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On intra-regular ordered Γ-semigroups

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Abstract. We use the definition of intra-regularity (left regularity) of *po*-Γ-semigroups introduced in 2016 in Armenian Journal of Mathematics. Being able to describe the form of the elements of the principal filter by using this definition, we study the decomposition of an intra-regular *po*-Γ-semigroup into simple components. Then we prove that a *po*-Γ-semigroup M is intra-regular and the ideals of M form a chain if and only if M is a chain of simple semigroups. Moreover, a *po*-Γ-semigroup M is intra-regular and the ideals of M form a chain if and only if the ideals of M are prime. Finally, for an intra-regular *po*-Γ-semigroup M, the set $\{(x)_{\mathcal{N}} \mid x \in M\}$ coincides with the set of all maximal simple subsemigroups of M. A decomposition of some left regular *po*-Γ-semigroups into their left simple components is also given.

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1. Introduction and prerequisites

The notion of a Γ-ring, a generalization of the concept of associative rings, has been introduced and studied by Nobusawa in [11]. Γ-rings have been also studied by Barnes in [1]. Luh studied many properties of simple Γ-rings and primitive Γ-rings in [10]. The concept of a Γ-semigroup has been introduced by Sen in 1981 as follows: Given two nonempty sets S and Γ, S is called a Γ-semigroup if the following assertions are satisfied:

(1) $a\alpha b \in S$ and $\alpha a\beta \in \Gamma$ and

(2) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$

for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$ [13]. In 1986 Sen and Saha gave a second definition of Γ-semigroups as follows: Let $S = \{a, b, c, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then S is called a Γ-semigroup if

(1) $a\alpha b \in S$ and

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$

for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$ [14] (the sets S and Γ should not be denumerable). One can find this definition of Γ-semigroups in [17] where the notion of a radical in Γ-semigroups and the notion of ΓS-act over a Γ-semigroup have been introduced and in [15]

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and [16] where the notions of regular and orthodox Γ-semigroups have been introduced and studied. But still we cannot say that Γ is a set of binary operations on the set M. Probably this is why Saha defines in [12] the Γ-semigroup as follows: Given two nonempty sets S and Γ, S is called a Γ-semigroup if there exists a mapping

$$S \times \Gamma \times S \to S \mid (a, \gamma, b) \to a\gamma b$$

such that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$ and remarks that most usual semigroup concepts, in particular regular and inverse semigroups, have their analogous for Γ-semigroups. Defining the Γ-semigroup via mappings, in an expression of the form $a_1\gamma_1 a_2\gamma_2 a_3 \ldots. a_n\gamma_n$, we can put parentheses in any place beginning with some a_i and ending in some a_j. The ordered Γ-semigroups have been first considered by Sen and Seth in [18].

Let M and Γ be two nonempty sets. Denote by $M\Gamma M$ the set of (all) elements of the form $a\gamma b$, where $a, b \in M$ and $\gamma \in \Gamma$. That is,

$$M\Gamma M := \{a\gamma b \mid a, b \in M, \gamma \in \Gamma\}.$$

Then M is called a Γ-semigroup [5, 6] if the following assertions are satisfied:

(1) $M\Gamma M \subseteq M$;
(2) if $a, b, c, d \in M$, $\gamma, \mu \in \Gamma$, $a = b$, $\gamma = \mu$ and $c = d$, then $a\gamma c = b\mu d$;
(3) $a\gamma(b\mu c) = (a\gamma b)\mu c \ \forall\, a, b, c \in M \ \forall\, \gamma, \mu \in \Gamma$.

In other words, Γ is a set of binary operations on M and the following condition is satisfied: $a\gamma(b\mu c) = (a\gamma b)\mu c \ \forall\, a, b, c \in M \ \forall\, \gamma, \mu \in \Gamma$.

A Γ-semigroup endowed with an order relation "$\leq$" such that $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for every $c \in M$ and every $\gamma \in \Gamma$ is called an ordered Γ-semigroup (shortly, *po*-Γ-semigroup). For a *po*-Γ-semigroup M and a subset H of M we denote by $(H]$ the subset of M defined by $(H] = \{t \in M \mid t \leq a \text{ for some } a \in H\}$. We have $M = (M]$, and for any two subsets A, B of M, we have $A \subseteq (A]$; if A is a right (or left) ideal of M, then $A = (A]$; $(A]\Gamma(B] \subseteq (A\Gamma B]$; if $A \subseteq B$, then $(A] \subseteq (B]$; $((A]] = (A]$. Let M be a *po*-Γ-semigroup. A nonempty subset A of M is called a *subsemigroup* of M if for every $a, b \in A$ and every $\gamma \in \Gamma$ we have $a\gamma b \in A$, that is, if $A\Gamma A \subseteq A$. A nonempty subset A of M is called a *left* (resp. *right*) *ideal* of M if (1) $M\Gamma A \subseteq A$ and (2) if $a \in A$ and $M \ni b \leq a$, then $b \in A$. It is called an *ideal* of M if it is both a left ideal and a right ideal of M. Clearly, every left (resp. right) ideal of M is a subsemigroup of M. For an element a of M, we denote by $L(a)$, $R(a)$ and $I(a)$ the left ideal, the right ideal and the ideal of M, respectively, generated by a, and we have $L(a) = (a \cup M\Gamma a]$, $R(a) = (a \cup a\Gamma M]$, and $I(a) = (a \cup M\Gamma a \cup a\Gamma M \cup M\Gamma a\Gamma M]$. We denote by $\mathcal{L}$ the equivalence relation on M defined by $a\mathcal{L}b$ if and only if $L(a) = L(b)$, by $\mathcal{R}$ the equivalence relation on M defined by $a\mathcal{R}b$ if and only if $R(a) = R(b)$ and by $\mathcal{I}$ the equivalence relation on M defined by $a\mathcal{I}b$ if and only if $I(a) = I(b)$. A subsemigroup F of M is called a *filter* of M if (1) $a, b \in F$ and $\gamma \in \Gamma$ such that $a\gamma b \in F$ implies $a \in F$ and $b \in F$ and (2) if $a \in F$ and $M \ni c \geq a$, then $c \in F$. An equivalence relation σ on M is called *congruence* if $(a, b) \in \sigma$ implies $(a\gamma c, b\gamma c) \in \sigma$ and $(c\gamma a, c\gamma b) \in \sigma$ for any $c \in M$ and any $\gamma \in \Gamma$. A congruence σ on M

is called *semilattice congruence* if $(a\gamma b, b\gamma a) \in \sigma$ and $(a\gamma a, a) \in \sigma$ for every $a, b \in M$ and every $\gamma \in \Gamma$. If σ is a semilattice congruence on M, then the σ-class $(a)_\sigma$ of M containing a is a subsemigroup of M for every $a \in M$. A semilattice congruence σ on M is called *complete* if $a \le b$ implies $(a, a\gamma b) \in \sigma$ for every $\gamma \in \Gamma$. We denote by $\mathcal{N}$ the relation on M defined by $a\mathcal{N}b$ if and only if the filters of M generated by the elements a and b of M coincide. As in semigroups, the relation $\mathcal{N}$ is a semilattice congruence on M. So, if $z \in M$ and $\gamma \in \Gamma$, then we have $(z\gamma z, z) \in \mathcal{N}$, $(z\gamma z\gamma z, z\gamma z) \in \mathcal{N}$, $(z\gamma z\gamma z\gamma z, z\gamma z\gamma z) \in \mathcal{N}$ and so on. In particular, exactly as in ordered semigroups, the relation $\mathcal{N}$ is a complete semilattice congruence on M. A subsemigroup T of a *po*-Γ-semigroup M is called *left* (resp. *right*) *simple* if for every left (resp. right) ideal A of T we have $A = T$, that is if T is the only left (resp. right) ideal of T. It is called *simple* if it is both left and right simple. A subsemigroup T of a *po*-Γ-semigroup M is called *maximal simple* if for any simple subsemigroup A of M such that $A \supseteq T$, we have $A = T$. A *po*-Γ-semigroup M is said to be a *semilattice of simple* (resp. *left simple*) *semigroups* if there exists a semilattice congruence σ on M such that the σ-class $(x)_\sigma$ of M containing x is a simple (resp. left simple) subsemigroup of M for every $x \in M$. A *po*-Γ-semigroup M is called a *chain of simple semigroups* if there exists a semilattice congruence σ on M such that $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$ and the set M/σ of all σ-classes of M endowed with the order relation $(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (x\gamma y)_\sigma \ \forall\, \gamma \in \Gamma$ is a chain.

Many results on Γ-semigroups (or *po*-Γ-semigroups) can be obtained from semigroups (ordered semigroups) just putting a "Gamma" in the appropriate place. But there are also results for which the transfer is not so easy. Both for a Γ-semigroup or *po*-Γ-semigroup M the filter of M generated by an element a of M plays an essential role in the structure, in particular, in the decomposition of M. So it is important to get the form of its elements. The definition of intra-regularity of a *po*-Γ-semigroup in the bibliography was as follows: A *po*-Γ-semigroup M is intra-regular if $a \in (M\Gamma a\Gamma a\Gamma M]$ for every $a \in M$. With this definition is not possible to describe the form of the elements of the $N(x)$ $(x \in S)$. To overcome this difficulty, the following new concept of intra-regularity has been introduced in [9]: We say that a *po*-Γ-semigroup M is intra-regular if $a \in (M\Gamma a\gamma a\Gamma M]$ for every $a \in M$ and every $\gamma \in \Gamma$. Using this definition, we first give a structure theorem referring to the decomposition of a *po*-Γ-semigroup into simple components. Then, for a *po*-Γ-semigroup M, we prove the following: The ideals of M are weakly prime if and only if they are idempotent and they form a chain. The ideals of M are prime if and only if they form a chain and M is intra-regular. M is intra-regular and the ideals of M form a chain if and only if M is a chain of simple semigroups. For an intra-regular *po*-Γ-semigroup M the set $\{(x)_\mathcal{N} \mid x \in M\}$ coincides with the set of all maximal simple subsemigroups of M. Keeping the new definition of left (right) regularity of *po*-Γ-semigroups introduced in [9], we also give a structure theorem related to the decomposition of a *po*-Γ-semigroup M which is left regular and satisfies the relation $(x\Gamma M] \subseteq (M\Gamma x]$ for every $x \in M$ into left simple components. The results of this paper are based on the corresponding results on ordered semigroups considered in [3] and [4], and the aim of writing this paper is to show the importance of these new concepts of intra-regularity and left (right) regularity in the investigation.

2. Main results

Let M be a *po*-Γ-semigroup. A subset A of M is called *idempotent*, if $A = (A\Gamma A]$. A subset T of M is called *prime* if $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$ implies $a \in T$ or $b \in T$. The set T is called *semiprime* if $a \in M$ and $\gamma \in \Gamma$ such that $a\gamma a \in T$ implies $a \in T$ [9]. A subset T of M is called *weakly prime* if the following assertion is satisfied:

If A, B are ideals of M such that $A\Gamma B \subseteq T$, then $A \subseteq T$ or $B \subseteq T$.

For a subset T of M, we consider the statements:

(1) $a, b \in M$, $\gamma \in \Gamma$, $a\gamma b \in T \Longrightarrow a \in T$ or $b \in T$.

(2) $A, B \subseteq M$, $A\Gamma B \subseteq T \Longrightarrow A \subseteq T$ or $B \subseteq T$.

Then (1) $\Rightarrow$ (2). In fact: Let $A, B \subseteq M$, $A\Gamma B \subseteq T$, $A \nsubseteq T$ and $b \in B$. Take an element $a \in A$ such that $a \notin T$ and an element $\gamma \in \Gamma$ ($\Gamma \neq \emptyset$). Since $a\gamma b \in A\Gamma B \subseteq T$, by (1), we have $a \in T$ or $b \in T$. Since $a \notin T$, we get $b \in T$.

We have the following:

(a) If T is a prime subset of M, then T is a semiprime subset of M.

(b) If T is a prime subset of M, then T is a weakly prime subset of M.

Definition 1. [9] A *po*-Γ-semigroup M is called *intra-regular* if

$$x \in (M\Gamma x\gamma x\Gamma M]$$

for every $x \in M$ and every $\gamma \in \Gamma$.

Proposition 2. *If M is an intra-regular po-Γ-semigroup then, for every $x, y \in M$ and every $\gamma \in \Gamma$, we have $(M\Gamma x\gamma y\Gamma M] = (M\Gamma y\gamma x\Gamma M]$.*

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Since $x\gamma y \in M\Gamma M \subseteq M$ and M is intra-regular, we have

$$\begin{aligned} x\gamma y &\in \Big(M\Gamma(x\gamma y)\gamma(x\gamma y)\Gamma M\Big] \subseteq \Big((M\Gamma M)\Gamma(y\gamma x)\Gamma(M\Gamma M)\Big] \\ &\subseteq (M\Gamma y\gamma x\Gamma M]. \end{aligned}$$

Then we have

$$\begin{aligned} M\Gamma(x\gamma y)\Gamma M &\subseteq (M]\Gamma(M\Gamma y\gamma x\Gamma M]\Gamma(M] \subseteq (M\Gamma M\Gamma y\gamma x\Gamma M\Gamma M] \\ &\subseteq (M\Gamma y\gamma x\Gamma M], \end{aligned}$$

from which $(M\Gamma(x\gamma y)\Gamma M] \subseteq \Big((M\Gamma y\gamma x\Gamma M]\Big] = (M\Gamma y\gamma x\Gamma M]$. Since M is intra-regular and $y\gamma x \in M$, by symmetry, we get $(M\Gamma y\gamma x\Gamma M] \subseteq (M\Gamma x\gamma y\Gamma M]$, thus we have $(M\Gamma x\gamma y\Gamma M] = (M\Gamma y\gamma x\Gamma M]$. □

Lemma 3. [9] *A po-Γ-semigroup M is intra-regular if and only if, for every $x \in M$, we have*

$$N(x) = \{y \in M \mid x \in (M\Gamma y\Gamma M]\}.$$

In a similar way as in [2] and [7], we can prove the following lemma.

Lemma 4. *If M is a po-Γ-semigroup, then $\mathcal{I} \subseteq \mathcal{N}$ and $\mathcal{I} \subseteq \mathcal{L}$.*

Lemma 5. [9] *A po-Γ-semigroup M is intra-regular if and only if the ideals of M are semiprime.*
The proof of the following lemma is easy.

Lemma 6. *If M is a po-Γ-semigroup, then the set $(M\Gamma a\Gamma M]$ (resp. $(M\Gamma a]$) is an ideal (resp. left ideal) of M, and the set $(a\Gamma M]$ is a right ideal of M* for every $a \in M$.

Definition 7. A *po*-Γ-semigroup S is said to be a *semilattice of simple* (resp. *left simple*) *semigroups* if there exists a semilattice congruence σ on M such that the σ-class $(x)_\sigma$ of M containing x is a simple (resp. left simple) subsemigroup of M for every $x \in M$.

Theorem 8. *Let M be a po-Γ-semigroup. The following are equivalent:*
(1) *M is intra-regular.*
(2) *$N(x) = \{y \in M \mid x \in (M\Gamma y\Gamma M]\}$ for every $x \in M$.*
(3) *$\mathcal{N} = \mathcal{I}$.*
(4) *For every ideal I of M, we have $I = \bigcup\limits_{x\in I} (x)_\mathcal{N}$.*
(5) *$(x)_\mathcal{N}$ is a simple subsemigroup of M for every $x \in M$.*
(6) *M is a semilattice of simple semigroups.*
(7) *Every ideal of M is semiprime.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 3, the proof of (3) $\Rightarrow$ (4) is similar with the corresponding result for semigroups without order in [8], (5) $\Rightarrow$ (6) since $\mathcal{N}$ is a semilattice congruence on M and (7) $\Rightarrow$ (1) by Lemma 5.

(2) $\Longrightarrow$ (3). Let $(a, b) \in \mathcal{N}$. Since $a \in N(a) = N(b)$, by (2), we have $b \in (M\Gamma a\Gamma M] \subseteq (a\cup M\Gamma a\cup a\Gamma M\cup M\Gamma a\Gamma M] = I(a)$, so $I(b) \subseteq I(a)$. Since $b \in N(a)$, by symmetry, we get $I(a) \subseteq I(b)$, so $I(a) = I(b)$, and $(a, b) \in \mathcal{I}$. Then $\mathcal{N} \subseteq \mathcal{I}$, on the other hand by Lemma 4, we have $\mathcal{I} \subseteq \mathcal{N}$, thus $\mathcal{I} = \mathcal{N}$.

(4) $\Longrightarrow$ (5). Let $x \in M$. Since $\mathcal{N}$ is a semilattice congruence on M, $(x)_\mathcal{N}$ is a subsemigroup of M. Let I be an ideal of $(x)_\mathcal{N}$. Then $I = (x)_\mathcal{N}$. Indeed: Let $y \in (x)_\mathcal{N}$. Take an element $z \in I$ and an element $\gamma \in \Gamma$ ($I, \Gamma \neq \emptyset$). Since $z\gamma z\gamma z \in (M\Gamma M)\Gamma M \subseteq M\Gamma M \subseteq M$, by Lemma 6, $(M\Gamma z\gamma z\gamma z\Gamma M]$ is an ideal of M. By hypothesis, we have $(M\Gamma z\gamma z\gamma z\Gamma M] = \bigcup\limits_{t\in(M\Gamma z\gamma z\gamma z\Gamma M]} (t)_\mathcal{N}$. Since $z \in I \subseteq (x)_\mathcal{N}$, we have $(x)_\mathcal{N} = (z)_\mathcal{N}$. Since $y \in (x)_\mathcal{N} = (z)_\mathcal{N} = (z\gamma z\gamma z\gamma z\gamma z)_\mathcal{N} \subseteq (M\Gamma z\gamma z\gamma z\Gamma M]$, we have

$$y \leq a\delta z\gamma z\gamma z\xi b = (a\delta z)\gamma z\gamma(z\xi b) \text{ for some } a, b \in M,\ \delta, \xi \in \Gamma.$$

Using the fact that $\mathcal{N}$ is a complete semilattice congruence on M, in a similar way as in [8], we prove that $a\delta z \in (x)_\mathcal{N}$ and $z\xi b \in (x)_\mathcal{N}$. Then, since I is an ideal of $(x)_\mathcal{N}$ and $z \in I$, we have we have $(a\delta z)\gamma z\gamma(z\xi b) \in (x)_\mathcal{N}\Gamma I\Gamma(x)_\mathcal{N} \subseteq I$, and $y \in I$. Hence $(x)_\mathcal{N} \subseteq I$, and so $I = (x)_\mathcal{N}$.

(6) $\Longrightarrow$ (7). Let σ be a semilattice congruence on M such that $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$. Let I be an ideal of M, $x \in M$ and $\gamma \in \Gamma$ such that $x\gamma x \in I$. The set $I \cap (x)_\sigma$ is an ideal of $(x)_\sigma$. Indeed: Taking into account the proof of the implication (6) $\Rightarrow$ (7) in [8], it is enough to prove the following: Let $a \in I \cap (x)_\sigma$ and

$(x)_\sigma \ni b \le a$, then $b \in I \cap (x)_\sigma$. Since $M \in b \le a \in I$ and I is an ideal of M, we have $b \in I$, then $b \in I \cap (x)_\sigma$. Since $(x)_\sigma$ is a simple subsemigroup of M, we have $I \cap (x)_\sigma = (x)_\sigma$, then $x \in I$. Thus M is semiprime. □

Lemma 9. *Let M be a po-Γ-semigroup. The ideals of M are idempotent if and only if for any ideals A, B of M, we have $A \cap B = (A\Gamma B]$.*

Proof. $\Longrightarrow$. Let A, B be ideals of M. Then $(A\Gamma B] \subseteq (A\Gamma M] \subseteq (A] = A$ and $(A\Gamma B] \subseteq (M\Gamma B] \subseteq (B] = B$, thus $(A\Gamma B] \subseteq A \cap B$. On the other hand, $A \cap B$ is an ideal of M. Indeed: Take an element $a \in A$, an element $b \in B$ and an element $\gamma \in \Gamma$ $(A, B, \Gamma \neq \emptyset)$. Then $a\gamma b \in A\Gamma B \subseteq A\Gamma M \subseteq A$ and $a\gamma b \in A\Gamma B \subseteq M\Gamma B \subseteq B$, so $a\gamma b \in A\cap B$, so $A\cap B$ is a nonempty subset of M. We also have $(A\cap B)\Gamma M \subseteq A\Gamma M \subseteq A$, $M\Gamma(A\cap B) \subseteq M\Gamma B \subseteq B$, and if $x \in A \cap B$ and $M \ni y \le x$ then, since $x \in A$ we have $y \in A$ and since $x \in B$ we have $y \in B$, so $y \in A \cap B$. Since $A \cap B$ is an ideal of M, by hypothesis, we have $A \cap B = \Big((A \cap B)\Gamma(A \cap B)\Big] \subseteq (A\Gamma B]$. Hence we have $A \cap B = (A\Gamma B]$.
$\Longleftarrow$. Let A be an ideal of M. By hypothesis, we have $A = (A\Gamma A]$, so A is idempotent. □

Theorem 10. *Let M be a po-Γ-semigroup. The ideals of M are weakly prime if and only if they are idempotent and they form a chain.*

Proof. $\Longrightarrow$. Let A, B be ideals of M. One can easily prove that $(A\Gamma B]$ is an ideal of M. Since $A, B, (A\Gamma B]$ are ideals of M, $A\Gamma B \subseteq (A\Gamma B]$ and $(A\Gamma B]$ is weakly prime, we have $A \subseteq (A\Gamma B] \subseteq (M\Gamma B] \subseteq (B] = B$ or $B \subseteq (A\Gamma B] \subseteq (A\Gamma M] \subseteq (A] = A$, thus the ideals of M form a chain. Furthermore, since A and $(A\Gamma A]$ are ideals of M, $A\Gamma A \subseteq (A\Gamma A]$ and $(A\Gamma A]$ is weakly prime, we have $A \subseteq (A\Gamma A] \subseteq (M\Gamma A] \subseteq (A] = A$, thus we get $A = (A\Gamma A]$, and A is idempotent.
$\Longleftarrow$. Let A, B, T be ideals of M such that $A\Gamma B \subseteq T$. By hypothesis, we have $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$ then, by Lemma 9, $A = A \cap B = (A\Gamma B] \subseteq (T] = T$. If $B \subseteq A$, then $B = A \cap B = (A\Gamma B] \subseteq (T] = T$, thus M is weakly prime. □

Lemma 11. *Let M be a po-Γ-semigroup. If M is intra-regular, then*

$$I(x) = (M\Gamma x\Gamma M] \textit{ for every } x \in M.$$

Proof. Let $x \in M$. Since $(M\Gamma x\Gamma M]$ is an ideal of M, by Lemma 5, it is semiprime. Take an element $\gamma \in \Gamma$ $(\Gamma \neq \emptyset)$. Since $x\gamma x \in M$, $(x\gamma x)\gamma(x\gamma x) \in (M\Gamma x\Gamma x]$ and $(M\Gamma x\Gamma M]$ is a semiprime subset of M, we have $x\gamma x \in (M\Gamma x\Gamma x]$, then $x \in (M\Gamma x\Gamma x]$, and $I(x) \subseteq (M\Gamma x\Gamma x]$. On the other hand, $(M\Gamma x\Gamma x] \subseteq I(x)$, thus we get $I(x) = (M\Gamma x\Gamma M]$. □

Lemma 12. *If M is a po-Γ-semigroup, $x, y \in M$ and $\gamma \in \Gamma$, then*

$$I(x\gamma y) \subseteq I(x) \cap I(y).$$

In particular, if M is intra-regular, then $I(x\gamma y) = I(x) \cap I(y)$.

Proof. Let $x, y \in M$ and $\gamma \in \Gamma$. Since $I(x)$ is an ideal of M, we have $x\gamma y \in I(x)\Gamma M \subseteq I(x)$ and $x\gamma y \in M\Gamma I(y) \subseteq I(y)$. Thus we have $I(x\gamma y) \subseteq I(x) \cap I(y)$.

Let now M be intra-regular and $t \in I(x) \cap I(y)$. Then, by Lemma 11, we have $t \in (M\Gamma x\Gamma M]$ and $t \in (M\Gamma y\Gamma M]$. Thus we have

$$t \le a\mu x\rho b \text{ and } t \le c\xi y\zeta d \text{ for some } a,b,c,d \in M,\ \mu,\rho,\xi,\zeta \in \Gamma.$$

Then $t\gamma t \le (c\xi y\zeta d)\gamma(a\mu x\rho b) = c\xi(y\zeta d\gamma a\mu x)\rho b$. In addition, we have $y\zeta d\gamma a\mu x \in I(x\gamma y)$. Indeed, by Lemma 11,

$$(y\zeta d\gamma a\mu x)\gamma(y\zeta d\gamma a\mu x) \in M\Gamma(x\gamma y)\Gamma M \subseteq \Big(M\Gamma(x\gamma y)\Gamma M\Big] = I(x\gamma y).$$

Since M is intra-regular and $I(x\gamma y)$ is an ideal of M, by Lemma 5, $I(x\gamma y)$ is semiprime. So we get $y\zeta d\gamma a\mu x \in I(x\gamma y)$. Since $I(x\gamma y)$ is an ideal of M, we have $c\xi(y\zeta d\gamma a\mu x)\rho b \in M\Gamma I(x\gamma y)\Gamma M \subseteq I(x\gamma y)\Gamma M \subseteq I(x\gamma y)$, then $t\gamma t \in I(x\gamma y)$. Since $I(x\gamma y)$ is semiprime, we have $t \in I(x\gamma y)$. Thus we get $I(x) \cap I(y) \subseteq I(x\gamma y)$ and so $I(x\gamma y) = I(x) \cap I(y)$. □

Theorem 13. *Let M be a po-Γ-semigroup. The ideals of M are prime if and only if they form a chain and M is intra-regular.*

Proof. $\Longrightarrow$. The ideals of M are prime, so they are weakly prime and semiprime. Since they are weakly prime, by Theorem 10, they form a chain. Let now $a \in M$ and $\gamma \in \Gamma$. Since $(M\Gamma a\gamma a\Gamma M]$ is an ideal of M, $(a\gamma a)\gamma(a\gamma a) \in (M\Gamma a\gamma a\Gamma M]$ and $(M\Gamma a\gamma a\Gamma M]$ is semiprime, we have $a\gamma a \in (M\Gamma a\gamma a\Gamma M]$, then $a \in (M\Gamma a\gamma a\Gamma M]$, thus M is intra-regular.
$\Longleftarrow$. Suppose M is intra-regular and the ideals of M form a chain. Let now T be an ideal of M, $a,b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in T$. We have $I(a) \subseteq I(b)$ or $I(b) \subseteq I(a)$. If $I(a) \subseteq I(b)$ then, by Lemma 12, we have $a \in I(a) = I(a) \cap I(b) = I(a\gamma b) \subseteq I(T) = T$. If $I(b) \subseteq I(a)$, then $b \in I(b) = I(a) \cap I(b) = I(a\gamma b) \subseteq T$. Thus the ideals of M are prime. □

Proposition 14. *Let M be an intra-regular po-Γ-semigroup such that the ideals of M form a chain. Then, for every $x,y \in M$ and every $\gamma \in \Gamma$, we have*

$$x \in (M\Gamma x\gamma y\Gamma M] \text{ or } y \in (M\Gamma x\gamma y\Gamma M].$$

Proof. Let $x,y \in M$ and $\gamma \in \Gamma$. Since M is intra-regular and the ideals of M form a chain, by Theorem 13, the ideals of M are prime. Since $(M\Gamma x\gamma y\Gamma M]$ is an ideal of M, $(M\Gamma x\gamma y\Gamma M]$ is prime. Since $(x\gamma x)\gamma(y\gamma y) \in (M\Gamma x\gamma y\Gamma M]$, we have $x\gamma x \in (M\Gamma x\gamma y\Gamma M]$ or $y\gamma y \in (M\Gamma x\gamma y\Gamma M]$. If $x\gamma x \in (M\Gamma x\gamma y\Gamma M]$ then, since $(M\Gamma x\gamma y\Gamma M]$ is prime, we have $x \in (M\Gamma x\gamma y\Gamma M]$. If $y\gamma y \in (M\Gamma x\gamma y\Gamma M]$, then $y \in (M\Gamma x\gamma y\Gamma M]$. □

Definition 15. A *po*-Γ-semigroup M is called a *chain of simple semigroups* if there exists a semilattice congruence σ on M such that $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$ and the set M/σ of all σ-classes of M endowed with the order relation

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (x\gamma y)_\sigma \ \forall\ \gamma \in \Gamma$$

is a chain. In other words, for any $x,y \in M$ and any $\gamma \in \Gamma$ we have

$$(x)_\sigma = (x\gamma y)_\sigma \text{ or } (y)_\sigma = (x\gamma y)_\sigma.$$

Theorem 16. *A po-Γ-semigroup M is intra-regular and the ideals of M form a chain if and only if M is chain of simple semigroups.*

Proof. $\Longrightarrow$. Since M is intra-regular and $\mathcal{N}$ is a semilattice congruence on M, by Theorem 8(1)$\Rightarrow$ (5), $(x)_{\mathcal{N}}$ is a simple subsemigroup of M for every $x \in M$, so M is a semilattice of simple semigroups. Let now $x, y \in M$ and $\gamma \in \Gamma$. Then $(x)_{\mathcal{N}} = (x\gamma y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} = (x\gamma y)_{\mathcal{N}}$. In fact: Since M is intra-regular and the ideals of M form a chain, by Proposition 14, we have $x \in (M\Gamma x\gamma y\Gamma M]$ or $y \in (M\Gamma x\gamma y\Gamma M]$. If $x \in (M\Gamma x\gamma y\Gamma M]$, then

$$N(x) \ni x \le a\mu(x\gamma y)\rho b \text{ for some } a, b \in M,\ \mu, \rho \in \Gamma.$$

Since $N(x)$ is a filter of M, we have $a\mu(x\gamma y)\rho b \in N(x)$, then $(x\gamma y)\rho b \in N(x)$, $x\gamma y \in N(x)$ and $N(x\gamma y) \subseteq N(x)$. If $y \in (M\Gamma x\gamma y\Gamma M]$, then

$$N(y) \ni y \le c\xi(x\gamma y)\zeta d \text{ for some } c, d \in M,\ \xi, \zeta \in \Gamma,$$

then $c\xi(x\gamma y)\zeta d \in N(y)$, $x\gamma y \in N(y)$, and $N(x\gamma y) \subseteq N(y)$. On the other hand, since $x\gamma y \in N(x\gamma y)$, we have $x \in N(x\gamma y)$ and $y \in N(x\gamma y)$, so $N(x) \subseteq N(x\gamma y)$ and $N(y) \subseteq N(x\gamma y)$. Thus we get $N(x) = N(x\gamma y)$ or $N(y) = N(x\gamma y)$, then $(x)_{\mathcal{N}} = (x\gamma y)_{\mathcal{N}}$ or $(y)_{\mathcal{N}} = (x\gamma y)_{\mathcal{N}}$. Therefore M is a chain of simple semigroups.

$\Longleftarrow$. Let σ be a semilattice congruence on M such that $(x)_\sigma$ is a simple subsemigroup of M for every $x \in M$ and $(M/\sigma, \preceq)$ is a chain. By Theorem 13 it is enough to prove that the ideals of M are prime. So, let I be an ideal of M, $a, b \in M$ and $\gamma \in \Gamma$ such that $a\gamma b \in I$. The set $(a\gamma b)_\sigma \cap I$ is an ideal of $(a\gamma b)_\sigma$. Indeed:

$\emptyset \neq (a\gamma b)_\sigma \cap I \subseteq (a\gamma b)_\sigma$ $(a\gamma b \in (a\gamma b)_\sigma,\ a\gamma b \in I)$,

$$\begin{aligned}\Big((a\gamma b)_\sigma \cap I\Big)\Gamma(a\gamma b)_\sigma &\subseteq (a\gamma b)_\sigma\Gamma(a\gamma b)_\sigma \cap I\Gamma(a\gamma b)_\sigma \subseteq (a\gamma b)_\sigma \cap I\Gamma(a\gamma b)_\sigma \\ &\subseteq (a\gamma b)_\sigma \cap I\Gamma M \subseteq (a\gamma b)_\sigma \cap I,\end{aligned}$$

$$\begin{aligned}(a\gamma b)_\sigma\Gamma\Big((a\gamma b)_\sigma \cap I\Big) &\subseteq (a\gamma b)_\sigma\Gamma(a\gamma b)_\sigma \cap (a\gamma b)_\sigma\Gamma I \subseteq (a\gamma b)_\sigma \cap M\Gamma I \\ &\subseteq (a\gamma b)_\sigma \cap I.\end{aligned}$$

Let $x \in (a\gamma b)_\sigma \cap I$ and $(a\gamma b)_\sigma \ni y \le x$. Since $M \ni y \le x \in I$ and I is an ideal of M, we have $y \in I$, then $y \in (a\gamma b)_\sigma \cap I$. Since $(a\gamma b)_\sigma$ is simple, we have $(a\gamma b)_\sigma \cap I = (a\gamma b)_\sigma$. By hypothesis, $(a)_\sigma = (a\gamma b)_\sigma$ or $(b)_\sigma = (a\gamma b)_\sigma$. Then we have $a \in I$ or $b \in I$, thus I is a prime. □

Lemma 17. *Let M be a po-Γ-semigroup, T a subsemigroup of M and $x \in T$. Then the set $(M\Gamma x\Gamma M] \cap T$ is an ideal of T.*

Proof. First of all, the set $(M\Gamma x\Gamma M] \cap T$ is a nonempty subset of T. Indeed: Take an element $\gamma \in \Gamma$ $(\Gamma \neq \emptyset)$. Then we have $x\gamma x\gamma x \in M\Gamma x\Gamma M$ and $x\gamma x\gamma x \in (T\Gamma T)\Gamma T \subseteq T\Gamma T \subseteq T$. Moreover,

$$\Big((M\Gamma x\Gamma M] \cap T\Big)\Gamma T \subseteq (M\Gamma x\Gamma M]\Gamma T \cap T\Gamma T \subseteq (M\Gamma x\Gamma M]\Gamma(M] \cap T$$

$$\subseteq \Big(M\Gamma x\Gamma(M\Gamma M)\Big] \cap T \subseteq (M\Gamma x\Gamma M] \cap T.$$

In a similar way, we have $T\Gamma\Big((M\Gamma x\Gamma M]\cap T\Big) \subseteq (M\Gamma x\Gamma M]\cap T$. Let now $a \in (M\Gamma x\Gamma M]\cap T$ and $T \ni b \le a$. Since $a \in (M\Gamma x\Gamma M]$, there exist $u, v \in M$ and $\xi, \zeta \in \Gamma$ such that $a \le u\xi x\zeta v$. Then we have $b \le u\xi x\zeta v \in M\Gamma x\Gamma M$, and $b \in (M\Gamma x\Gamma M]$, thus $b \in (M\Gamma x\Gamma M]\cap T$. □

Theorem 18. *Let M be an intra-regular po-Γ-semigroup. Then the set $(x)_{\mathcal{N}}$ is a maximal simple subsemigroup of M for every $x \in M$. Moreover, if T is a maximal simple subsemigroup of M, then there exists $x \in M$ such that $T = (x)_{\mathcal{N}}$.*

Proof. Let $x \in M$. By the Theorem 8(1)⇒ (5), the set $(x)_{\mathcal{N}}$ is a simple subsemigroup of M. Let now T be a simple subsemigroup of M such that $T \supseteq (x)_{\mathcal{N}}$. Then $T = (x)_{\mathcal{N}}$. Indeed: Let $y \in T$. Since $x \in T$ and T is a subsemigroup of M, by Lemma 17, the set $(M\Gamma x\Gamma M] \cap T$ is an ideal of T. Since T is a simple subsemigroup of M, we have $(M\Gamma x\Gamma M] \cap T = T$, then $y \in (M\Gamma x\Gamma M]$. Since M is intra-regular, by Lemma 3, we have $x \in N(y)$, then $N(x) \subseteq N(y)$. On the other hand, since $y \in T$ and T is a subsemigroup of M, by Lemma 17, the set $(M\Gamma y\Gamma M] \cap T$ is an ideal of T. So $(M\Gamma y\Gamma M] \cap T = T$, $x \in (M\Gamma y\Gamma M]$, $y \in N(x)$, and $N(y) \subseteq N(x)$. Therefore we have $N(x) = N(y)$, then $y \in (x)_{\mathcal{N}}$, and $T \subseteq (x)_{\mathcal{N}}$. Then we have $T = (x)_{\mathcal{N}}$, thus the class $(x)_{\mathcal{N}}$ is a maximal simple subsemigroup of M.

Let now T be a maximal simple subsemigroup of M. Take an element $x \in T$ ($T \neq \emptyset$). Exactly as in the proof of the "⇒"-part of the theorem given above, we prove that $T \subseteq (x)_{\mathcal{N}}$. Since M is intra-regular, by Theorem 8(1)⇒ (5), $(x)_{\mathcal{N}}$ is a simple subsemigroup of M. Since $T \subseteq (x)_{\mathcal{N}}$ and T is a maximal simple subsemigroup of M, we have $T = (x)_{\mathcal{N}}$. □

Corollary 19. *For an intra-regular po-Γ-semigroup M, the set $\{(x)_{\mathcal{N}} \mid x \in M\}$ coincides with the set of all maximal simple subsemigroup of M.*

Definition 20. [9] A Γ-semigroup M is called *left* (resp. *right*) *regular* if

$$x \in (M\Gamma x\gamma x] \text{ (resp. } x \in (x\gamma x\Gamma M])$$

for every $x \in M$ and every $\gamma \in \Gamma$.

Taking into account the Theorem 6 in [8], in a similar way as in the Theorem 8 above, we can prove the following theorem

Theorem 21. *Let M be a po-Γ-semigroup. The following are equivalent:*

(1) *M is left regular and $(x\Gamma M] \subseteq (M\Gamma x]$ for every $x \in M$.*
(2) *$N(x) = \{y \in M \mid x \in (M\Gamma y]\}$ for every $x \in M$.*
(3) *$\mathcal{N} = \mathcal{L}$.*
(4) *For every left ideal L of M, we have $L = \bigcup_{x\in L} (x)_{\mathcal{N}}$.*
(5) *$(x)_{\mathcal{N}}$ is a left simple subsemigroup of M for every $x \in M$.*
(6) *M is a semilattice of left simple semigroups.*

(7) *Every left ideal of M is semiprime and two-sided.*

The right analogue of the above theorem also holds.

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