Intuitionistic Fuzzy Zweier I-convergent Double Sequence Spaces Defined by Orlicz Function

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Abstract. The purpose of this paper is to introduce the intuitionistic fuzzy Zweier I-convergent double sequence spaces $_2\mathcal{Z}^I_{(\mu,\nu)}(M)$ and $_2\mathcal{Z}^I_{0(\mu,\nu)}(M)$ defined by Orlicz function and study the fuzzy topology on the said spaces.

Key Words and Phrases: Ideal, filter, double I-convergence, intuitionistic fuzzy normed spaces.

1. Introduction and Preliminaries

After the pioneering work of Zadeh [29], a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy set theory is a powerful hand set for modelling uncertainty and vagueness in various problems arising in field of science and engineering. It has a wide range of applications in various fields: population dynamics [3], chaos control [5], computer programming [6], nonlinear dynamical system [7], etc. Fuzzy topology is one of the most important and useful tools and it proves to be very useful for dealing with such situations where the use of classical theories breaks down. The concept of intuitionistic fuzzy normed space [25] and of intuitionistic fuzzy 2-normed space [21] are the latest developments in fuzzy topology. Recently V. A. Khan and Yasmeen([12], [13]) studied the intuitionistic fuzzy Zweier I-convergent sequence spaces defined by modulus function and Orlicz function.

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems/matrices(double sequences) through the concept of density. The notion of I-convergence, which is a generalization of statistical convergence [4], was introduced by Kostyrko, Salat and Wilczynski [14] by using the idea of I of subsets of the set of natural numbers $\mathbb{N}$ and further studied in [22]. Recently, the notion of statistical convergence of double sequences $x = (x_{ij})$ has been defined and studied by Mursaleen and Edely [20]; and for fuzzy numbers by Savaş and Mursaleen [26]. Quite

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recently, Das et al. [15] studied the notion of I and I^*- convergence of double sequences in $\mathbb{R}$.

We recall some notations and basic definitions used in this paper.

Definition 1. *Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be I-convergent to a number L if, for every $\epsilon > 0$, the set $\{k \in \mathbb{N} :\mid x_k - L \mid \geq \epsilon\} \in I$.*

Definition 2. *Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. Then a sequence $x = (x_k)$ is said to be I-Cauchy if, for each $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that the set $\{k \in \mathbb{N} :\mid x_k - x_N \mid \geq \epsilon\} \in I$.*

Definition 3. *The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space(for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:*
   (a) $\mu(x, t) + \nu(x, t) \leq 1$,
   (b) $\mu(x, t) > 0$,
   (c) $\mu(x, t) = 1$ if and only if $x = 0$,
   (d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
   (e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,
   (f) $\mu(x, .) : (0, \infty) \to [0, 1]$ is continuous,
   (g) $\lim_{t \to \infty} \mu(x, t) = 1$ and $\lim_{t \to 0} \mu(x, t) = 0$,
   (h) $\nu(x, t) < 1$,
   (i) $\nu(x, t) = 0$ if and only if $x = 0$,
   (j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,
   (k) $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, t + s)$,
   (l) $\nu(x, .) : (0, \infty) \to [0, 1]$ is continuous,
   (m) $\lim_{t \to \infty} \nu(x, t) = 0$ and $\lim_{t \to 0} \nu(x, t) = 1$.
In this case (μ, ν) is called an intuitionistic fuzzy norm.

Definition 4. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - L, t) > 1 - \epsilon$ and $\nu(x_k - L, t) < \epsilon$ for all $k \geq k_0$. In this case we write $(\mu, \nu) - \lim x = L$.*

Definition 5. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then a sequence $x = (x_k)$ is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_l, t) < \epsilon$ and $\nu(x_k - x_l, t) < \epsilon$ for all $k, l \geq k_0$.*

Definition 6. *Let K be the subset of natural numbers $\mathbb{N}$. Then the asymptotic density of K, denoted by $\delta(K)$, is defined as $\delta(K) = \lim_n \frac{1}{n}|\{k \leq n : k \in K\}|$, where the vertical bars denotes the cardinality of the enclosed set.*

A number sequence $x = (x_k)$ is said to be statistically convergent to a number ℓ if, for each $\epsilon > 0$, the set $K(\epsilon) = \{k \le n :\mid x_k - \ell \mid > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} |\{k \le n :\mid x_k - \ell \mid > \epsilon\}| = 0.$$

In this case we write $st - \lim x = \ell$.

Definition 7. *A number sequence $x = (x_k)$ is said to be statistically Cauchy sequence if, for every $\epsilon > 0$, there exists a number $N = N(\epsilon)$ such that*

$$\lim_n \frac{1}{n} |\{j \le n :\mid x_j - x_N \mid \ge \epsilon\}| = 0.$$

The concepts of statistical convergence and statistical Cauchy for double sequences in intuitionistic fuzzy normed spaces have been studied by Mursaleen and Mohiuddine[14].

Definition 8. *Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal and $(X, \mu, \nu, *, \diamond)$ be an IFNS. A sequence $x = (x_k)$ of elements of X is said to be I-convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if for every $\epsilon > 0$ and $t > 0$, the set*

$$\{k \in \mathbb{N} : \mu(x_k - L, t) \ge 1 - \epsilon \ or \ \nu(x_k - L, t) \le \epsilon\} \in I.$$

In this case L is called the I-limit of the sequence (x_k) with respect to the intuitionistic fuzzy norm (μ, ν) and we write $I_{(\mu,\nu)} - \lim x_k = L$.

2. I_2-Convergence in an IFNS

Definition 9. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{ij})$ is said to be statistically convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and $t > 0$,*

$$\delta(\{(i, j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \le 1 - \epsilon \ or \ \nu(x_{ij} - L, t) \ge \epsilon\}) = 0.$$

or equivalently

$$\lim_{mn} \frac{1}{mn} |\{i \le m, j \le n, : \mu(x_{ij} - L, t) \le 1 - \epsilon \ or \ \nu(x_{ij} - L, t) \ge \epsilon\}| = 0.$$

In this case we write $st^2_{(\mu,\nu)} - \lim x = L$.

Definition 9.2 *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a double sequence $x = (x_{ij})$ is said to be statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν) if, for every*

$\epsilon > 0$ *and* $t > 0$, *there exist* $N = N(\epsilon)$ *and* $M = M(\epsilon)$ *such that for all* $i, p \geq N$ *and* $j, q \geq M$,

$$\delta(\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - x_{pq}, t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - x_{pq}, t) \geq \epsilon\}) = 0.$$

Definition 10. *Let* I_2 *be a non trivial ideal of* $\mathbb{N} \times \mathbb{N}$ *and* $(X, \mu, \nu, *, \diamond)$ *be an intuitionistic fuzzy normed space. A double sequence* $x = (x_{ij})$ *of elements of* X *is said to be* I_2 *convergent to* $L \in X$ *with respect to the intuitionistic fuzzy norm* (μ, ν) *if, for each* $\epsilon > 0$ *and* $t > 0$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : \mu(x_{ij} - L, t) \leq 1 - \epsilon \text{ or } \nu(x_{ij} - L, t) \geq \epsilon\} \in I_2.$$

In this case we write $I_2^{(\mu,\nu)} - \lim x = L.$

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar, Mursaleen [1], Malkowsky [19] Ng and Lee [23], and Wang [28]. Şeng*önü*l [27] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transformation of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & \text{if } (i = k), \\ 1 - p, & (i - 1 = k); (i, k \in \mathbb{N}) \\ 0, & \text{otherwise} . \end{cases}$$

Analogous to Başar and Altay [2], Şeng*önü*l [27] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c\};$$
$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : \mathcal{Z}^p x \in c_0\}.$$

Khan, Ebadullah and Yasmeen [8] introduced the following classes of sequences:

$$\mathcal{Z}^I = \{(x_k) \in \omega : \exists L \in \mathbb{C} \text{ such that for a given } \epsilon > 0, \{k \in \mathbb{N} : \mid x_k^{/} - L \mid \geq \epsilon\} \in I\};$$

$$\mathcal{Z}_0^I = \{(x_k) \in \omega : \text{ for a given } \epsilon > 0; \{k \in \mathbb{N} : \mid x_k^{/} \mid \geq \epsilon\} \in I\}$$

,

where $(x_k^/) = (Z^p x)$.
Recently V.A. Khan and Yasmeen [13] introduced the following sequence spaces:

$$\mathcal{Z}_{(\mu,\nu)}^{I}(M) = \left\{(x_k) \in \omega : \left\{k \in \mathbb{N} : M\left(\frac{\mu(x_k^/ - L,t)}{\rho}\right) \leq 1 - \epsilon \text{ or } M\left(\frac{\nu(x_k^/ - L,t)}{\rho}\right) \geq \epsilon\right\} \in I\right\},$$

$$\mathcal{Z}_{0(\mu,\nu)}^{I}(M) = \left\{(x_k) \in \omega : \left\{k \in \mathbb{N} : M\left(\frac{\mu(x_k^/,t)}{\rho}\right) \leq 1 - \epsilon \text{ or } M\left(\frac{\nu(x_k^/,t)}{\rho}\right) \geq \epsilon\right\} \in I\right\}.$$

In this article we introduce the intuitionistic Zweier I-convergent double sequence spaces defined by Orlicz function as follows:

$$_2\mathcal{Z}_{(\mu,\nu)}^{I}(M) = \left\{(x_{ij}) \in {}_2\omega : \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu(x_{ij}^{//} - L,t)}{\rho}\right) \leq 1 - \epsilon \text{ or } M\left(\frac{\nu(x_{ij}^{//} - L,t)}{\rho}\right) \geq \epsilon\right\} \in I_2\right\};$$

$$_2\mathcal{Z}_{0(\mu,\nu)}^{I}(M) = \left\{(x_{ij}) \in {}_2\omega : \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu(x_{ij}^{//},t)}{\rho}\right) \leq 1 - \epsilon \text{ or } M\left(\frac{\nu(x_{ij}^{//},t)}{\rho}\right) \geq \epsilon\right\} \in I_2\right\}.$$

3. Main results

Theorem 1. *The spaces $_2\mathcal{Z}_{(\mu,\nu)}^{I}(M)$ and $_2\mathcal{Z}_{0(\mu,\nu)}^{I}(M)$ are linear spaces.*

Proof. We prove the result for $_2\mathcal{Z}_{(\mu,\nu)}^{I}(M)$. Similarly the result can be proved for $_2\mathcal{Z}_{0(\mu,\nu)}^{I}(M)$. Let $(x_{ij}^{//})$, $(y_{ij}^{//}) \in {}_2\mathcal{Z}_{(\mu,\nu)}^{I}(M)$ and let α, β be scalars. Then for a given $\epsilon > 0$, we have

$$A_1 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu\left(x_{ij}^{//} - L_1, \frac{t}{2|\alpha|}\right)}{\rho_1}\right) \leq 1 - \epsilon \text{ or } \left(\frac{\nu\left(x_{ij}^{//} - L_1, \frac{t}{2|\alpha|}\right)}{\rho_1}\right) \geq \epsilon\right\} \in I_2;$$

$$A_2 = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu\left(y_{ij}^{//} - L_2, \frac{t}{2|\beta|}\right)}{\rho_2}\right) \leq 1 - \epsilon \text{ or } M\left(\frac{\nu\left(y_{ij}^{//} - L_2, \frac{t}{2|\beta|}\right)}{\rho_2}\right) \geq \epsilon\right\} \in I_2. \text{ Thus}$$

$$A_1^c = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu\left(x_{ij}^{//} - L_1, \frac{t}{2|\alpha|}\right)}{\rho_1}\right) > 1 - \epsilon \text{ or } M\left(\frac{\nu\left(x_{ij}^{//} - L_1, \frac{t}{2|\alpha|}\right)}{\rho_1}\right) < \epsilon\right\} \in \mathcal{F}(I_2);$$

$$A_2^c = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu\left(y_{ij}^{//} - L_2, \frac{t}{2|\beta|}\right)}{\rho_2}\right) > 1 - \epsilon \text{ or } M\left(\frac{\nu\left(y_{ij}^{//} - L_2, \frac{t}{2|\beta|}\right)}{\rho_2}\right) < \epsilon\right\} \in \mathcal{F}(I_2).$$

Define the set $A_3 = A_1 \cup A_2$, so that $A_3 \in I_2$. It follows that A_3^c is a non-empty set in $\mathcal{F}(I_2)$.
We shall show that for $\rho_3 = max\{2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2\}$ and for each $(x_{ij}^{//})$, $(y_{ij}^{//}) \in {}_2\mathcal{Z}_{(\mu,\nu)}^{I}(M)$,

$$A_3^c \subset \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu\left((\alpha x_{ij}^{//} + \beta y_{ij}^{//}) - (\alpha L_1 + \beta L_2), t\right)}{\rho_3}\right) > 1 - \epsilon \text{ or }\right.$$

$$\left.M\left(\frac{\nu\left((\alpha x_{ij}^{//} + \beta y_{ij}^{//}) - (\alpha L_1 + \beta L_2), t\right)}{\rho_3}\right) < \epsilon\right\}.$$

Let $(m,n) \in A_3^I$. In this case

$$M\Big(\frac{\mu\big(x_{mn}^{//}-L_1,\frac{t}{2|\alpha|}\big)}{\rho_3}\Big) > 1-\epsilon \text{ or } M\Big(\frac{\nu\big(x_{mn}^{//}-L_1,\frac{t}{2|\alpha|}\big)}{\rho_3}\Big) < \epsilon$$

and

$$M\Big(\frac{\mu\big(y_{mn}^{//}-L_2,\frac{t}{2|\beta|}\big)}{\rho_3}\Big) > 1-\epsilon \text{ or } M\Big(\frac{\nu\big(y_{mn}^{//}-L_2,\frac{t}{2|\beta|}\big)}{\rho_3}\Big) < \epsilon.$$

we have

$$M\Big(\frac{\mu\big((\alpha x_{mn}^{//}+\beta y_{mn}^{//})-(\alpha L_1+\beta L_2),t\big)}{\rho_3}\Big)$$
$$\geq M\Big(\frac{\mu\big(\alpha x_{mn}^{//}-\alpha L_1,\frac{t}{2}\big)}{\rho_3}\Big) * M\Big(\frac{\mu\big(\beta y_{mn}^{//}-\beta L_2,\frac{t}{2}\big)}{\rho_3}\Big)$$
$$= M\Big(\frac{\mu\big(x_{mn}^{//}-L_1,\frac{t}{2|\alpha|}\big)}{\rho_3}\Big) * M\Big(\frac{\mu\big(y_{mn}^{//}-L_2,\frac{t}{2|\beta|}\big)}{\rho_3}\Big)$$
$$> (1-\epsilon)*(1-\epsilon) = (1-\epsilon)$$

and

$$M\Big(\frac{\nu\big((\alpha x_{mn}^{//}+\beta y_{mn}^{//})-(\alpha L_1+\beta L_2),t\big)}{\rho_3}\Big)$$
$$\leq M\Big(\frac{\nu\big(\alpha x_{mn}^{//}-\alpha L_1,\frac{t}{2}\big)}{\rho_3}\Big) \diamond M\Big(\frac{\nu\big(\beta y_{mn}^{//}-\beta L_2,\frac{t}{2}\big)}{\rho_3}\Big)$$
$$= M\Big(\frac{\nu\big(x_{mn}^{//}-L_1,\frac{t}{2|\alpha|}\big)}{\rho_3}\Big) \diamond M\Big(\frac{\nu\big(y_{mn}^{//}-L_2,\frac{t}{2|\beta|}\big)}{\rho_3}\Big)$$
$$> \epsilon \diamond \epsilon = \epsilon.$$

This implies that

$$A_3^c \subset \Big\{(i,j) \in \mathbb{N}\times\mathbb{N} : M\Big(\frac{\mu\big((\alpha x_{ij}^{//}+\beta y_{ij}^{//})-(\alpha L_1+\beta L_2),t\big)}{\rho_3}\Big) > 1-\epsilon \text{ or}$$

$$M\Big(\frac{\nu\big((\alpha x_{ij}^{//}+\beta y_{ij}^{//})-(\alpha L_1+\beta L_2),t\big)}{\rho_3}\Big) < \epsilon\Big\}.$$

Hence $_2\mathcal{Z}_{(\mu,\nu)}^I(M)$ is a linear space.

Theorem 2. *Every open ball $_2B_{x^{//}}(r,t)(M)$ is an open set in $_2\mathcal{Z}_{(\mu,\nu)}^I(M)$.*

Proof. Let $_2B_{x^{//}}(r,t)(M)$ be an open ball with centre $x^{//}$ and radius r with respect to t.
That is

$$_2B_{x^{//}}(r,t)(M) = \{(i,j)\in\mathbb{N}\times\mathbb{N} : M\Big(\frac{\mu(x_{ij}^{//}-L,t)}{\rho}\Big) \leq 1-r \text{ or } M\Big(\frac{\nu(x_{ij}^{//}-L,t)}{\rho}\Big) \geq r\} \in I_2.$$

Let $y^{//} \in {}_2B_{x^{//}}^c(r,t)(M)$. Then

$$M\Big(\frac{\mu(x^{//}-y^{//},t)}{\rho}\Big) > 1-r \text{ and } M\Big(\frac{\nu(x^{//}-y^{//},t)}{\rho}\Big) < r$$

.

Since $M\left(\frac{\mu(x^{//}-y^{//},t)}{\rho}\right) > 1 - r$, there exists $t_0 \in (0,1)$ such that

$$M\left(\frac{\mu(x^{//}-y^{//},t_0)}{\rho}\right) > 1 - r \quad \text{and} \quad M\left(\frac{\nu(x^{//}-y^{//},t_0)}{\rho}\right) < r.$$

Putting $r_0 = M\left(\frac{\mu(x^{//}-y^{//},t_0)}{\rho}\right)$. We have $r_0 > 1 - r$, there exists $s \in (0,1)$ such that $r_0 > 1 - s > 1 - r$.

For $r_0 > 1 - s$, we have $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1 - s$ and $(1-r_0) \diamond (1-r_2) \leq s$.
Putting $r_3 = max\{r_1, r_2\}$, consider the ball $_2 B_{y^{//}}^c(1-r_3, t-t_0)(M)$. We prove that
$_2 B_{y^{//}}^c(1-r_3, t-t_0)(M) \subset {}_2 B_{x^{//}}^c(r,t)(M)$.
Let $z^{//} \in {}_2 B_{y^{//}}^c(1-r_3, t-t_0)(M)$.
$M\left(\frac{\mu(y^{//}-z^{//},t-t_0)}{\rho}\right) > r_3$ and $M\left(\frac{\nu(y^{//}-z^{//},t-t_0)}{\rho}\right) < r_3$.
Therefore,

$$M\left(\frac{\mu(x^{//}-z^{//},t)}{\rho}\right) \geq M\left(\frac{\mu(x^{//}-y^{//},t_0)}{\rho}\right) * M\left(\frac{\mu(y^{//}-z^{//},t-t_0)}{\rho}\right)$$

$$\geq (r_0 * r_3) \geq (r_0 * r_1) \geq (1-s) > (1-r).$$

and

$$M\left(\frac{\nu(x^{//}-z^{//},t)}{\rho}\right) \leq M\left(\frac{\nu(x^{//}-y^{//},t_0)}{\rho}\right) \diamond M\left(\frac{\nu(y^{//}-z^{//},t-t_0)}{\rho}\right)$$

$$\leq (1-r_0) \diamond (1-r_3) \leq (1-r_0) > (1-r_2) < s < r.$$

Thus $z^{//} \in {}_2 B_{x^{//}}^c(r,t)(M)$ and hence $_2 B_{y^{//}}^c(1-r_3, t-t_0)(M) \subset {}_2 B_{x^{//}}^c(r,t)(M)$.

Remark 1. $_2 \mathcal{Z}_{(\mu,\nu)}^I(M)$ *is IFNS.*

Define
$_2 \tau_{(\mu,\nu)}(M) = \{A \subset {}_2 \mathcal{Z}_{(\mu,\nu)}^I(M) : \text{ for each } x \in A \text{ there exists } t > 0 \text{ and } r \in (0,1) \text{ s. t. } _2 B_{x^{//}}^c(r,t)(M) \subset A\}$
Then $_2 \tau_{(\mu,\nu)}(M)$ is a topology on $_2 \mathcal{Z}_{(\mu,\nu)}^I(M)$.

Theorem 3. *The topology $_2 \tau_{(\mu,\nu)}(M)$ on $_2 \mathcal{Z}_{(\mu,\nu)}^I(M)$ is first countable.*

Proof. $\left\{_2 B_{x^{//}}\left(\frac{1}{n}, \frac{1}{n}\right)(M) : n = 1,2,3, ...\right\}$ is a local base at $x^{//}$ the topology $_2 \tau_{(\mu,\nu)}(M)$ on $_2 \mathcal{Z}_{(\mu,\nu)}^I(M)$ is first countable.

Theorem 4. $_2 \mathcal{Z}_{(\mu,\nu)}^I(M)$ *and* $_2 \mathcal{Z}_{0(\mu,\nu)}^I(M)$ *are Housdorff spaces.*

Proof. We prove the result for $_2\mathcal{Z}^I_{(\mu,\nu)}(M)$. Similarly the result can be proved for $_2\mathcal{Z}^I_{0(\mu,\nu)}(M)$.

Let $x^{//}, y^{//} \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(M)$ such that $x^{//} \neq y^{//}$.

Then $0 < M\left(\frac{\mu(x^{//}-y^{//},t)}{\rho}\right) < 1$

and $0 < M\left(\frac{\nu(x^{//}-y^{//},t)}{\rho}\right) < 1$.

Putting $r_1 = M\left(\frac{\mu(x^{//}-y^{//},t)}{\rho}\right)$ and $r_2 = M\left(\frac{\nu(x^{//}-y^{//},t)}{\rho}\right)$ and $r = max\{r_1, 1-r_2\}$.

For each $r_0 \in (r,1)$, there exists r_3 and r_4 such that

$$r_3 * r_4 \geq r_0 \quad \text{and} \quad (1-r_3) \diamond (1-r_4) \leq (1-r_0).$$

Putting $r_5 = max\{r_3, 1-r_4\}$ and consider the open balls $_2B_{x^{//}}(1-r_5, \frac{t}{2})(M)$ and $_2B_{y^{//}}(1-r_5, \frac{t}{2})(M)$.

Then clearly $_2B_{x^{//}}(1-r_5, \frac{t}{2}) \quad \cap {}_2B_{y^{//}}(1-r_5, \frac{t}{2})(M) = \phi$.

For if there exists $z^{//} \in {}_2B_{x^{//}}(1-r_5, \frac{t}{2}) \quad \cap {}_2B_{y^{//}}(1-r_5, \frac{t}{2})(M)$, then

$$r_1 = M\left(\frac{\mu(x^{//}-y^{//},t)}{\rho}\right) \geq M\left(\frac{\mu(x^{//}-z^{//}, \frac{t}{2})}{\rho}\right) * M\left(\frac{\mu(z^{//}-y^{//}, \frac{t}{2})}{\rho}\right)$$

$$\geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1.$$

and

$$r_2 = M\left(\frac{\nu(x^{//}-y^{//},t)}{\rho}\right) \leq M\left(\frac{\mu(x^{//}-z^{//}, \frac{t}{2})}{\rho}\right) \diamond M\left(\frac{\nu(z^{//}-y^{//}, \frac{t}{2})}{\rho}\right)$$

$$\leq (1-r_5) \diamond (1-r_5) \leq (1-r_4) \diamond (1-r_4) \leq (1-r_0) < r,$$

which is a contradiction.

Hence $_2\mathcal{Z}^I_{(\mu,\nu)}(M)$ is Housdorff.

Theorem 5. $_2\mathcal{Z}^I_{(\mu,\nu)}(m)$ *is an IFNS.* $_2\tau_{(\mu,\nu)}(M)$ *is a topology on* $_2\mathcal{Z}^I_{(\mu,\nu)}(M)$. *Then a sequence* $(x^{//}_{ij}) \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(M)$, $x^{//}_{ij} \to x^{//}$ *if and if* $\mu(x^{//}_{ij} - x^{//},t)(M)$ *and*

$$M\left(\frac{\nu(x^{//}_{ij} - x^{//},t)}{\rho}\right) \to 0$$

as $i \to \infty, j \to \infty$.

Proof. Fix $t_0 > 0$. Suppose $x^{//}_{ij} \to x^{//}$. Then for $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $x^{//}_{ij} \in {}_2B_{x^{//}}(r,t)(M)$ for all $i \geq n_0, j \geq n_0$.

$$_2B_{x^{//}}(r,t)(M) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\left(\frac{\mu(x^{//}_{ij} - x^{//},t)}{\rho}\right) \leq 1-r \text{ or } M\left(\frac{\nu(x^{//}_{ij} - x^{//},t)}{\rho}\right) \geq r\} \in I_2$$

such that $_2B^c_{x^{//}}(r,t)(M) \in \mathcal{F}(I_2)$.
Then

$$1 - M\Big(\frac{\mu(x^{//}_{ij} - x^{//}, t)}{\rho}\Big) < r$$

and $M\Big(\frac{\nu(x^{//}_{ij} - x^{//}, t)}{\rho}\Big) \to 0$ as $i \to \infty, j \to \infty$

Conversely, if for each $t > 0$, $M\Big(\frac{\mu(x^{//}_{ij} - x^{//}, t)}{\rho}\Big) \to 1$ and

$$M\Big(\frac{\nu(x^{//}_{ij} - x^{//}, t)}{\rho}\Big) \to 0$$

as $i \to \infty, j \to \infty, then for r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - M\Big(\frac{\mu(x^{//}_{ij} - x^{//}, t)}{\rho}\Big) < r$ for all $i \geq n_0, j \geq n_0$.
  Thus $x^{//}_{ij} \in {}_2B^c_{x_{ij}}(r,t)(M)$ for all $i \geq n_0, j_0 \geq n_0$ and hence $x^{//}_{ij} \to x^{//}$.

Theorem 6. *A double sequence $x = (x^{//}_{ij}) \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(M)$. I-converges if and only if for every $\epsilon > 0$ and $t > 0$ there exists a number $M = M(x, \epsilon, t), N = N(x, \epsilon, t)$ such that*
$$\{(M,N) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{\mu(x^{//}_{MN} - L, \frac{t}{2})}{\rho}\Big) > 1 - \epsilon \text{ or } M\Big(\frac{\nu(x^{//}_{MN} - L, \frac{t}{2})}{\rho}\Big) < \epsilon\} \in \mathcal{F}(I_2).$$

  Proof. Suppose that $_2I_{(\mu,\nu)} - \lim x = L$ and let $\epsilon > 0$ and $t > 0$. For a given $\epsilon > 0$ choose, $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < s$. Then for each $x \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(M)$,

$$A_x(\epsilon, t)(M) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{\mu(x^{//}_{ij} - L, \frac{t}{2})}{\rho}\Big) \leq 1 - \epsilon \text{ or } M\Big(\frac{\nu(x^{//}_{ij} - L, \frac{t}{2})}{\rho}\Big) \geq \epsilon\} \in I_2$$
which implies that
$$A^c_x(\epsilon, t)(M) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{\mu(x^{//}_{ij} - L, \frac{t}{2})}{\rho}\Big) > 1 - \epsilon \text{ or } M\Big(\frac{\nu(x^{//}_{ij} - L, \frac{t}{2})}{\rho}\Big) < \epsilon\} \in \mathcal{F}(I_2).$$
Conversely let us choose $N \in A^c_x(\epsilon, t)(M)$. Then

$$M\Big(\frac{\mu(x^{//}_N - L, \frac{t}{2})}{\rho}\Big) > 1 - \epsilon \text{ or } M\Big(\frac{\nu(x^{//}_N - L, \frac{t}{2})}{\rho}\Big) < \epsilon.$$

  Now we want to show that there exists a number $N = N(x, \epsilon, t)$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{\mu(x^{//}_{ij} - x^{//}_N, t)}{\rho}\Big) \leq 1 - s \text{ or } M\Big(\frac{\nu(x^{//}_{ij} - x^{//}_N, t)}{\rho}\Big) \geq s\} \in I_2.$$

  For this, define for each $x \in {}_2\mathcal{Z}^I_{(\mu,\nu)}(M)$.

$$_2B_x(\epsilon, t)(M) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : M\Big(\frac{\mu(x^{//}_{ij} - x^{//}_N, t)}{\rho}\Big) \leq 1 - s \text{ or } M\Big(\frac{\nu(x^{//}_{ij} - x^{//}_N, t)}{\rho}\Big) \geq s\} \in I_2.$$

Now we show that $_2B_x(\epsilon, t)(M) \subset {}_2A_x(\epsilon, t)(M)$.

Suppose that $_2B_x(\epsilon,t)(M) \subset {_2A_x}(\epsilon,t)(M)$.

Then there exists $(m,n) \in {_2B_x}(\epsilon,t)(M)$ and $(m,n) \notin {_2A_x}(\epsilon,t)(M)$. Therefore we have

$$M\Big(\frac{\mu(x_{ij}^{//} - x_N^{//}, t)}{\rho}\Big) < 1 - s$$

and $M\Big(\frac{\mu(x_{ij}^{//} - L, \frac{t}{2})}{\rho}\Big) > 1 - \epsilon$.

In particular $M\Big(\frac{\mu(x_N^{//} - L, \frac{t}{2})}{\rho}\Big) > 1 - \epsilon$.

Therefore we have

$$1 - s \geq M\Big(\frac{\mu(x_{mn}^{//} - x_N^{//}, t)}{\rho}\Big) \geq M\Big(\frac{\mu(x_{mn}^{//} - L, \frac{t}{2})}{\rho}\Big) * M\Big(\frac{\mu(x_N^{//} - L, \frac{t}{2})}{\rho}\Big)$$

$$\geq (1 - \epsilon) * (1 - \epsilon) > 1 - s,$$

which is not possible.

On the other hand $M\Big(\frac{\nu(x_{ij}^{//} - x_N^{//}, t)}{\rho}\Big) \geq s$ and $M\Big(\frac{\nu(x_{ij}^{//} - L, \frac{t}{2})}{\rho}\Big) < \epsilon$.

In particular $M\Big(\frac{\nu(x_N^{//} - L, \frac{t}{2})}{\rho}\Big) < \epsilon$.

Therefore we have

$$s \leq M\Big(\frac{\nu(x_{mn}^{//} - x_N^{//}, t)}{\rho}\Big) \leq M\Big(\frac{\nu(x_{mn}^{//} - L, \frac{t}{2})}{\rho}\Big) \diamond M\Big(\frac{\nu(x_N^{//} - L, \frac{t}{2})}{\rho}\Big) \leq \epsilon \diamond \epsilon < s,$$

which is not possible.

Hence $_2B_x(\epsilon,t)(M) \subset {_2A_x}(\epsilon,t)(M)$, $_2A_x(\epsilon,t)(M) \in I_2$.

This implies that $_2B_x(\epsilon,t)(M) \in I_2$. Hence proved.

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