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Common fixed point theorems for three maps in cone pentagonal metric spaces

Abba Auwalu[1,*], Evren Hınçal[1]

[1] *Department of Mathematics, Near East University, Nicosia-TRNC, Turkey*

Abstract. In this paper, we prove some common fixed point theorems of three self mappings in non-normal cone pentagonal metric spaces. Our results extend and improve the recent results announced by many authors.

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1. Introduction

The concept of metric space was introduced by Fŕechet [8]. Let (X, d) be a metric space and $S : X \to X$ be a mapping. Then S is called Banach contraction if there exists $\alpha \in [0, 1)$ such that

$$d(Sx, Sy) \le \alpha d(x, y), \quad \text{for all } x, y \in X. \tag{1}$$

Banach [7] proved that if X is complete, then every Banach contraction mapping has a fixed point. The mapping S is called Kannan contraction if there exists $\alpha \in [0, 1/2)$ such that

$$d(Sx, Sy) \le \alpha\big[d(x, Sx) + d(y, Sy)\big], \quad \text{for all } x, y \in X. \tag{2}$$

Kannan [14] proved that if X is complete, then every Kannan contraction has a fixed point. He further showed that the conditions (1) and (2) are independent of each other (see, [14, 15]).
The study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction and Kannan contraction principles in various generalized metric spaces (e.g., see [4, 5, 6, 9, 10, 11, 13, 18]).

Long-Guang and Xian [11] introduced the concept of a cone metric space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many

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*Corresponding author.

Email addresses: abba.auwalu@neu.edu.tr, abbaauwalu@yahoo.com (A. Auwalu), evren.hincal@neu.edu.tr, evrenhincal@yahoo.co.uk (E. Hınçal)

authors have (for e.g., [1, 6, 9, 12, 17, 19]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Recently, Garg and Agarwal [9] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Motivated and inspired by the results of [9, 17, 16], it is our purpose in this paper to continue the study of common fixed points of a three self mappings in non-normal cone pentagonal metric space setting. Our results extend and improve the results of [2, 3, 6, 9, 13, 18, 17, 16], and many others.

2. Preliminaries

The following definitions and Lemmas, introduced in [1, 3, 6, 9, 11], are needed in the sequel.

Definition 1. *Let E be a real Banach space and P subset of E. P is called a cone if and only if:*

(i) P *is closed, nonempty, and* $P \neq \{0\}$;

(ii) $a, b \in \mathbb{R}, \quad a, b \geq 0$ *and* $x, y \in P \Longrightarrow ax + by \in P$;

(iii) $x \in P$ *and* $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

A cone P is called normal if there is a number $k \geq 1$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \tag{3}$$

The least positive number k satisfying (3) is called the normal constant of P.
In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2. *Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:*

(i) $0 < \rho(x, y)$ *for all* $x, y \in X$ *and* $\rho(x, y) = 0$ *if and only if* $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$;

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ *for all* $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [11]).

Definition 3. *Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:*

(i) $0 < \rho(x, y)$ *for all* $x, y \in X$ *and* $\rho(x, y) = 0$ *if and only if* $x = y$;

(ii) $\rho(x, y) = \rho(y, x)$ *for all* $x, y \in X$;

(iii) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ *for all* $x, y, z \in X$ *and for all distinct points* $w, z \in X - \{x, y\}$ *[Rectangular property].*

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 1. *Every cone metric space is cone rectangular metric space. The converse is not necessarily true* (e.g., see [6]).

Definition 4. *Let X be a non empty set. Suppose the mapping $d : X \times X \to E$ satisfies:*

(i) $0 < d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for* $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ *for all* $x, y, z, w, u \in X$ *and for all distinct points* $z, w, u, \in X - \{x, y\}$ *[Pentagonal property].*

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Remark 2. *Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true* (e.g., see [9]).

Let (X, d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in X. If every Cauchy sequence is convergent in X, then X is called a complete cone pentagonal metric space.

Definition 5. *Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi : P \to P$ satisfying:*

(i) $0 < \varphi(t) < t$ *for all* $t \in P \setminus \{0\}$,

(ii) the series $\sum_{n\geq 0} \varphi^n(t)$ *converge for all* $t \in P \setminus \{0\}$

From (i), *we have* $\varphi(0) = 0$, *and from* (ii), *we have* $\lim_{n\to 0} \varphi^n(t) = 0$ *for all* $t \in P \setminus \{0\}$.

Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S. Also, T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

Lemma 1. *Let T and S be weakly compatible self mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

Lemma 2. *Let (X, d) be a cone metric space with cone P not necessary to be normal. Then for $a, c, u, v, w \in E$, we have*

(i) If $a \leq ha$ and $h \in [0, 1)$, then $a = 0$.

(ii) If $0 \leq u \ll c$ for each $0 \ll c$, then $u = 0$.

(iii) If $u \leq v$ and $v \ll w$, then $u \ll w$.

Lemma 3. *Let (X, d) be a complete cone pentagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:*

(i) $x_n \neq x_m$ for all $n, m > N$;

(ii) x_n, x are distinct points in X for all $n > N$;

(iii) x_n, y are distinct points in X for all $n > N$;

(iv) $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Then $x = y$.

3. Main Results

In this section, we prove Banach type and Kannan type contraction principles in cone pentagonal metric spaces of a three self mappings. We give some examples to illustrate the results.

Theorem 1. *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, fy) \leq \varphi\big(d(gx, gy)\big), \tag{4}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \cup f(X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X, then the mappings S, f and g have a unique point of coincidence in X. Moreover, if (S, g) and (f, g) are weakly compatible then S, f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $S(X) \cup f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $gx_1 = Sx_0$. Also we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} such that

$$gx_{n+1} = Sx_n \text{ and } gx_{n+2} = fx_{n+1}, \text{ for all } n = 0, 1, 2, \cdots .$$

If $gx_n = gx_{n+1}$, then $gx_n = Sx_n = fx_n$, and x_n is a coincidence point of S, f and g. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, from (4), it follows that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= \varphi\big(d(Sx_{n-1}, fx_n)\big) \\ &\leq \varphi\big(d(gx_{n-1}, gx_n)\big) \\ &\leq \varphi^2\big(d(gx_{n-2}, gx_{n-1})\big) \\ &\vdots \\ &\leq \varphi^n\big(d(gx_0, gx_1)\big). \end{aligned} \tag{5}$$

In similar way, it again follows that

$$d(gx_n, gx_{n+2}) \leq \varphi^n\big(d(gx_0, gx_2)\big), \tag{6}$$

$$d(gx_n, gx_{n+3}) \leq \varphi^n\big(d(gx_0, gx_3)\big). \tag{7}$$

Similarly, for $k = 1, 2, 3, \cdots$, it further follows that

$$d(gx_n, gx_{n+3k+1}) \leq \varphi^n\big(d(gx_0, gx_{3k+1})\big), \tag{8}$$

$$d(gx_n, gx_{n+3k+2}) \leq \varphi^n\big(d(gx_0, gx_{3k+2})\big), \tag{9}$$

$$d(gx_n, gx_{n+3k+3}) \leq \varphi^n\big(d(gx_0, gx_{3k+3})\big). \tag{10}$$

By pentagonal property and (5), we have

$$\begin{aligned} d(gx_0, gx_4) &\leq d(gx_0, gx_1) + d(gx_1, gx_2) + d(gx_2, gx_3) + d(gx_3, gx_4) \\ &\leq d(gx_0, gx_1) + \varphi\big(d(gx_0, gx_1)\big) + \varphi^2\big(d(gx_0, gx_1)\big) + \varphi^3\big(d(gx_0, gx_1)\big) \\ &\leq \sum_{i=0}^{3} \varphi^i\big(d(gx_0, gx_1)\big), \end{aligned}$$

and

$$\begin{aligned} d(gx_0, gx_7) &\leq d(gx_0, gx_1) + d(gx_1, gx_2) + d(gx_2, gx_3) + d(gx_3, gx_4) \\ &\quad + d(gx_4, gx_5) + d(gx_5, gx_6) + d(gx_6, gx_7) \\ &\leq \sum_{i=0}^{6} \varphi^i\big(d(gx_0, gx_1)\big). \end{aligned}$$

Now, by induction, we obtain for each $k = 1, 2, 3, \cdots$

$$d(gx_0, gx_{3k+1}) \leq \sum_{i=0}^{3k} \varphi^i\big(d(gx_0, gx_1)\big). \tag{11}$$

Also, using (5), (6), and pentagonal property, we have that

$$d(gx_0, gx_5) \leq \sum_{i=0}^{2} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^3\big(d(gx_0, gx_2)\big),$$

and

$$d(gx_0, gx_8) \leq \sum_{i=0}^{5} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^6\big(d(gx_0, gx_2)\big).$$

By induction, we obtain for each $k = 1, 2, 3, \cdots$

$$d(gx_0, gx_{3k+2}) \leq \sum_{i=0}^{3k-1} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^{3k}\big(d(gx_0, gx_2)\big). \tag{12}$$

Again, using (5), (7), and pentagonal property, we have that

$$d(gx_0, gx_6) \leq \sum_{i=0}^{2} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^3\big(d(gx_0, gx_3)\big),$$

and

$$d(gx_0, gx_9) \leq \sum_{i=0}^{5} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^6\big(d(gx_0, gx_3)\big).$$

By induction, we obtain for each $k = 1, 2, 3, \cdots$

$$d(gx_0, gx_{3k+3}) \leq \sum_{i=0}^{3k-1} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^{3k}\big(d(gx_0, gx_3)\big). \tag{13}$$

Using (8) and (11), for $k = 1, 2, 3, \cdots$, we have

$$\begin{aligned} d(gx_n, gx_{n+3k+1}) &\leq \varphi^n \sum_{i=0}^{3k} \varphi^i\big(d(gx_0, gx_1)\big) \\ &\leq \varphi^n \Big[\sum_{i=0}^{3k} \varphi^i\big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3)\big)\Big] \\ &\leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3)\big)\Big]. \end{aligned} \tag{14}$$

Similarly for $k = 1, 2, 3, \cdots$, (9) and (12) implies that

$$\begin{aligned} d(gx_n, gx_{n+3k+2}) &\leq \varphi^n \Big[\sum_{i=0}^{3k-1} \varphi^i\big(d(gx_0, gx_1)\big) + \varphi^{3k}\big(d(gx_0, gx_2)\big)\Big] \\ &\leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3)\big)\Big]. \end{aligned} \tag{15}$$

Again, for $k = 1, 2, 3, \cdots$, (10) and (13) implies that

$$d(gx_n, gx_{n+3k+3}) \leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i\big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3)\big)\Big]. \tag{16}$$

Thus, by (14), (15), and (16), we have, for each m,

$$d(gx_n, gx_{n+m}) \leq \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \big) \Big]. \tag{17}$$

Since $\sum_{i=0}^{\infty} \varphi^i \big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \big)$ converges (by definition 5), where $d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \in P \setminus \{0\}$, and P is closed, then $\sum_{i=0}^{\infty} \varphi^i \big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \big) \in P \setminus \{0\}$. Hence

$$\lim_{n\to\infty} \varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \big) \Big] = 0.$$

Then, for given $c \gg 0$, there is a natural number N_1 such that

$$\varphi^n \Big[\sum_{i=0}^{\infty} \varphi^i \big(d(gx_0, gx_1) + d(gx_0, gx_2) + d(gx_0, gx_3) \big) \Big] \ll c, \quad \forall n \geq N_1. \tag{18}$$

Thus, from (17) and (18), we have

$$d(gx_n, gx_{n+m}) \ll c, \text{ for all } n \geq N_1.$$

Therefore, $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is a complete subspace of X, there exists a points $u, v \in g(X)$ such that $\lim_{n\to\infty} gx_n = v = gu$.

Now, we show that $gu = Su$. Given $c \gg 0$, we choose a natural numbers N_2, N_3 such that $d(v, gx_n) \ll \frac{c}{4}, \quad \forall n \geq N_2$, and $d(gx_n, gx_{n+1}) \ll \frac{c}{4}, \quad \forall n \geq N_3$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property, we have that

$$\begin{aligned} d(gu, Su) &\leq d(gu, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, Su) \\ &= d(v, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(Su, fx_{n+1}) \\ &\leq d(v, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \varphi\big(d(gu, gx_{n+1})\big) \\ &< d(v, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(v, gx_{n+1}) \\ &\ll \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} = c, \text{ for all } n \geq N, \end{aligned}$$

where $N := \max\{N_2, N_3\}$. Since c is arbitrary, we have $d(gu, Su) \ll \frac{c}{m}, \quad \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gu, Su) \to -d(gu, Su)$ as $m \to \infty$. Since P is closed, $-d(gu, Su) \in P$. Hence $d(gu, Su) \in P \cap -P$. By definition of cone we get that $d(gu, Su) = 0$, and so $gu = Su = v$. Hence, v is a coincidence point of S and g. Similarly, we can prove that $gu = fu = v$, which implies that v is a point of coincidence of S, f and g, i.e. $gu = fu = Su = v$.

Next, we show that v is unique. For suppose v' be another point of coincidence of S, f and g, that is $Su' = fu' = gu' = v'$, for some $u' \in X$, then

$$d(v, v') = d(Su, fu') \leq \varphi\big(d(gu, gu')\big) = \varphi\big(d(v, v')\big) < d(v, v').$$

Hence $v = v'$. Since (S, g) and (f, g) are weakly compatible, by Lemma 1, v is the unique common fixed point of S, f and g. This completes the proof of the theorem.

Example 1. Let $X = \{1, 2, 3, 4, 5\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $d : X \times X \to E$ as follows:

$$\begin{aligned} d(x,x) &= 0, \forall x \in X; \\ d(1,2) = d(2,1) &= (4,8); \\ d(1,3) = d(3,1) = d(3,4) = d(4,3) = d(2,4) = d(4,2) &= (1,2); \\ d(1,5) = d(5,1) = d(2,5) = d(5,2) = d(3,5) = d(5,3) = d(4,5) = d(5,4) &= (3,6). \end{aligned}$$

Then (X, d) is a cone pentagonal metric space, but (X, d) is not a cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned} (4,8) = d(1,2) &> d(1,3) + d(3,4) + d(4,2) \\ &= (1,2) + (1,2) + (1,2) \\ &= (3,6) \text{ as } (4,8) - (3,6) = (1,2) \in P. \end{aligned}$$

Define a mapping S, f and $g : X \to X$ as follows:

$$S(x) = 4, \ \forall x \in X.$$

$$f(x) = \begin{cases} 4, & \text{if } x \neq 5; \\ 2, & \text{if } x = 5. \end{cases}$$

$$g(x) = x, \ \forall x \in X.$$

Clearly $S(X) \cup f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X. Also, the pairs (S, g) and (f, g) are weakly compatibles. The conditions of Theorem 1 holds for all $x, y \in X$, where $\varphi(t) = \frac{1}{3}t$, and 4 is the unique common fixed point of the mappings S, f and g.

Corollary 1. *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, fy) \leq \lambda d(gx, gy),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $S(X) \cup f(X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X, then the mappings S, f and g have a unique point of coincidence in X. Moreover, if (S, g) and (f, g) are weakly compatible then S, f and g have a unique common fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition 5. Hence the results follows from Theorem 1.

Corollary 2. (see [4]) *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \varphi\big(d(gx, gy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq g(X)$, and $g(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and g have a unique point of coincidence in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. Putting $f = S$ in Theorem 1. This completes the proof.

Corollary 3. *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda d(gx, gy),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $S(X) \subseteq g(X)$, and $g(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and g have a unique point of coincidence in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. Putting $f = S$ in Theorem 1. The results follows from Corollary 1.

Corollary 4. (see [16]) *Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, f, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, fy) \leq \lambda d(gx, gy),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Suppose that $S(X) \cup f(X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X, then the mappings S, f and g have a unique point of coincidence in X. Moreover, if (S, g) and (f, g) are weakly compatible then S, f and g have a unique common fixed point in X.

Proof. This follows from the Remark 2 and Theorem 1.

Corollary 5. (see [17]) *Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \varphi\big(d(gx, gy)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Suppose that $S(X) \subseteq g(X)$, and $g(X)$ or $S(X)$ is a complete subspace of X, then the mappings S and g have a unique point of coincidence in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. This follows from the Remark 2 and Corollary 2.

Corollary 6. (see [2]) *Let (X, d) be a cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \preceq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. Putting $g = I$ in Corollary 2, where I is the identity mapping. This completes the proof.

Corollary 7. (see [17]) *Let (X, d) be a cone rectangular metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \preceq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2 and Putting $g = I$ in Corollary 2.

Corollary 8. (see [9]) *Let (X, d) be a cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Putting $g = I$ in Corollary 3, where I is the identity mapping. This completes the proof.

Corollary 9. (see [6]) *Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Putting $g = I$ in Corollary 3 and Remark 2, the results follows.

Theorem 2. *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, f, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, fy) \leq \lambda\big[d(gx, Sx) + d(gy, fy)\big], \tag{19}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $S(X) \cup f(X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X, then the mappings S, f and g have a unique point of coincidence in X. Moreover, if (S, g) and (f, g) are weakly compatible then S, f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define, like in Theorem 1, a sequence $\{gx_n\}$ in X such that

$$gx_{n+1} = Sx_n \text{ and } gx_{n+2} = fx_{n+1}, \text{ for all } n = 0, 1, 2, \cdots .$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from (19), it follows that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(Sx_{n-1}, fx_n) \\ &\leq \lambda\big(d(gx_{n-1}, Sx_{n-1}) + d(gx_n, fx_n)\big) \\ &\leq \lambda\big(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\big). \end{aligned}$$

So that,

$$\begin{aligned} d(gx_n, gx_{n+1}) &\leq \frac{\lambda}{1-\lambda} d(gx_{n-1}, gx_n) \\ &\leq rd(gx_{n-1}, gx_n), \text{ where } r = \frac{\lambda}{1-\lambda} \in [0, 1) \\ &\leq r^2 d(gx_{n-2}, gx_{n-1}) \\ &\vdots \\ &\leq r^n\big(d(gx_0, gx_1)\big). \end{aligned} \tag{20}$$

In similar way, it again follows that

$$d(gx_n, gx_{n+2}) \leq r^n\big(d(gx_0, gx_2)\big), \tag{21}$$

and

$$d(gx_n, gx_{n+3}) \leq r^n\big(d(gx_0, gx_3)\big). \tag{22}$$

Similarly, for $k = 1, 2, 3, \cdots$, It further follows that

$$d(gx_n, gx_{n+3k+1}) \leq r^n\big(d(gx_0, gx_{3k+1})\big), \tag{23}$$

$$d(gx_n, gx_{n+3k+2}) \leq r^n\big(d(gx_0, gx_{3k+2})\big), \tag{24}$$

$$d(gx_n, gx_{n+3k+3}) \leq r^n\big(d(gx_0, gx_{3k+3})\big). \tag{25}$$

Using the same argument in the proof of Theorem 1, we can show that $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is a complete subspace of X, there exists a points $u, v \in g(X)$ such that $\lim_{n\to\infty} gx_n = v = gu$.

Now, we show that $gu = Su$. Given $c \gg 0$, we choose a natural numbers M_1, M_2, M_3 such that $d(v, gx_n) \ll \frac{c(1-\lambda)}{3}$, $\forall n \geq M_1$, $d(gx_n, gx_{n+1}) \ll \frac{c(1-\lambda)}{3}$, $\forall n \geq M_2$ and $d(gx_{n+1}, gx_{n+2}) \ll \frac{c(1-\lambda)}{3(1+\lambda)}$, $\forall n \geq M_3$. Since $x_n \neq x_m$ for $n \neq m$, by pentagonal property, we have that

$$d(gu, Su) \leq d(gu, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(gx_{n+2}, Su)$$

$$\begin{aligned}
&\leq d(v, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + d(fx_{n+1}, Su) \\
&\leq d(v, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \lambda\big(d(gu, Su) + d(gx_{n+1}, fx_{n+1})\big) \\
&< d(v, gx_n) + d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \lambda\big(d(gu, Su) + d(gx_{n+1}, gx_{n+2})\big)
\end{aligned}$$

$$\begin{aligned}
d(gu, Su) &\leq \frac{1}{1-\lambda}\big(d(v, gx_n) + d(gx_n, gx_{n+1}) + (1+\lambda)d(gx_{n+1}, gx_{n+2})\big) \\
&\ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c, \text{ for all } n \geq M,
\end{aligned}$$

where $M := \max\{M_1, M_2, M_3\}$. Since c is arbitrary, we have $d(gu, Su) \ll \frac{c}{m}, \quad \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gu, Su) \to -d(gu, Su)$ as $m \to \infty$. Since P is closed, $-d(gu, Su) \in P$. Hence $d(gu, Su) \in P \cap -P$. By definition of cone we get that $d(gu, Su) = 0$, and so $gu = Su = v$. Hence, v is a point of coincidence of S and g. Similarly, we can prove that $gu = fu = v$, which implies that v is a point of coincidence of S, f and g, i.e. $gu = fu = Su = v$.

Next, we show that v is unique. For suppose v' be another point of coincidence, that is $gu' = fu' = Su' = v'$, for some $u' \in X$, then

$$d(v, v') = d(Su, fu') \leq \lambda\big(d(gu, Su) + d(gu', fu')\big) \leq \lambda\big(d(v, v) + d(v', v')\big).$$

Hence $v = v'$. Since (S, g) and (f, g) are weakly compatible, by Lemma 1, v is the unique common fixed point of S, f and g. This completes the proof of the theorem.

Corollary 10. (see [5]) *Let (X, d) be a cone pentagonal metric space. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big[d(gx, Sx) + d(gy, Sy)\big],$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $S(X) \subseteq g(X)$, and $S(X)$ or $g(X)$ is a complete subspace of X, then the mappings S and g have a unique point of coincidence in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. Putting $f = S$ in Theorem 2. This completes the proof.

Corollary 11. (see [16]) *Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, f, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, fy) \leq \lambda\big[d(gx, Sx) + d(gy, fy)\big],$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $S(X) \cup f(X) \subseteq g(X)$, and $g(X)$ is a complete subspace of X, then the mappings S, f and g have a unique point of coincidence in X. Moreover, if (S, g) and (f, g) are weakly compatible then S, f and g have a unique common fixed point in X.

Proof. This follows from the Remark 2 and Theorem 2.

Corollary 12. (see [3]) *Let (X, d) be a complete cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big[d(x, Sx) + d(y, Sy)\big], \tag{26}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

(i) S has a unique fixed point in X.

(ii) For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. Putting $g = I$ in Corollary 10. This completes the proof.

Corollary 13. (see [18]) *Let (X, d) be a cone rectangular metric space. Suppose the mappings $S, g : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big[d(gx, Sx) + d(gy, Sy)\big],$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $S(X) \subseteq g(X)$, and $S(X)$ or $g(X)$ is a complete subspace of X, then the mappings S and g have a unique point of coincidence in X. Moreover, if S and g are weakly compatible then S and g have a unique common fixed point in X.

Proof. This follows from the Remark 2 and Corollary 10.

Corollary 14. (see [13]) *Let (X, d) be a complete cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $S : X \to X$ satisfies the contractive condition:*

$$d(Sx, Sy) \leq \lambda\big[d(x, Sx) + d(y, Sy)\big], \tag{27}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

(i) S has a unique fixed point in X.

(ii) For any $x \in X$, the iterative sequence $\{S^n x\}$ converges to the fixed point.

Proof. Putting $g = I$ in Corollary 10 and Remark 2. This completes the proof.

Example 2. Let $X = \{1, 2, 3, 4, 5\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $d : X \times X \to E$ as follows:

$$d(x, x) = 0, \forall x \in X;$$
$$d(1, 2) = d(2, 1) = (4, 8);$$
$$d(1, 3) = d(3, 1) = d(3, 4) = d(4, 3) = d(2, 4) = d(4, 2) = (1, 2);$$
$$d(1, 5) = d(5, 1) = d(2, 5) = d(5, 2) = d(3, 5) = d(5, 3) = d(4, 5) = d(5, 4) = (3, 6).$$

Then (X, d) is a cone pentagonal metric space, but (X, d) is not a cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned}(4,8) = d(1,2) &> d(1,3) + d(3,4) + d(4,2)\\ &= (1,2) + (1,2) + (1,2)\\ &= (3,6) \text{ as } (4,8) - (3,6) = (1,2) \in P.\end{aligned}$$

Define a mapping S, f and $g : X \to X$ as follows:

$$S(x) = 4, \ \forall x \in X.$$

$$f(x) = \begin{cases} 4, & \text{if } x \neq 5;\\ 2, & \text{if } x = 5.\end{cases}$$

$$g(x) = \begin{cases} 3, & \text{if } x = 1;\\ 1, & \text{if } x = 2;\\ 2, & \text{if } x = 3;\\ 4, & \text{if } x = 4;\\ 5, & \text{if } x = 5.\end{cases}$$

Clearly $S(X) \cup f(X) \subseteq g(X)$, $g(X)$ is a complete subspace of X. Also, the pairs (S, g) and (f, g) are weakly compatibles. The conditions of Theorem 2 holds for all $x, y \in X$, where $\lambda = \frac{1}{3}$, and 4 is the unique common fixed point of the mappings S, f and g.

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