,V)$, $b = S(U,V) = S(V,U) = c$ , since $g(U,V) = 0$. Therefore only one of $b$ or $c$ is sufficient to define a mixed quasi Einstein space. A mixed quasi Einstein space may be defined as a Riemannian manifold in which ricci tensor $S$ satisfies

$$S(X,Y) = ag(X,Y) + b(A(X)B(Y) + B(X)A(Y)),$$

It is well known that for a unit vector field $X$, $S(X,X)$ is the ricci curvature in the direction of $X$. Now if $X$ is a unit vector field in the section spanned by $U$ and $V$, then we have

$$1 = g(X,X) = g(\alpha U + \beta V, \alpha U + \beta V) = \alpha^2 + \beta^2,$$

since $g(U,V)=0$ and $g(U,U)=g(V,V)=1$. Now

$$\begin{aligned} S(X,X) &= S(\alpha U+\beta V,\alpha U+\beta V) \\ &= a+2bA(X)B(X). \end{aligned}$$

Thus we can state that

**Theorem 4.** *In a $N(k)-(MQE)_n$ manifold, the ricci curvature in the direction of both $U$ and $V$ is 'a' and the ricci curvature in all other directions of the section of $U$ and $V$ is $a+2bA(X)B(X)$.*

Let $(M^n,g)$ be a $N(k)-(MQE)_n$ manifold.
Then $S(U,U)=S(V,V)=a$ from which we get $g(QU,U)=g(QV,V)=a$.
Since $U,V\in N_p(k)$, we have,

$$g(R(X,Y)U,W)=k\left\{A(Y)g(X,W)-A(X)g(Y,W)\right\}.$$

Putting $X=W=e_i$ and taking summation over $i, 1\le i\le n$ , we obtain

$$S(Y,U)=(n-1)kA(X) \tag{16}$$

Similarly we can get

$$S(Y,V)=(n-1)kB(X) \tag{17}$$

From (1), we have

$$S(X,U)=aA(X)+bB(X) \tag{18}$$

$$S(X,V)=bA(X)+aB(X) \tag{19}$$

Substracting (17) from (16) and (19) from (18), and comparing the resulting equations, we obtain

$$k=\frac{a-b}{n-1}.$$

Therefore

$$S(X,U)=(a-b)g(X,U)$$

and

$$S(X,V)=(a-b)g(X,V).$$

Therefore $U$ and $V$ are eigen vectors corresponding to the eigen value $(a-b)$.

## 5. Semi Symmetric and Ricci Symmetric $N(k)-(MQE)_n$ Manifolds

A Riemannian manifold $(M^n,g)$ is semi symmetric if $R(X,Y).R=0,\forall X,Y\in TM$ Since $U$ and $V$ are in $N_p(k)$ , we have

$$R(X,Y)U=k\left(A(Y)X-A(X)Y\right) \tag{20}$$

$$R(X,Y)V=k\left(B(Y)X-B(X)Y\right) \tag{21}$$

The equation (20) is equivalent to

$$R(U,Y)Z = k\left(g(Y,Z)U - A(Z)Y\right) \tag{22}$$

$$R(X,U)Z = k\left(A(Z)X - g(X,Z)U\right) \tag{23}$$

The equation (21) is equivalent to

$$R(V,Y)Z = k\left(g(Y,Z)V - B(Z)Y\right) \tag{24}$$

$$R(X,V)Z = k\left(B(Z)X - g(X,Z)V\right) \tag{25}$$

If $(M^n, g)$ is semi symmetric then we have

$$R(X,Y)R(Z,W)T - R(R(X,Y)Z,W)T - R(Z,R(X,Y)W)T - R(Z,W)R(X,Y)T = 0 \tag{26}$$

Putting $X = U$ and $T = V$ in (26), then using (21) and (22), we get

$$k^2\left\{2A(Z)B(Y)W + A(W)B(Z)Y - 2B(Z)g(Y,W)U\right\} = 0 \tag{27}$$

From (27),we have
If $k \neq 0$, then $2A(Z)B(Y)W + A(W)B(Z)Y = 2B(Z)g(Y,W)U, \forall Y,Z,W \in TM$ holds.
Putting $Z = V$ in the above equation, we get

$$g(Y,W)U = A(W)Y$$

Taking covariant derivative on both sides of the above equation with respect to $Z$, we obtain

$$g(X,Y)\nabla_Z U = (ZA(Y)X - A(Y))\,\nabla_Z X, \forall X,Y$$

Putting $Y = V$, we get $B(X)\nabla_Z U = 0$.
Since $B(X) \neq 0$, we obtain $\nabla_Z U = 0$.
i.e. $U$ is a parallel vector field.
Similarly by taking $X = V$ and $T = U$ in (26),
we obtain $\nabla_Z V = 0$.
i.e. $V$ is a parallel vector field.

Conversely suppose that $U$ and $V$ are parallel vector fields. Then $\nabla_Z U = 0$ and $\nabla_Z V = 0$, which then imply that
$R(X,Y)U = 0$ and $R(X,Y)V = 0$.
Substituting this in (26) with $X = U$, we obtain $R(U,X).R = 0$.
Similarly we get $R(V,X).R = 0$.
Thus we can state that

**Theorem 5.** *A $N(k) - (MQE)_n$ manifold with $k \neq 0$ satisfies $R(U,X).R = 0$ $(orR(V,X).R = 0)$ if and only if $U\,(orV)$ is a parallel vector field.*

Let $(M^n, g)$ be a $N(k)-(MQE)_n$ ricci semi symmetric manifold. Then we have

$$S(R(X,Y)Z,W)+S(Z,R(X,Y)W=0 \tag{28}$$

Putting $X = V$ in (28) we obtain

$$k\left\{g(Y,Z)S(V,W)-B(Z)S(Y,W)+g(Y,W)S(Z,V)-B(W)S(Z,Y)\right\}=0$$

Putting $W = V$ in the above equation, we get

$$k\left[S(Z,Y)-ag(Y,Z)+bA(Y)B(Z)-bA(Z)B(Y)\right]=0$$

If $k \neq 0$ then we have S(Z, Y) = a g(Y, Z) - b A(Y) B(Z) + b A(Z) B(Y).
Comparing this with (1), we obtain $b+c=0$.
But we have $b-c=0$, { section 4 }
Therefore $b=0$ and $c=0$. i.e. $(M^n, g)$ reduces to Einstein space which it is not.
Therefore we must have $k=0$.
Conversely suppose $k=0$. Then we obtain $R(V,X)Y=0$ which implies $R(V,X).S=0$. Similarly, we have, $R(U,X).S=0$. if and only if $k=0$.
Thus we have,

**Theorem 6.** *A $N(k)-(MQE)_n$ manifold satisfies $R(V,X).S=0$. and $R(U,X).S=0$ if and only if $k=0$.*

## 6. Ricci Recurrent $N(k)-(MQE)_n$ Manifolds

Let $(M^n, g)$ be a $N(k)-(MQE)_n$ manifold. If $U$ and $V$ are parallel vector fields, then $\nabla_X U=0$ and $\nabla_X V=0$.
From which we get that $R(X,Y)U=0$ and $R(X,Y)U=0$. Therefore

$$S(X,U)=0, S(X,V)=0 \tag{29}$$

From (1), we have

$$S(X,U)=aA(X)+bB(X)\text{and} \tag{30}$$

$$S(X,V)=aB(X)+bA(X) \tag{31}$$

From (29), (30) and (31), we have $a=b$.
Therefore we can rewrite the equation (1) in the following form:

$$S(X,Y)=a\left\{g(X,Y)+A(X)B(Y)+B(X)A(Y)\right\}.$$

Taking the covariant derivative of the above equation with respect to $Z$, we obtain

$$\nabla_Z S(X,Y)=da(Z)\left\{g(X,Y)+A(X)B(Y)+B(X)A(Y)\right\}$$

since $\nabla_X U = 0$ and $\nabla_X V = 0$ imply that $\nabla_Z A(X) = 0$ and $\nabla_Z B(X) = 0$.
Therefore $(\nabla_Z S)(X,Y) = \frac{da(Z)}{a} S(X,Y)$,
i.e. the manifold $(M^n, g)$ is ricci recurrent.
Conversely, suppose that $N(k) - (MQE)_n$ manifold is ricci recurrent. Then

$$\left(\nabla_X S\right)(Y,Z) = D(X)S(Y,Z), D(X) \neq 0.$$

But

$$\left(\nabla_X S\right)(Y,Z) = XS(Y,Z) - S\left(\nabla_X Y, Z\right) - S\left(Y, \nabla_X Z\right)$$

Therefore

$$D(X)S(Y,Z) = XS(Y,Z) - S\left(\nabla_X Y, Z\right) - S\left(Y, \nabla_X Z\right)$$

Putting $Y = Z = U$, we obtain
$Xa - aD(X) = 2a\left(g\left(\nabla_X U, U\right) + B(\nabla_X U)\right)$
i.e. $(da - aD)X = 2aB(\nabla_X U)$. since $g(U,U) = 1$ implies $g(\nabla_X U, U) = 0$
Therefore $B(\nabla_X U) = 0$ if and only if

$$(da)(X) = aD(X) \tag{32}$$

But $B(\nabla_X U) = 0$ implies that either $U$ is a parallel vector field or $\nabla_X U \perp V$.
Similarly we have, if (32) holds then either $V$ is a parallel vector field or $\nabla_X V \perp U$.
Thus we can state that

**Theorem 7.** *A $N(k)(MQE)_n$ manifold, where the generators $U$ and $V$ are parallel is a ricci recurrent manifold.Conversely suppose that $N(k) - (MQE)_n$ manifold is ricci recurrent, then either the vector field U ( or V) is parallel or $\nabla_X U \perp V$ ( or $\nabla_X V \perp U$).*

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