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Certain Classes of fuzzy graphs

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Abstract. In this paper, several classes of fuzzy graphs are characterized and we provide two new operations on fuzzy graphs; namely parallel connection and series connection. We show that parallel connection and series connection of balanced fuzzy graphs need not be balanced and they are balanced in case the original graphs are induced by cycles. Finally, we study the notions of strong and complete via these operations.

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1. Introduction

A graph is a nice way of representing information involving relationships between objects where objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or relationships, it is natural that we think of it as what is called 'Fuzzy Graph Model'. Applications of fuzzy relations are widespread and important in many fields; especially in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph. The notion of fuzzy relation was introduced by Zadeh [17] in his landmark paper "Fuzzy sets" in 1965. Fuzzy graph and several fuzzy analogs of graph theoretic concepts were introduced by Rosenfeld [15] in 1975. After that, the theory of fuzzy graph started to finding an increasing number of applications in several fields.

Mordeson and Peng [9] defined the concept of fuzzy graph complement and introduced several operations on fuzzy graphs. In [16], the definition of complement of a fuzzy graph was modified to agree with the classical graph case. Moreover, some properties of self-complementary fuzzy graphs and complements of several operations of fuzzy graphs that were introduced in [9] were discussed. Al-Hawary [1] introduced and explored the new notion of balanced fuzzy graphs and several new operations. After that, balanced concept was studied by many authors. The idea of balanced came from matroids, see [2, 3, 4, 6].

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For a complete background on the previous notions and the following ones, the reader is referred to [1, 5, 7, 9, 10, 11, 12, 13, 14, 15, 16].

A fuzzy subset of a non-empty set V is a mapping $\sigma : V \to [0,1]$ and a fuzzy relation μ on a fuzzy subset σ, is a fuzzy subset of $V \times V$. All throughout this paper, we assume that σ is reflexive, μ is symmetric and V is finite.

Definition 1. *[15] A* fuzzy graph *with V as the underlying set is a pair $G : (\sigma, \mu)$ where $\sigma : V \to [0,1]$ is a fuzzy subset and $\mu : V \times V \to [0,1]$ is a fuzzy relation on σ such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum. The underlying crisp graph of G is denoted by $G^* : (\sigma^*, \mu^*)$ where $\sigma^* = \sup p(\sigma) = \{x \in V : \sigma(x) > 0\}$ and $\mu^* = \sup p(\mu) = \{(x,y) \in V \times V : \mu(x,y) > 0\}.H = (\sigma', \mu')$ is a* fuzzy subgraph *of G if there exists $X \subseteq V$ such that, $\sigma' : X \to [0,1]$ is a fuzzy subset and $\mu' : X \times X \to [0,1]$ is a fuzzy relation on σ' such that $\mu(x,y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in X$.*

Definition 2. *[14] A fuzzy graph $G : (\sigma, \mu)$ with underlying graph $G : (V, E)$ is* complete *if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.*

Definition 3. *[14] A fuzzy graph $G : (\sigma, \mu)$ with underlying graph $G : (V, E)$ is* strong *if $\mu(x,y) = \sigma(x) \wedge \sigma(y)$ for all $\{x, y\} \in E$.*

Definition 4. *[8] Two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$ are* isomorphic *if there exists a bijection $h : V_1 \to V_2$ such that $\sigma_1(x) = \sigma_2(h(x))$ and $\mu_1(x,y) = \mu_2(h(x), h(y))$ for all $x, y \in V_1$.*

Next, we recall the following definition of balanced fuzzy graphs that will be our main concern in this paper. It was motivated and deeply studied by Al-Hawary in [1, 5, 7].

Definition 5. *[1] The* density *of a fuzzy graph $G : (\sigma, \mu)$ is*

$$D(G) = 2\Big(\sum_{u,v \in V} \mu(u,v)\Big) / \Big(\sum_{\substack{u,v \in V \\ u \neq v}} (\sigma(u) \wedge \sigma(v))\Big).$$

G is balanced *if. $D(H) \leq D(G)$ for all fuzzy non-empty subgraphs H of G.*

In this paper, several classes of balanced fuzzy graphs are characterized; namely the classes of complete fuzzy graphs, fuzzy graphs induced by the complete graph, fuzzy graphs with balanced components, fuzzy graphs induced by trees and fuzzy graphs induced by cycles. Moreover, we provide two new operations on fuzzy graphs, namely parallel connection and series connection. We show that parallel connection and series connection of balanced fuzzy graphs need not be balanced and they are balanced in case the original graphs are induced by cycles. Finally, we study the notions of strong and complete via these operations.

2. Classes of balanced fuzzy graphs

In this section, several classes of balanced fuzzy graphs are provided. The first class of balanced fuzzy graphs is the class of complete fuzzy graphs.

Theorem 1. *[1] Any complete fuzzy graph is balanced.*

By a *fuzzy graph* $G : (\sigma, \mu)$ *induced by a graph* $G : (V, E)$, we mean the the graph G after assigning values $\sigma(x)$ for all $x \in V$ and $\mu(x, y)$ for all $\{x, y\} \in E$ where each $\sigma(x)$ and $\mu(x, y)$ are in the interval $[0, 1]$.

Theorem 2. *A fuzzy graph G induced by the complete graph K_n is balanced.*

Proof. Let H be a fuzzy subgraph of G with m vertices. We claim that the many possible cases of H reduce to the case that H is is induced by a complete graph K_m. If this is true, then as $\sum_{x,y\in V(K_m)} \mu(x, y) \leq \sum_{x,y\in V(K_n)} \mu(x, y)$ and $\sum_{x,y\in V(K_n)}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y)) \leq \sum_{x,y\in V(K_m)}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y))$, we have

$$D(H) = \frac{2\sum_{x,y\in V(K_m)} \mu(x, y)}{\sum_{x,y\in V(K_m)}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y))} \leq \frac{2\sum_{x,y\in V(K_n)} \mu(x, y)}{\sum_{x,y\in V(K_n)}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y))} = D(G).$$

Therefore, G is balanced.

To show the claim is true, let $H\prime$ be a fuzzy subgraph of with k vertices. Obviously, $D(H\prime) \leq D(H)$ since the corresponding graphs of both have same vertices and that of H has more edges.

Lemma 3. *The union of two balanced fuzzy graphs is balanced.*

Proof. If $G_1 : (\sigma_1, \mu_1)$ is a fuzzy balanced graph with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ is a fuzzy balanced graph with crisp graph $G_2^* : (V_2, E_2)$,where we assume that $V_1 \cap V_2 = \emptyset$, then for any fuzzy subgraph H of $G_1 \cup G_2$, $H \simeq H_1 \cup H_2$ for some fuzzy subgraphs H_1 of G_1 and H_2 of G_2. Now $\sum_{x,y\in V(H_1)\cup V(H_2)} \mu(x, y) \leq \sum_{x,y\in V_1\cup V_2} \mu(x, y)$ and $\sum_{x,y\in V_1\cup V_2}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y)) \leq \sum_{x,y\in V(H_1)\cup V(H_2))}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y))$ and so

$$D(H) = \frac{2\sum_{x,y\in V(H_1)\cup V(H_2)} \mu(x, y)}{\sum_{x,y\in V(H_1)\cup V(H_2))}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y))} \leq \frac{2\sum_{x,y\in V_1\cup V_2} \mu(x, y)}{\sum_{x,y\in V_1\cup V_2}(\sigma_2^{'}(x) \wedge \sigma_2^{'}(y))} = D(G_1 \cup G_2).$$

Therefore, $G_1 \cup G_2$ is balanced.

Although the union of two balanced fuzzy graphs is balanced, the intersection (join) of two balanced fuzzy graphs needs not be balanced.

Example 1. *Consider the fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ with crisp graph $G_1^* : (V_1, E_1)$ where $V_1 = \{v_1, v_2\}$, $\sigma_1(v_1) = 1/2$, $\sigma_1(v_2) = 1/3$ and $\mu_1(v_1, v_2) = 1/3$ and $G_2 : (\sigma_2, \mu_2)$ with crisp graph $G_2^* : (V_2, E_2)$ where $V_2 = \{w_1, w_2\}$, $\sigma_2(w_1) = 1$, $\sigma_2(w_2) = 1$ and $\mu_2(w_1, w_2) = 1$. Then G_1 and G_2 are balanced, while the intersection (join) of G_1 and G_2 is not balanced as the join has density equals to 71/90 while the fuzzy subgraph G_1 of the join has density equals to 2.*

The following result can be proved by induction on the number of components of a fuzzy graph G.

Corollary 1. *If $G : (\sigma, \mu)$ is a fuzzy graph with each of its component is balanced, then G is balanced.*

Theorem 4. *A fuzzy graph G induced by a tree with n vertices is balanced.*

Proof. Let H be a fuzzy subgraph of G with m vertices. Since all subgraphs of trees are forests, we have the following two case:

Case 1: H is induced by a tree T_m. Then as $\sum_{x,y \in V(T_m)} \mu(x, y) \leq \sum_{x,y \in V(T_n)} \mu(x, y)$ and $\sum_{x,y \in V(T_n)}(\sigma_2'(x) \wedge \sigma_2'(y)) \leq \sum_{x,y \in V(T_m)}(\sigma_2'(x) \wedge \sigma_2'(y))$, we have

$$D(H) = \frac{2 \sum_{x,y \in V(T_m)} \mu(x, y)}{\sum_{x,y \in V(T_m)}(\sigma_2'(x) \wedge \sigma_2'(y))} \leq \frac{2 \sum_{x,y \in V(T_n)} \mu(x, y)}{\sum_{x,y \in V(T_n)}(\sigma_2'(x) \wedge \sigma_2'(y))} = D(G).$$

Case 2: H is induced by a forest K with m vertices and $p \geq 2$ components. Let the number of vertices in each tree be n_i for $i = 1, 2, ..., p$. Since a forest is a disjoint union of trees, by Lemma 3, G is balanced.

Corollary 2. *A fuzzy graph G induced by a forest is balanced.*

Theorem 5. *A fuzzy graph G induced by the cycle C_n is balanced.*

Proof. Let H be a proper fuzzy subgraph of G . It is easy to see that H is induced by forests. Now by Corollary 2, forests are balanced and by Lemma 3, G is balanced as a a union of balanced fuzzy graphs.

3. Series and parallel connections

The operations of joining electrical components in parallel and in series are very important in electrical network theory. Graphs may be also constructed through an operation known as the parallel connection. This can be defined as adding elements in parallel to some existing element. That is as doubling one or more elements. It is also can be constructed through an operation known as the series connection. Analogously, in this section, we introduce two new operations on fuzzy graphs; namely series and parallel connections. To describe these operations, let $G_1 : (\sigma_1, \mu_1)$ be a fuzzy graph with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ be a fuzzy graph with crisp graph $G_2^* : (V_2, E_2)$ and let $e_1 = \{v_1, v_1'\} \in E_1$ and $e_2 = \{v_2, v_2'\} \in E_2$. For $i = 1, 2$, arbitrary assign a direction to e_i and say its tail v_i and its head v_i' .

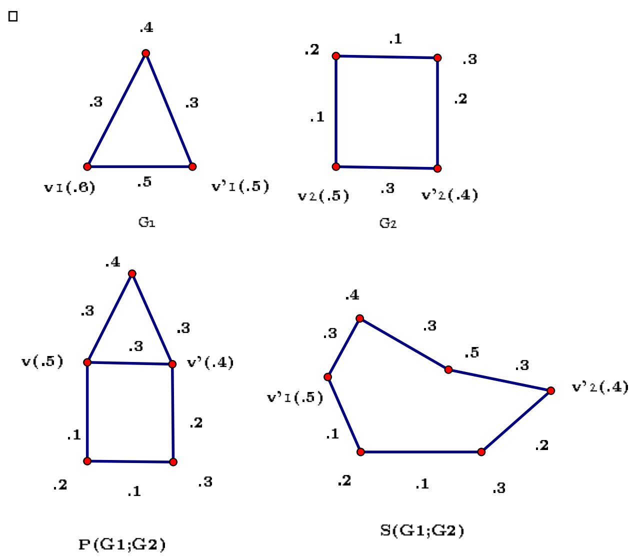
Definition 6. *Let $G_1 : (\sigma_1, \mu_1)$ be a fuzzy graph with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ be a fuzzy graph with crisp graph $G_2^* : (V_2, E_2)$. The* parallel connection *of G_1 and G_2 with respect to the directed edges e_1 and e_2 is the fuzzy graph $P((G_1, e_1); (G_2, e_2))$*

obtained by deleting the edge e_1 *from* G_1 *and the edge* e_2 *from* G_2 *and then identify the vertices* v_1, v_2 *as the vertex* v *with* $\sigma(v) = \sigma_1(v_1) \wedge \sigma_2(v_2)$ *and* v_1', v_2' *as the vertex* v' *with* $\sigma(v') = \sigma_1(v_1') \wedge \sigma_2(v_2')$ *and finally adding a new edge* e *joining* v *and* v' *with* $\mu(v, v') = \mu_1(v_1, v_1') \wedge \mu_2(v_2, v_2')$.

Definition 7. *Let* $G_1 : (\sigma_1, \mu_1)$ *be a fuzzy graph with crisp graph* $G_1^* : (V_1, E_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *be a fuzzy graph with crisp graph* $G_2^* : (V_2, E_2)$. *The* series connection *of* G_1 *and* G_2 *with respect to the directed edges* e_1 *and* e_2 *is the fuzzy graph* $S((G_1, e_1); (G_2, e_2))$ *obtained by deleting the edge* e_1 *from* G_1 *and the edge* e_2 *from* G_2 *and then identify the vertices* v_1, v_2 *as the vertex* v *with* $\sigma(v) = \sigma_1(v_1) \wedge \sigma_2(v_2)$ *and finally adding a new edge* e *joining* v_1', v_2' *with* $\mu(v_1', v_2') = \mu_1(v_1, v_1') \wedge \mu_2(v_2, v_2')$.

The connecting edges in $P((G_1, e_1); (G_2, e_2))$ and $S((G_1, e_1); (G_2, e_2))$ are usually arbitrary and so we instead write $P(G_1; G_2)$ and $S(G_1; G_2)$, respectively. To illustrate these definitions, we offer the following example:

Example 2. *Consider the fuzzy graphs* G_1, G_2, $P(G_1; G_2)$ *and* $S(G_1; G_2)$ *in Figure 1.*

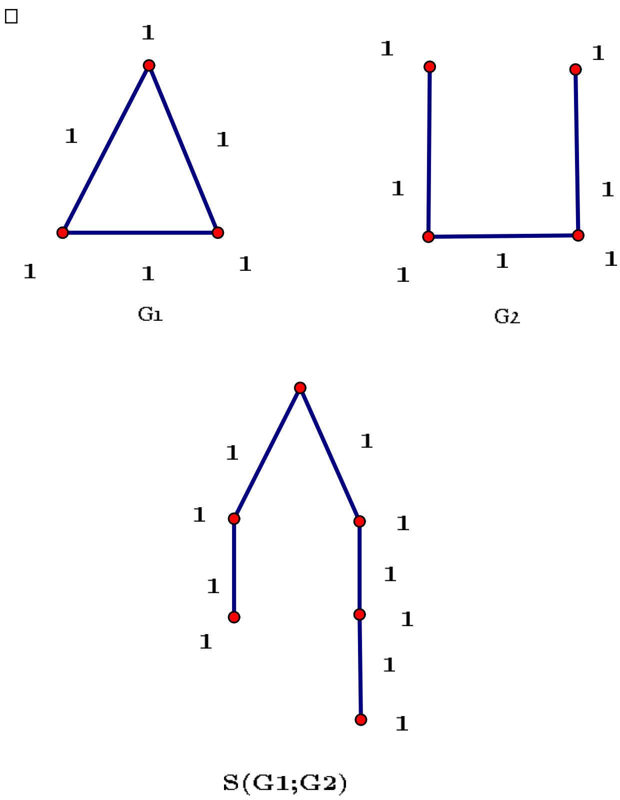


As an application, we provide the following result which is immediate on the operation of union that was defined in [16]:

Theorem 6. *Let $G_1 : (\sigma_1, \mu_1)$ be a fuzzy graph with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ be a fuzzy graph with crisp graph $G_2^* : (V_2, E_2)$. Then*
a) If $E_1 \cap E_2 = \emptyset$, then $G_1 \cup G_2 = G_1 \oplus G_2$.
b) If $E_1 \cap E_2 = \{e\}$, then $G_1 \cup G_2 = S(G_1; G_2)$.

In general, the series connection of two fuzzy graphs needs not be balanced.

Example 3. *Consider the fuzzy graphs G_1, G_2 and $S(G_1; G_2)$ in Figure 2. The first graph with one edge removed is subgraph of $S(G_1; G_2)$ that has density 1 which is greater than$\frac{10}{12} = D(S(G_1; G_2))$.Thus $S(G_1; G_2)$ is not balanced.*



We remark that in the preceding example, G_1 and G_2 are balanced. Thus the series connection of two balanced fuzzy graphs needs not be balanced.

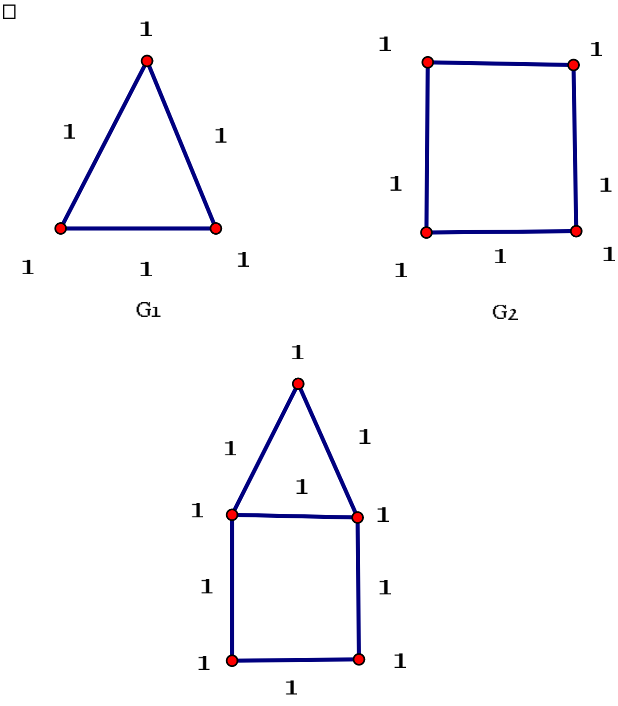
Theorem 7. *The series connection of a fuzzy graph $G_1 : (\sigma_1, \mu_1)$ induced by C_n and a fuzzy graph $G_2 : (\sigma_2, \mu_2)$ induced by C_m is balanced.*

Proof. From the definition of series connection of two fuzzy graphs, it is obvious that $S(G_1; G_2)$ is isomorphic to a fuzzy graph induced by the cycle C_{n+m-1}. Hence by Theorem 5, $S(G_1; G_2)$ is balanced.

We remark that the above result needs not be true in the case of parallel connection.

Example 4. *Consider the fuzzy graphs G_1, G_2 and $P(G_1; G_2)$ in Figure 3. Now G_1 is a subgraph of $P(G_1; G_2)$ and $D(G_1) = 2 > .8 = D(P(G_1; G_2))$. Thus $P(G_1; G_2)$ is not balanced.*

□



P(G1;G2)

Note that in the preceding example, G_1 and G_2 are complete and balanced while $S(G_1; G_2)$ is not complete and not balanced.

Theorem 8. *Let $G_1 : (\sigma_1, \mu_1)$ be a strong fuzzy graph with crisp graph $G_1^* : (V_1, E_1)$ and $G_2 : (\sigma_2, \mu_2)$ be a strong fuzzy graph with crisp graph $G_2^* : (V_2, E_2)$. Then $P(G_1; G_2)$ and $S(G_1; G_2)$ are strong.*

Proof. We only prove the case of parallel connection. The case of series connection is similar.

Any edge $\{x, y\}$ in $P(G_1; G_2)$ is either an edge from G_1 or an edge from G_2 or it is the new edge. In the first two cases, the result is obvious. In the third case, $\mu(x, y) = \mu_1(x_1, x_2) \wedge \mu_2(y_1, y_2)$ for some $\{x_1, x_2\} \in E_1$ and $\{y_1, y_2\} \in E_2$. Thus $\mu(x, y) = \sigma_1(x_1) \wedge \sigma_1(x_2) \wedge \sigma_2(y_1) \wedge \sigma_2(y_2)$. As x_1 and x_2 are identified by x and y_1 and y_2 are identified by y and by definition of $P(G_1; G_2)$, $\mu(x, y) = \sigma(x) \wedge \sigma(y)$. Therefore, $P(G_1; G_2)$ is strong.

We end this section with the following immediate result:

Corollary 3. *Let G_1 and G_2 be complete fuzzy graphs. Then $P(G_1; G_2)$ and $S(G_1; G_2)$ are complete.*

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