**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 10, No. 3, 2017, 495-505
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

# Hermite-Hadamard type fractional integral inequalities for generalized $(r; s, m, \varphi)$-preinvex functions

Artion Kashuri[1,*], Rozana Liko[1]

[1] *Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania*

**Abstract.** In the present paper, a new class of generalized $(r; s, m, \varphi)$-preinvex functions is introduced and some new integral inequalities for the left hand side of Gauss-Jacobi type quadrature formula involving generalized $(r; s, m, \varphi)$-preinvex functions are given. Moreover, some generalizations of Hermite-Hadamard type inequalities for generalized $(r; s, m, \varphi)$-preinvex functions via Riemann-Liouville fractional integrals are established. These results not only extend the results appeared in the literature (see [1], [2]), but also provide new estimates on these types.

**2010 Mathematics Subject Classifications**: Primary: 26A51. Secondary: 26A33, 26D07, 26D10, 26D15.

**Key Words and Phrases**: Hermite-Hadamard type inequality, Hölder's inequality, Minkowski's inequality, Cauchy's inequality, power mean inequality, Riemann-Liouville fractional integral, $s$-convex function in the second sense, $m$-invex, $P$-function.

## 1. Introduction and Preliminaries

The following notations are used throughout this paper. We use $I$ to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and $I^\circ$ to denote the interior of $I$. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of $K$. $\mathbb{R}^n$ is used to denote a generic $n$-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.** *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

*Corresponding author.

*Email addresses:* `artionkashuri@gmail.com` (A. Kashuri), `rozanaliko86@gmail.com` (R. Liko)

 

Fractional calculus (see [14]) and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

**Definition 1.** *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

*and*

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

*where* $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. *Here* $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.
*In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.*

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [13], [14]) and the references cited therein.
Now, let us recall some definitions of various convex functions.

**Definition 2.** *(see [4]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_{\circ}$ is said to be $P$-function or $P$-convex, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0,1].$$

**Definition 3.** *(see [5]) A function $f : \mathbb{R}_{\circ} \longrightarrow \mathbb{R}$ is said to be $s$-convex in the second sense, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2}$$

*for all $x, y \in \mathbb{R}_{\circ}$, $\lambda \in [0,1]$ and $s \in (0,1]$.*

It is clear that a 1-convex function must be convex on $\mathbb{R}_{\circ}$ as usual. The $s$-convex functions in the second sense have been investigated in (see [5]).

**Definition 4.** *(see [6]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please see (see [6], [7]) and the references therein.

**Definition 5.** *(see [8]) The function $f$ defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect $\eta$, if for every $x, y \in K$ and $t \in [0,1]$, we have that*

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^{\star}|f|, \tag{3}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^{\star}|f|$ (see [9]).
Recently, Liu (see [10]) obtained several integral inequalities for the left hand side of (3) under the Definition 2 of $P$-function.
Also in (see [11]), Özdemir et al. established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized $(r;s,m,\varphi)$-preinvex function is introduced and some new integral inequalities for the left hand side of (3) involving generalized $(r;s,m,\varphi)$-preinvex functions are given. In Section 3, some generalizations of Hermite-Hadamard type inequalities for generalized $(r;s,m,\varphi)$-preinvex functions via fractional integrals are given. These general inequalities give us some new estimates for the left hand side of Gauss-Jacobi type quadrature formula and Hermite-Hadamard type fractional integral inequalities.

## 2. New integral inequalities for generalized $(r;s,m,\varphi)$-preinvex functions

**Definition 6.** *(see [3]) A set $K \subseteq \mathbb{R}^n$ is said to be $m$-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.*

**Remark 1.** *In Definition 6, under certain conditions, the mapping $\eta(y,x,m)$ could reduce to $\eta(y,x)$. For example when $m = 1$, then the $m$-invex set degenerates an invex set on $K$.*

**Definition 7.** *(see [12]) A positive function $f$ on the invex set $K$ is said to be logarithmically preinvex, if*

$$f(u + t\eta(v,u)) \leq f^{1-t}(u) f^t(v)$$

*for all $u, v \in K$ and $t \in [0,1]$.*

**Definition 8.** *(see [12]) The function $f$ on the invex set $K$ is said to be $r$-preinvex with respect to $\eta$, if*

$$f(u + t\eta(v,u)) \leq M_r(f(u), f(v); t)$$

*holds for all $u, v \in K$ and $t \in [0,1]$, where*

$$M_r(x,y;t) = \begin{cases} \left[(1-t)x^r + ty^r\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ x^{1-t}y^t, & \text{if } r = 0, \end{cases}$$

*is the weighted power mean of order $r$ for positive numbers $x$ and $y$.*

We next give new definition, to be referred as generalized $(r; s, m, \varphi)$-preinvex function.

**Definition 9.** *Let $K \subseteq \mathbb{R}^n$ be an open $m$-invex set with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}^n$, and $\varphi : I \longrightarrow K$ is a continuous increasing function. The function $f : K \longrightarrow (0,\infty)$ is said to be generalized $(r; s, m, \varphi)$-preinvex with respect to $\eta$, if*

$$f\left(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\right) \leq M_r(f(\varphi(x)), f(\varphi(y)), m, s; t) \quad (4)$$

*holds for any fixed $s, m \in (0,1]$ and for all $x, y \in I, t \in [0,1]$, where*

$$M_r(f(\varphi(x)), f(\varphi(y)), m, s; t) = \begin{cases} \left[m(1-t)^s f^r(\varphi(x)) + t^s f^r(\varphi(y))\right]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ f(\varphi(x))^{m(1-t)^s} f(\varphi(y))^{t^s}, & \text{if } r = 0, \end{cases}$$

*is the weighted power mean of order $r$ for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.*

**Remark 2.** *In Definition 9, it is worthwhile to note that the class of generalized $(r; s, m, \varphi)$-preinvex function is a generalization of the class of $s$-convex in the second sense function given in Definition 3. Also, for $r = 1$ and $\varphi(x) = x$, $\forall x \in I$, we get the notion of generalized $(s, m)$-preinvex function (see [3]).*

**Example 1.** *Let $f(x) = |x|$, $\varphi(x) = x$, $r = s = 1$ and*

$$\eta(y, x, m) = \begin{cases} y - mx, & \text{if } x \geq 0,\ y \geq 0; \\ y - mx, & \text{if } x \leq 0,\ y \leq 0; \\ mx - y, & \text{if } x \geq 0,\ y \leq 0; \\ mx - y, & \text{if } x \leq 0,\ y \geq 0. \end{cases}$$

*Then $f(x)$ is a generalized $(1; 1, m, x)$-preinvex function of with respect to $\eta : \mathbb{R}\times\mathbb{R}\times(0,1] \longrightarrow \mathbb{R}$ and any fixed $m \in (0,1]$. However, it is obvious that $f(x) = |x|$ is not a convex function on $\mathbb{R}$.*

In this section, in order to prove our main results regarding some new integral inequalities involving generalized $(r; s, m, \varphi)$-preinvex functions, we need the following new Lemma:

**Lemma 1.** *Let $\varphi : I \longrightarrow K$ be a continuous increasing function. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers $K^\circ$ with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$, for $m\varphi(a) < m\varphi(a)+\eta(\varphi(b), \varphi(a), m)$. Then for any fixed $m \in (0,1]$ and $p, q > 0$, we have*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x - m\varphi(a))^p (m\varphi(a) + \eta(\varphi(b), \varphi(a), m) - x)^q f(x)dx$$

$$= \eta(\varphi(b), \varphi(a), m)^{p+q+1} \int_0^1 t^p(1-t)^q f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt.$$

*Proof.* It is easy to observe that

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$= \eta(\varphi(b),\varphi(a),m)\int_0^1 (m\varphi(a)+t\eta(\varphi(b),\varphi(a),m)-m\varphi(a))^p$$
$$\times(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-m\varphi(a)-t\eta(\varphi(b),\varphi(a),m))^q$$
$$\times f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$
$$= \eta(\varphi(b),\varphi(a),m)^{p+q+1}\int_0^1 t^p(1-t)^q f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt.$$

The following definition will be used in the sequel.

**Definition 10.** *The Euler Beta function is defined for $x, y > 0$ as*

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

**Theorem 2.** *Let $\varphi : I \longrightarrow K$ be a continuous increasing function. Assume that $f : K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,\infty)$ is a continuous function on the interval of real numbers $K^\circ$ with $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $k > 1$ and $0 < r \le 1$. If $f^{\frac{k}{k-1}}$ is a generalized $(r;s,m,\varphi)$-preinvex function on an open $m$-invex set $K$ with respect to $\eta : K\times K\times(0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$, then for any fixed $p, q > 0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\le |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left(\frac{r}{s+r}\right)^{\frac{k-1}{k}}\beta^{\frac{1}{k}}(kp+1,kq+1)$$
$$\times\left[mf^{\frac{rk}{k-1}}(\varphi(a))+f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}}. \tag{5}$$

*Proof.* Let $k > 1$ and $0 < r \le 1$. Since $f^{\frac{k}{k-1}}$ is a generalized $(r;s,m,\varphi)$-preinvex function on $K$, combining with Lemma 1, Hölder inequality and Minkowski inequality for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\le |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}}$$

$$\times\left[\int_0^1 f^{\frac{k}{k-1}}(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{k-1}{k}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times\left[\int_0^1 \left(m(1-t)^s f^r(\varphi(a))^{\frac{k}{k-1}}+t^s f^r(\varphi(b))^{\frac{k}{k-1}}\right)^{\frac{1}{r}}dt\right]^{\frac{k-1}{k}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}}(1-t)^{\frac{s}{r}}f^{\frac{k}{k-1}}(\varphi(a))dt\right)^r+\left(\int_0^1 t^{\frac{s}{r}}f^{\frac{k}{k-1}}(\varphi(b))dt\right)^r\right]^{\frac{k-1}{rk}}$$

$$=|\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left(\frac{r}{s+r}\right)^{\frac{k-1}{k}}\beta^{\frac{1}{k}}(kp+1,kq+1)$$

$$\times\left[mf^{\frac{rk}{k-1}}(\varphi(a))+f^{\frac{rk}{k-1}}(\varphi(b))\right]^{\frac{k-1}{rk}}.$$

**Corollary 1.** *Under the same conditions as in Theorem 2 for $r=1$, we get (see [1], Theorem 2.2).*

**Theorem 3.** *Let $\varphi: I \longrightarrow K$ be a continuous increasing function. Assume that $f: K=[m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,\infty)$ is a continuous function on the interval of real numbers $K^\circ$ with $m\varphi(a) < m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Let $l \geq 1$ and $0 < r \leq 1$. If $f^l$ is a generalized $(r;s,m,\varphi)$-preinvex function on an open $m$-invex set $K$ with respect to $\eta: K\times K\times(0,1] \longrightarrow \mathbb{R}$ for any fixed $s,m\in(0,1]$, then for any fixed $p,q>0$,*

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[mf^{rl}(\varphi(a))\beta^r\left(p+1,q+\frac{s}{r}+1\right)+f^{rl}(\varphi(b))\beta^r\left(p+\frac{s}{r}+1,q+1\right)\right]^{\frac{1}{rl}}. \tag{6}$$

*Proof.* Let $l \geq 1$ and $0<r\leq 1$. Since $f^l$ is a generalized $(r;s,m,\varphi)$-preinvex function on $K$, combining with Lemma 1, the well-known power mean inequality and Minkowski inequality for all $t\in[0,1]$ and for any fixed $s,m\in(0,1]$, we get

$$\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}(x-m\varphi(a))^p(m\varphi(a)+\eta(\varphi(b),\varphi(a),m)-x)^q f(x)dx$$

$$=\eta(\varphi(b),\varphi(a),m)^{p+q+1}$$

$$\times \int_0^1 \left[t^p(1-t)^q\right]^{\frac{l-1}{l}} \left[t^p(1-t)^q\right]^{\frac{1}{l}} f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 t^p(1-t)^q f^l(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt\right]^{\frac{1}{l}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[\int_0^1 t^p(1-t)^q \left(m(1-t)^s f^r(\varphi(a))^l + t^s f^r(\varphi(b))^l\right)^{\frac{1}{r}} dt\right]^{\frac{1}{l}}$$

$$\leq |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}} t^p(1-t)^{q+\frac{s}{r}} f^l(\varphi(a))dt\right)^r + \left(\int_0^1 t^{p+\frac{s}{r}}(1-t)^q f^l(\varphi(b))dt\right)^r\right]^{\frac{1}{rl}}$$

$$= |\eta(\varphi(b),\varphi(a),m)|^{p+q+1}\beta^{\frac{l-1}{l}}(p+1,q+1)$$

$$\times\left[m f^{rl}(\varphi(a))\beta^r\left(p+1,q+\frac{s}{r}+1\right) + f^{rl}(\varphi(b))\beta^r\left(p+\frac{s}{r}+1,q+1\right)\right]^{\frac{1}{rl}}.$$

**Corollary 2.** *Under the same conditions as in Theorem 3 for $r = 1$, we get (see [1], Theorem 2.3).*

## 3. Hermite-Hadamard type fractional integral inequalities for generalized $(r; s, m, \varphi)$-preinvex functions

In this section, we prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized $(r; s, m, \varphi)$-preinvex functions via fractional integrals.

**Theorem 4.** *Let $\varphi : I \longrightarrow K$ be a continuous increasing function. Suppose $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0, 1]$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b), \varphi(a), m)$. Assume that $f : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \longrightarrow (0, \infty)$ be a generalized $(r; s, m, \varphi)$-preinvex function on an open $m$-invex set $K^\circ$. Then for $\alpha > 0$ and $0 < r \leq 1$, we have*

$$\frac{\Gamma(\alpha)}{\eta^\alpha(\varphi(b),\varphi(a),m)} J^\alpha_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-} f(m\varphi(a))$$

$$\leq \left[m f^r(\varphi(a))\beta^r\left(\alpha,\frac{s}{r}+1\right) + f^r(\varphi(b))\left(\frac{r}{\alpha r+s}\right)^r\right]^{\frac{1}{r}}. \quad (7)$$

*Proof.* Let $0 < r \leq 1$. Since $f$ is a generalized $(r; s, m, \varphi)$-preinvex function on an open $m$-invex set $K^{\circ}$, combining with Minkowski inequality for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} J^{\alpha}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-} f(m\varphi(a))$$

$$= \int_0^1 t^{\alpha-1} f(m\varphi(a) + t\eta(\varphi(b),\varphi(a),m))dt$$

$$\leq \int_0^1 t^{\alpha-1}\Big[m(1-t)^s f^r(\varphi(a)) + t^s f^r(\varphi(b))\Big]^{\frac{1}{r}} dt$$

$$\leq \left\{ \left[\int_0^1 t^{\alpha-1+\frac{s}{r}} f(\varphi(b))dt\right]^r + \left[\int_0^1 m^{\frac{1}{r}} t^{\alpha-1}(1-t)^{\frac{s}{r}} f(\varphi(a))dt\right]^r \right\}^{\frac{1}{r}}$$

$$= \left[ m f^r(\varphi(a))\beta^r\left(\alpha, \frac{s}{r}+1\right) + f^r(\varphi(b))\left(\frac{r}{\alpha r + s}\right)^r \right]^{\frac{1}{r}}.$$

**Corollary 3.** *Under the same conditions as in Theorem 4 for $m = s = 1, \varphi(x) = x$ and $\eta(\varphi(b),\varphi(a),m) = \eta(b,a)$, we get (see [2], Theorem 3.1).*

**Theorem 5.** *Let $\varphi : I \longrightarrow K$ be a continuous increasing function. Suppose $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for any fixed $s, m \in (0,1]$ with $m\varphi(a) < m\varphi(a) + \eta(\varphi(b),\varphi(a),m)$. Assume that $f, h : K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b),\varphi(a),m)] \longrightarrow (0,\infty)$ are respectively generalized $(r; s, m, \varphi)$-preinvex function and generalized $(l; s, m, \varphi)$-preinvex function on an open $m$-invex set $K^{\circ}$. Then for $\alpha > 0, r > 1$ and $r^{-1} + l^{-1} = 1$, we have*

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} J^{\alpha}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-} f(m\varphi(a))h(m\varphi(a))$$

$$\leq \frac{1}{2}\left\{ \left[ m f^r(\varphi(a))\beta^{\frac{r}{2}}\left(\frac{2(\alpha-1)}{r}+1, \frac{2s}{r}+1\right) + f^r(\varphi(b))\left(\frac{r}{2(\alpha-1+s)+r}\right)^{\frac{r}{2}} \right]^{\frac{2}{r}} \right. \quad (8)$$

$$\left. + \left[ m h^l(\varphi(a))\beta^{\frac{l}{2}}\left(\frac{2(\alpha-1)}{l}+1, \frac{2s}{l}+1\right) + h^l(\varphi(b))\left(\frac{l}{2(\alpha-1+s)+l}\right)^{\frac{l}{2}} \right]^{\frac{2}{l}} \right\}.$$

*Proof.* Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since $f$ and $h$ are respectively generalized $(r; s, m, \varphi)$-preinvex function and generalized $(l; s, m, \varphi)$-preinvex function on an open $m$-invex set $K^{\circ}$, combining with Cauchy and Minkowski inequalities for all $t \in [0,1]$ and for any fixed $s, m \in (0,1]$, we get

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} J^{\alpha}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-} f(m\varphi(a))h(m\varphi(a))$$

$$= \int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)} f(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))$$
$$\times h(m\varphi(a)+t\eta(\varphi(b),\varphi(a),m))dt$$
$$\leq \int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)} \Big[m(1-t)^s f^r(\varphi(a))+t^s f^r(\varphi(b))\Big]^{\frac{1}{r}}$$
$$\times\Big[m(1-t)^s h^l(\varphi(a))+t^s h^l(\varphi(b))\Big]^{\frac{1}{l}} dt$$
$$\leq \frac{1}{2}\Bigg\{\int_0^1 \Big[t^{\alpha-1+s} f^r(\varphi(b))+mt^{\alpha-1}(1-t)^s f^r(\varphi(a))\Big]^{\frac{2}{r}} dt$$
$$+\int_0^1 \Big[t^{\alpha-1+s} h^l(\varphi(b))+mt^{\alpha-1}(1-t)^s h^l(\varphi(a))\Big]^{\frac{2}{l}} dt\Bigg\}$$
$$\leq \frac{1}{2}\Bigg[\Bigg\{\left(\int_0^1 t^{\frac{2(\alpha-1+s)}{r}} f^2(\varphi(b))dt\right)^{\frac{r}{2}}+\left(\int_0^1 m^{\frac{2}{r}} t^{\frac{2(\alpha-1)}{r}}(1-t)^{\frac{2s}{r}} f^2(\varphi(a))dt\right)^{\frac{r}{2}}\Bigg\}^{\frac{2}{r}}$$
$$+\Bigg\{\left(\int_0^1 t^{\frac{2(\alpha-1+s)}{l}} h^2(\varphi(b))dt\right)^{\frac{l}{2}}+\left(\int_0^1 m^{\frac{2}{l}} t^{\frac{2(\alpha-1)}{l}}(1-t)^{\frac{2s}{l}} h^2(\varphi(a))dt\right)^{\frac{l}{2}}\Bigg\}^{\frac{2}{l}}\Bigg]$$
$$= \frac{1}{2}\Bigg\{\left[mf^r(\varphi(a))\beta^{\frac{r}{2}}\left(\frac{2(\alpha-1)}{r}+1,\frac{2s}{r}+1\right)+f^r(\varphi(b))\left(\frac{r}{2(\alpha-1+s)+r}\right)^{\frac{r}{2}}\right]^{\frac{2}{r}}$$
$$+\left[mh^l(\varphi(a))\beta^{\frac{l}{2}}\left(\frac{2(\alpha-1)}{l}+1,\frac{2s}{l}+1\right)+h^l(\varphi(b))\left(\frac{l}{2(\alpha-1+s)+l}\right)^{\frac{l}{2}}\right]^{\frac{2}{l}}\Bigg\}.$$

**Corollary 4.** *Under the same conditions as in Theorem 5 for $m=s=1,\varphi(x)=x$ and $\eta(\varphi(b),\varphi(a),m)=\eta(b,a)$, we get (see [2], Theorem 3.3).*

**Theorem 6.** *Let $\varphi: I\longrightarrow K$ be a continuous increasing function. Suppose $K\subseteq\mathbb{R}$ be an open $m$-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for any fixed $s,m\in(0,1]$ with $m\varphi(a)<m\varphi(a)+\eta(\varphi(b),\varphi(a),m)$. Assume that $f,h: K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\longrightarrow(0,\infty)$ are respectively generalized $(r;s,m,\varphi)$-preinvex function and generalized $(l;s,m,\varphi)$-preinvex function on an open $m$-invex set $K^{\circ}$. Then for $\alpha>0,r>1$ and $r^{-1}+l^{-1}=1$, we have*

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(b),\varphi(a),m)} J^{\alpha}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-} f(m\varphi(a))h(m\varphi(a))$$
$$\leq \left\{\frac{f^r(\varphi(b))}{s+\alpha}+mf^r(\varphi(a))\beta(\alpha,s+1)\right\}^{\frac{1}{r}}+\left\{\frac{h^l(\varphi(b))}{s+\alpha}+mh^l(\varphi(a))\beta(\alpha,s+1)\right\}^{\frac{1}{l}}. \quad (9)$$

*Proof.* Let $r > 1$ and $r^{-1} + l^{-1} = 1$. Since $f$ and $h$ are respectively generalized $(r; s, m, \varphi)$-preinvex function and generalized $(l; s, m, \varphi)$-preinvex function on an open $m$-invex set $K^{\circ}$, combining with Hölder inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(\varphi(b), \varphi(a), m)} J^{\alpha}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-} f(m\varphi(a))h(m\varphi(a))$$

$$= \int_0^1 t^{(\alpha-1)\left(\frac{1}{r}+\frac{1}{l}\right)} f(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))$$

$$\times h(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))dt$$

$$\leq \Bigg\{ \int_0^1 \Big[t^{\alpha-1+s} f^r(\varphi(b)) + mt^{\alpha-1}(1-t)^s f^r(\varphi(a))\Big]^{\frac{1}{r}}$$

$$\times \Big[t^{\alpha-1+s} h^l(\varphi(b)) + mt^{\alpha-1}(1-t)^s h^l(\varphi(a))\Big]^{\frac{1}{l}} dt \Bigg\}$$

$$\leq \left\{ \int_0^1 \Big[t^{\alpha-1+s} f^r(\varphi(b)) + mt^{\alpha-1}(1-t)^s f^r(\varphi(a))\Big] dt \right\}^{\frac{1}{r}}$$

$$+ \left\{ \int_0^1 \Big[t^{\alpha-1+s} h^l(\varphi(b)) + mt^{\alpha-1}(1-t)^s h^l(\varphi(a))\Big] dt \right\}^{\frac{1}{l}}$$

$$= \left\{ \frac{f^r(\varphi(b))}{s+\alpha} + mf^r(\varphi(a))\beta(\alpha, s+1) \right\}^{\frac{1}{r}} + \left\{ \frac{h^l(\varphi(b))}{s+\alpha} + mh^l(\varphi(a))\beta(\alpha, s+1) \right\}^{\frac{1}{l}}.$$

**Corollary 5.** *Under the same conditions as in Theorem 6 for $m = s = 1, \varphi(x) = x$ and $\eta(\varphi(b), \varphi(a), m) = \eta(b, a)$, we get (see [2], Theorem 3.9).*

**Remark 3.** *For different choices of positive values $r, l = \frac{1}{2}, \frac{1}{3}, 2$, etc., for any fixed $s, m \in (0, 1]$ and a particular choices of a continuous increasing function $\varphi(x) = e^x$ for all $x \in \mathbb{R}$, $x^n$ for all $x > 0$ and for all $n \in \mathbb{N}$, etc., by Theorem 4, Theorem 5 and Theorem 6 we can get some special kinds of Hermite-Hadamard type fractional integral inequalities.*

## References


[1] A. Kashuri, R. Liko, Ostrowski type fractional integral inequalities for generalized $(s, m, \varphi)$-preinvex functions, *Aust. J. Math. Anal. Appl.*, **13**, 1 (2016), Article 16, 1-11.

[2] A. Akkurt, H. Yildirim, On some fractional integral inequalities of Hermite-Hadamard type for $r$-preinvex functions, *Khayyam J. Math.*, **2**, 2 (2016), 119-126.

[3] T. S. Du, J. G. Liao, Y. J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized $(s, m)$-preinvex functions, *J. Nonlinear Sci. Appl.*, **9**, (2016), 3112-3126.

[4] S. S. Dragomir, J. Pečarić, L. E. Persson, Some inequalities of Hadamard type, *Soochow J. Math.*, **21**, (1995), 335-341.

[5] H. Hudzik, L. Maligranda, Some remarks on $s$-convex functions, *Aequationes Math.*, **48**, (1994), 100-111.

[6] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.*, **60**, (2005), 1473-1484.

[7] X. M. Yang, X. Q. Yang, K. L. Teo, Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory Appl.*, **117**, (2003), 607-625.

[8] R. Pini, Invexity and generalized convexity, *Optimization.*, **22**, (1991), 513-525.

[9] D. D. Stancu, G. Coman, P. Blaga, Analiză numerică şi teoria aproximării, *Cluj-Napoca: Presa Universitară Clujeană.*, **2**, (2002).

[10] W. Liu, New integral inequalities involving beta function via $P$-convexity, *Miskolc Math Notes.*, **15**, 2 (2014), 585-591.

[11] M. E. Özdemir, E. Set, M. Alomari, Integral inequalities via several kinds of convexity, *Creat. Math. Inform.*, **20**, 1 (2011), 62-73.

[12] W. Dong Jiang, D. Wei Niu, F. Qi, Some Fractional Inequalties of Hermite-Hadamard type for $r$-$\varphi$-Preinvex Functions, *Tamkang J. Math.*, **45**, 1 (2014), 31-38.

[13] F. Qi, B. Y. Xi, Some integral inequalities of Simpson type for $GA-\epsilon$-convex functions, *Georgian Math. J.*, **20**, 5 (2013), 775-788.

[14] W. Liu, W. Wen, J. Park, Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals, *J. Nonlinear Sci. Appl.*, **9**, (2016), 766-777.