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Optimal reliability allocation for redundancy series-parallel systems

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Abstract. This paper examines the optimal reliability approaches to allocate the reliability values based on minimization of the total cost for a series-parallel systems. The problem is approached as a nonlinear programming problem and general costs formulas were suggested. The original results include: (i) submersion of a "series-parallel system" into a "series system", (ii) detailed analyse of a series-parallel system whose components of each subsystem have the same reliability; (iii) designing series-parallel systems by similarities with other engineering problems; (iv) dualities between reliability systems and electric circuits.

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1. Introduction

Redundancy is the provision of alternative means or parallel paths in a system for accomplishing a given task such that all means must fail before causing a system failure. System reliability and mean life can be increased by additional means by applying redundancy at various levels. The problem of reliability allocation and optimization has been widely treated by many authors. Although most of the attention to this issue has been given to the redundancy allocation problem [9],[10], a different approach to the problem is taken in this paper. A series-parallel system can be improved by four methods (Wang, 1992): (1) use more reliable components; (2) increase redundant components in parallel; (3) utilize both (1) and (2); and (4) enable repeatedly the allocation of entire system framework. Prasad and Kuo (2000) pointed out that Misra algorithm sometimes cannot yield an optimal solution, and suggested a method of searching for the upper limit of reliability's objective function. The total cost of a system can be minimized subject to the resource constraints to determine the optimum number of redundant components for each

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stage, when the reliability of each component is known. In other situations, the reliability of the system can be maximized subject to the resource constraint to determine the reliability of the components in the system when the number of redundant units in each stage is known. As a result, addition of redundant components or increase components reliability leads to the increase of the system reliability.

The main objective of the reliability optimization is maximizing the reliability cost ratio, developing, mostly, two directions, as follows: (1) minimizing the system cost, but insuring a minimal reliability level; (2) maximizing the system reliability with respect of some costs constraints.

The redundancy allocation problem has previously been analyzed for many different system structures, objective functions and time-to-failure distributions. Generally, the problem domain has been limited to series-parallel systems with active redundancy or 'k-out-of-n' systems consisting of a single subsystem. A number of studies have examined such problems [1], [4]. The parameters of the proposed cost function can be altered, allowing the mathematicians/engineers to investigate different allocation scenarios. Thereafter, designers can decide and plan on how to achieve the assigned minimum required reliabilities for each of the components.

2. Optimization of series-parallel system

Until recently, for the purpose of modelling equilibrium flow reliability systems, it has often been assumed that all the particular cases must be solved separately. Now we underline that there are inter-related systems governed by a unique set of equilibrium criteria.

Hypothesis: components have two states: working or failed; the reliability of each component is known and is deterministic; failure of individual components are independent; failed components do not damage other components or the system, and the components are not repaired.

The theory in this Section can be applied to any simple or complex systems, and any number of redundant components can be added to the system to have maximum possible improvement.

2.1. From "series-parallel system" to "series system"

To a series system of n components characterized by reliabilities $0 \leq R_i \leq 1$, $i = 1, ..., n$, and by the admissible level of reliability of the whole system R_G, we associate the program

P: Find

$$\min_{R} \; C(R_1, ..., R_n)$$

subject to

$$R_s = \prod_{i=1}^{n} R_i \geq R_G, \; 0 < R_i \leq 1, \; i = 1, 2, ..., n.$$

Such kinds of programs were studied and solved in [14] and [15].

In order to formulate an optimal problem associated to a series-parallel system, we need additional notations: n number of subsystems; k_i number of different types of available components for the i-th subsystem, $i = 1, ..., n$; $0 \leq r_{ij} \leq 1$ reliability of the j-th component for the i-th subsystem, $i = 1, ..., n$, $j = 1, ..., k_i$; R_G admissible level of reliability of the whole system; $x_{ij} \in \mathbb{N}$ number of j components used in the i-th subsystem, $i = 1, ..., n$, $j = 1, ..., k_i$. Double-indexed numbers do not form matrices as "lines" would not have the same number of elements; they will be arranged like vectors.

The associated program is

p: Find

$$\min_r \ \mathcal{C}(r_{11}, ..., r_{1k_1}, ..., r_{n1}, ..., r_{nk_n}), \ r = (r_{11}, ..., r_{1k_1}, ..., r_{n1}, ..., r_{nk_n}),$$

subject to

$$R_s = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n} \left[1 - \prod_{j=1}^{k_i} (1 - r_{ij})^{x_{ij}}\right] \geq R_G,$$
$$0 < r_{ij} \leq 1, \ i = 1, 2, ..., n; \ j = 1, 2, ..., k_i.$$

Our goal is to determine the mutual relations between the program p and the program P. Passing from a series-parallel system to a series system is important when we need to decide if a given series-parallel system is optimal or not.

Theorem 1. *The image of the program p via a submersion R is the program P.*

Proof. The function R of components $R_i = 1 - \prod_{j=1}^{k_i} (1 - r_{ij})^{x_{ij}}, \ i = 1, 2, ..., n$, changes the program p into the program P. It is a submersion from $\mathbb{R}^{k_1 + ... + k_n}$ to $\mathbb{R}^n$ since

$$\frac{dR_i}{R_i - 1} = \sum_{j=1}^{k_i} x_{ij} \ \frac{dr_{ij}}{r_{ij} - 1}$$

and hence the differential is everywhere surjective. Also

$$R([0,1]^{k_1 + ... + k_n}) = [0,1]^n.$$

Moving from p to P is possible only when for $\mathcal{C}$ there exists a cost $C : [0,1]^n \subset \mathbb{R}^n \to \mathbb{R}$ such that the following costs diagram to be commutative:

$$\begin{array}{ccc} [0,1]^n \subset \mathbb{R}^n & \xrightarrow{C} & \mathbb{R} \\ R \nwarrow & & \nearrow \mathcal{C} = C \circ R \\ & [0,1]^{k_1 + ... + k_n} \subset \mathbb{R}^{k_1 + ... + k_n} & \end{array}$$

Remark 1. *A submersion locally looks like a projection $\mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$, while an immersion locally looks like an inclusion $\mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{n-m}$.*

Corollary 1. *To each optimal program P there corresponds an infinity of optimal programs p.*

Additional assumptions lead to more accurate design on subsystems.

2.2. Example of series-parallel system with additional assumptions

Let us analyse a series-parallel system with additional assumption that, in each subsystem, all components have the same reliability.

We use the notations: n number of subsystems; $0 \leq r_i \leq 1$ is the reliability of each component in subsystem i; $0 \leq R_i \leq 1$ is the reliability of subsystem i; $C_i(R_i)$ is the cost of each subsystem i; x_i = number of components in stage i; $\mathcal{C}(r_1, ..., r_n) = C \circ R(r_1, ..., r_n) = \sum_{i=1}^{n} a_i C_i(R_i)$ is the total system cost, where $a_i > 0$. In general the functionality of each subsystems can be unique, however there can be several choices for, many of the subsystems providing the same functionality, but differently reliability levels. The objective is to allocate reliability to all or some of the components of that system, in order to meet that goal with a minimum cost. An important problem P is formulated as a nonlinear programming problem, with *additively decomposable cost function* and a *nonlinear constraint*:

P: Find

$$\min_r \ \mathcal{C}(r_1, ..., r_n) = \sum_{i=1}^{n} a_i C_i(R_i),\ a_i > 0,\ r = (r_1, ..., r_n),$$

subject to

$$R_s = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n} \left[1 - (1 - r_i)^{x_i}\right] \geq R_G,\ 0 < r_i \leq 1,\ i = 1, 2, ..., n,$$

where R_s is the system reliability; R_G is the system reliability goal and $R_i = 1-(1-r_i)^{x_i}$. The foregoing formulation is designed to achieve a minimum total system cost, subject to R_G, a lower limit on the system reliability. The diffeomorphism $R_i = 1 - (1 - r_i)^{x_i}$, $i = 1, ..., n$, changes a program p into a program P, and p will have unique solution. The inverse is $r_i = 1 - (1 - R_i)^{\frac{1}{x_i}}$.

Remark 2. *The Kuhn-Tucker necessary conditions for previous program P were discussed in [13] and [14].*

3. Designing series-parallel systems by similarities

3.1. Similar to waste treatment plant

The problem of design a reliable system is similar to the design of an industrial waste treatment plant. Indeed the plant design incorporates several treatment processes in series (see, [3], pp. 494-495).

The cost formula for the i-th components has the form

$$C_i = c_i[1 - (1 - r_i)^{x_i}]^{a_i},\ i = 1, ..., n,$$

where C_i is the total annual cost, $c_i > 0$, a_i is a fixed negative exponent, r_i is reliability of i-th component.

For a design problem involving n components and $r = (r_1, ..., r_n)$, the minimization problem becomes

$$\min_r \sum_{i=1}^{n} c_i \left[1 - (1 - r_i)^{x_i}\right]^{a_i}$$

subject to

$$R_s = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n} \left[1 - (1 - r_i)^{x_i}\right] \geq R_G, \ 0 < r_i \leq 1, \ i = 1, 2, ..., n.$$

The positivity constraints $r_i > 0$ will be satisfied at a minimizing point because of the inverse relationship between process costs and the variables r_i, that is, as r_i approaches zero, the corresponding cost term $c_i[1 - (1 - r_i)^{x_i}]^{a_i}$ approaches positive infinity.

The diffeomorphism $R_i = 1 - (1 - r_i)^{x_i}$ changes the previous program into a geometric program.

3.2. Similar to transmission compressor design

The problem of design a reliable system is similar to transmission compressor design. Indeed a classical problem in compressor design is finding the interstage pressures for an adiabatic reversible compression of an ideal gas (see, [3], pp. 426-432; [11], pp. 180-181).

We seek to minimize the *energy consumption* of an $(n+1)$-stage system whose work is

$$E(R_1, \ldots, R_n) = c\left[\left(\frac{R_1}{c_1}\right)^{\alpha} + \left(\frac{R_2}{R_1}\right)^{\alpha} + \ldots + \left(\frac{R_n}{R_{n-1}}\right)^{\alpha} + \left(\frac{c_2}{R_n}\right)^{\alpha}\right],$$

where: c_1 = inlet reliability, c_2=outlet reliability, α = adiabatic index (the surrounding do not influence the reliability).

This free geometric program has zero degree of difficulty. The dual program is

$$\max \left(\frac{c\, c_1^{-\alpha}}{\delta_1}\right)^{\delta_1} \left(\frac{c}{\delta_2}\right)^{\delta_2} \cdots \left(\frac{c}{\delta_n}\right)^{\delta_n} \left(\frac{c\, c_2^{\alpha}}{\delta_{n+1}}\right)^{\delta_{n+1}},$$

with the constraints

$$\sum_{i=1}^{n+1} \delta_i = 1, \ \delta_1 = \delta_2, ..., \delta_n = \delta_{n+1}.$$

It follows $\delta_1 = ... = \delta_{n+1} = \frac{1}{n+1}$ and the searched minimum is $(n+1)c\left(\frac{c_2}{c_1}\right)^{\frac{\alpha}{n+1}}$. Since, at optimality, all dual variables are equal, the stages contribute equally to the minimizing energy policy, and all subsystem ratios must be equal.

The minimum point is the solution of the system

$$c\left(\frac{R_1}{c_1}\right)^{\alpha} = \ldots = c\left(\frac{c_2}{R_n}\right)^{\alpha} = c\left(\frac{c_2}{c_1}\right)^{\frac{\alpha}{n+1}},$$

i.e., $R_i = c_2^{\frac{i}{n+1}}\ c_1^{1-\frac{i}{n+1}}$, $i = 1, ..., n+1$.

Remark 3. *Let us point two interesting problems:*

$$\max_R E(R_1, \dots, R_n) \text{ subject to } \prod_{i=1}^n R_i \geq R_G,\ 0 < R_i \leq 1$$

and

$$\max_R E(R_1, \dots, R_n) \text{ subject to } H(R) = -\sum_{i=1}^n R_i \log_2 R_i \geq H_0,\ 0 < R_i < 1.$$

3.3. Similar to statistics metric

In statistics there exists a wide variety of metrics such as median, standard deviation, arithmetic mean, power mean, geometric mean and many others.

In many important problems, we seek to minimize the geometric mean

$$\left(\prod_{i=1}^n R_i\right)^{\frac{1}{n}} = \sqrt[n]{R_1 R_2 \cdots R_n}.$$

subject to some restrictions.

3.4. Symmetric polynomials of reliabilities

Let $R = (R_1, ..., R_n)$, with $0 \leq R_i \leq 1$. We build the symmetric polynomials $s_1(R) = \sum_{i=1}^n R_i$, $s_2(R) = \sum_{i<j} R_i R_j$, $s_3(R) = \sum_{i<j<k} R_i R_j R_k$, and so on. Then

$$\sup_{(R,n)} \frac{s_m(R)}{(s_1(R))^m} = \frac{1}{m!},$$

obtained for $R_1 = ... = R_n = 1$ and $n \to \infty$ (see Amer. Math. Monthly, 1976 and [11], p. 66).

The symmetric reliability polynomials determine canonically the standard polynomial

$$Q(R, t) = t^n - s_1(R)t^{n-1} + s_2(R)t^{n-2} - ... \pm s_n(R).$$

Elementary tricks shows that $\max s_n(R)$ *subject to* $s_1(R) = 1$ is attained for $R_1 = ... = R_n = \frac{1}{n}$. Moreover, $\max s_k(R)$, $k \geq 2$, *subject to* $s_1(R) = 1$ is attained also for $R_1 = ... = R_n = \frac{1}{n}$ and it is equal to $\frac{1}{n^k} C_n^k$. In this way, for $R_1 = ... = R_n = \frac{1}{n}$ we obtain an optimal polynomial

$$q(R, t) = t^n - t^{n-1} + [\max_R s_2(R)]t^{n-2} - ... \pm \max_R [s_n(R)] = \left(t - \frac{1}{n}\right)^n.$$

3.5. Other significant cost models

$$\min_R C(R) = R_1 \text{ subject to } R_1 \cdots R_n \geq R_G.$$

Full range Bregman reliability function

$$f(R) = \sum_{i=1}^{n} R_i \ln R_i,$$

with the convention $0 \ln 0 = 0$.

The *entropy function*

$$H(R) = -\sum_{i=1}^{n} R_i \log_2 R_i,\ 0 < R_i < 1,\ \sum_{i=1}^{n} R_i = 1.$$

One-parameter families of functions $F \in C^{\infty}([0,1]^n \times \mathbb{R}, \mathbb{R})$ on $[0,1]^n$:

$$F(R,t) = \frac{1}{3}R_1^3 + (t^2-1)R_1 + \sum_{i=2}^{n} R_i^2.$$

Here, the set of critical points

$$\text{Crit}\, F = \{(R,t) \in [0,1]^n \times \mathbb{R} \mid R_1^2 + t^2 = 1, R_i = 0, i = 2, ..., n\}$$

is a circle.

Example 1. *As an example, consider the problem of minimizing the geometric mean with a significative (non-trivial) constraint. Explicitly, find*

$$\min_R\ C(R_1, R_2, R_3) = (R_1 R_2 R_3)^{1/3},\ R = (R_1, R_2, R_3)$$

subject to

$$(1 + \max\{R_1, R_2\})\ (1 + R_1 + (0.1 R_1 R_3^{-0.5} + R_2^{1.6} R_3^{0.4})^{1.5})^{1.7} \leq 1,$$

with variables $R_1, R_2,$ and R_3.

We transform the previous problem into the problem

$$\min\ t_1^{1/3}$$

subject to

$$R_1 R_2 R_3 \leq t_1,\ t_2 \cdot t_3^{1.7} \leq 1$$

$$1 + R_1 \leq t_2,\ 1 + R_2 \leq t_2$$

$$1 + R_1 + t_4^{1.5} \leq t_3,\ 0.1 R_1 R_3^{-0.5} + R_2^{1.6} R_3^{0.4} \leq t_4,$$

where the variables are $R_1, R_2, R_3,\ t_1, t_2,\ t_3$ *and* t_4. *This method for handling positive fractional powers can be applied recursively. Using some simple transformations, we obtain the equivalent standard form as geometric program:*

$$\min\ t_1^{1/3}$$

subject to

$$t_1^{-1} R_1 R_2 R_3 \leq 1,\ t_2 \cdot t_3^{1.7} \leq 1$$
$$t_2^{-1}(1+R_1) \leq 1,\ t_2^{-1}(1+R_2) \leq 1$$
$$t_3^{-1}(1+R_1+t_4^{1.5}) \leq 1,\ t_4^{-1}(0.1 R_1 R_3^{-0.5} + R_2^{1.6} R_3^{0.4}) \leq 1.$$

The matrix of exponents is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & ... & 0 & 0 & 1.6 \\ 0 & 1 & 0 & 0 & ... & 0 & -0.5 & 0.4 \\ 1/3 & -1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & ... & 0 & 0 & 0 \\ 0 & 0 & 1.7 & 0 & ... & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & ... & 1.5 & -1 & -1 \end{pmatrix}.$$

Since the degree of difficulty is $12-7-1=4$, *the solution is not unique (depends on* 4 *parameters). The dual function is*

$$\begin{aligned} P(\lambda,\alpha) = & \left(\frac{1}{\alpha_1}\right)^{\alpha_1}\left(\frac{1}{\alpha_2}\right)^{\alpha_2}\left(\frac{1}{\alpha_3}\right)^{\alpha_3}\left(\frac{1}{\alpha_4}\right)^{\alpha_4}\left(\frac{1}{\alpha_5}\right)^{\alpha_5}\left(\frac{1}{\alpha_6}\right)^{\alpha_6}\left(\frac{1}{\alpha_7}\right)^{\alpha_7} \\ & \times\left(\frac{1}{\alpha_8}\right)^{\alpha_8}\left(\frac{1}{\alpha_9}\right)^{\alpha_9}\left(\frac{1}{\alpha_{10}}\right)^{\alpha_{10}}\left(\frac{0.1}{\alpha_{11}}\right)^{\alpha_{11}}\left(\frac{1}{\alpha_{12}}\right)^{\alpha_{12}} \\ & \times \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\alpha_3^{\alpha_3}\left(\alpha_4+\alpha_5\right)^{(\alpha_4+\alpha_5)}\left(\alpha_6+\alpha_7\right)^{(\alpha_6+\alpha_7)} \\ & \times\left(\alpha_8+\alpha_9+\alpha_{10}\right)^{(\alpha_8+\alpha_9+\alpha_{10})}\left(\alpha_{11}+\alpha_{12}\right)^{(\alpha_{11}+\alpha_{12})}. \end{aligned}$$

The associated dual program can be written

$$\max_{\alpha}\ P(\lambda,\alpha)$$

subject to

$$\alpha_1 = 1\ (normality),\ A\,\alpha = 0,\, \alpha = (\alpha_1, ..., \alpha_n)^T\ (orthogonality).$$

$$\alpha_2+\alpha_5+\alpha_9+\alpha_{11}=0,\ \alpha_2+\alpha_7+1.6\alpha_{12}=0,$$
$$\alpha_2-0.5\alpha_{11}+0.4\alpha_{12}=0,\ 1/3\alpha_1-\alpha_2=0,$$
$$\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7=0,\ 1.7\alpha_3-\alpha_8-\alpha_9-\alpha_{10}=0,\ 1.5\alpha_{10}-\alpha_{11}-\alpha_{12}=0.$$

Solving the previous system, we obtain $\alpha_1, \alpha_2, ..., \alpha_{12}$*. Now calculate* P^* *and*

$$u_1 = t_1^{1/3} = \alpha_1 P^*,\ u_2 = t_1^{-1} R_1 R_2 R_3 = \alpha_2 P^*,$$

$$u_3 = t_2 t_3^{1.7} = \alpha_3 P^*,\ u_4 = t_2^{-1} = \alpha_4 P^*,$$

$$u_5 = t_2^{-1} R_1 = \alpha_5 P^*,\ u_6 = t_2^{-1} = \alpha_6 P^*,$$

$$u_7 = t_2^{-1} R_2 = \alpha_7 P^*,\ u_8 = t_3^{-1} = \alpha_8 P^*,$$

$$u_9 = t_3^{-1} R_1 = \alpha_9 P^*,\ u_{10} = t_3^{-1} t_4^{1.5} = \alpha_{10} P^*,$$

$$u_{11} = 0.1 t_4^{-1} R_1 R_3^{-0.5} = \alpha_{11} P^*,\ u_{12} = t_4^{-1} R_2^{1.6} R_3^{0.4} = \alpha_{12} P^*.$$

To solve this system, we use the logarithm, and we denote $\omega_i = \ln R_i$*,* $\beta_i = \ln t_i$*:*

$$\left(\frac{1}{3}\right) \beta_1 = \ln(\alpha_1 P^*) = m_1,\ \omega_1 + \omega_2 + \omega_3 - \beta_1 = \ln(\alpha_2 P^*) = m_2,$$

$$\beta_1 + 1.7\beta_2 = \ln(\alpha_3 P^*) = m_3,\ -\beta_2 = \ln(\alpha_4 P^*) = m_4$$

$$\omega_1 - \beta_2 = \ln(\alpha_5 P^*) = m_5,\ -\beta_2 = \ln(\alpha_6 P^*) = m_6,$$

$$\omega_2 - \beta_2 = \ln(\alpha_7 P^*) = m_7,\ -\beta_3 = \ln(\alpha_8 P^*) = m_8,$$

$$\omega_1 - \beta_3 = \ln(\alpha_9 P^*) = m_9,\ 1.5\beta_4 - \beta_3 = \ln(\alpha_{10} P^*) = m_{10},$$

$$\omega_1 - 0.5\omega_3 - \beta_4 = \ln\left(\frac{\alpha_{11} P^*}{0.1}\right) = m_{11},\ 1.6\omega_2 + 0.4\omega_3 - \beta_4 = \ln(\alpha_{12} P^*) = m_{12}.$$

We find ω_1, ω_2, ω_3, β_1, β_2, β_3,β_4 *and*

$$R_1 = \exp(\omega_1),\ R_2 = \exp(\omega_2),\ R_3 = \exp(\omega_3),$$

$$t_1 = \exp(\beta_1),\ t_2 = \exp(\beta_2),\ t_3 = \exp(\beta_3),\ t_4 = \exp(\beta_4).$$

The problem has solution if and only if $\omega_1 < 0,\ \omega_2 < 0,\ \omega_3 < 0$.

4. Dualities between reliability systems and electric circuits

To identify a reliability system with an electric circuit, we use the following variables: R_i, $i = 1, ..., n$, are reliabilities for a series system; r_i, $i = 1, ..., n$, are reliabilities for a parallel system; ρ_i, $i = 1, ..., n$, are resistances for electrical series circuit; η_i, $i = 1, ..., n$, are resistances for electrical parallel circuit.

Mental experiments in this Section suggest a clear mathematical relationship between total circuit resistance and total reliability of a system.

Proposition 1. *The total reliability of a series/parallel system is a copy of the first/second Ohm law for an electric circuit, via logarithmic scale.*

Proof. To find dualities between reliability systems and electric circuits we compare some diagrams. We remark that the commutative set diagram

$$R_i = 1 - r_i \qquad \begin{array}{ccc} [0,1]^n = \{(R_1,...,R_n)\} & \overset{\rho_i = -\ln R_i}{\longrightarrow} & \mathbb{R}_+^n = \{(\rho_1,...,\rho_n)\} \\ \uparrow\downarrow & & \uparrow\downarrow \\ [0,1]^n = \{(r_1,...,r_n)\} & \overset{\eta_i = -\ln(1-r_i)}{\longrightarrow} & \mathbb{R}_+^n = (\eta_1,...,\eta_n) \end{array} \qquad \rho_i = \frac{1}{\eta_i}$$

induces a commutative function diagram

$$R = 1 - r \qquad \begin{array}{ccc} R_s = \prod_{i=1}^n R_i & \overset{\rho = -\ln R}{\longrightarrow} & \rho_s = \sum_{i=1}^n \rho_i \\ \uparrow\downarrow & & \uparrow\downarrow \\ R_p = 1 - \prod_{i=1}^n (1-r_i) & \overset{\eta = -\ln(1-r)}{\longrightarrow} & \eta_p = \frac{1}{\sum_{i=1}^n \frac{1}{\eta_i}} \end{array} \qquad \rho = \frac{1}{\eta}$$

We remark that the first line describes series systems - logarithmic scale - series circuits, while the second line describes parallel systems - logarithmic scale - parallel circuits, both connected by changing the variables (involutions).

Individually, we have the commutative diagrams

$$\rho_i = -\ln R_i \qquad \begin{array}{ccc} [0,1]^n & \overset{R_s = R_1 \cdots R_n}{\longrightarrow} & [0,1] \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathbb{R}_+^n & \overset{\rho_s = \rho_1 + \cdots + \rho_n}{\longrightarrow} & \mathbb{R}_+ \end{array} \qquad \rho_s = -\ln R_s$$

$$\rho_i = -\frac{1}{\ln(1-r_i)} \qquad \begin{array}{ccc} [0,1]^n & \overset{R_p = 1-(1-r_1)\cdots(1-r_n)}{\longrightarrow} & [0,1] \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathbb{R}_+^n & \overset{\rho_p = \frac{1}{\frac{1}{\rho_1} + \cdots + \frac{1}{\rho_n}}}{\longrightarrow} & \mathbb{R}_+ \end{array} \qquad \rho_p = -\frac{1}{\ln(1-R_p)}$$

Comparing these diagrams, we get the proof of the Proposition.

Example 2. *Let us find "reliability batteries" of minimum intensity as counterpart of electrical batteries of maximum intensity.*

Genuine example of batteries with maximum intensity

Let us give N identical electric cells, each with emf E and inner resistance r. Let us denote by R the exterior resistance. Tying in series n cells and then in parallel the groups so obtained, we form a battery. Determine n so that the battery to supply a current of maximum intensity.

Solution *The current intensity given by such battery is*

$$I = \frac{NnE}{NR + n^2 r}.$$

We think $n \to I(n)$ as a function on $(0, \infty)$ and we remark that

$$\max I = \frac{1}{\min \frac{1}{I}}.$$

On the other hand,

$$\min_{n>0} \frac{1}{I} = \min_{n>0} \left(\frac{R}{E} n^{-1} + \frac{r}{NE} n \right).$$

This is a posynomial geometric program with the solution $n = \sqrt{N \frac{R}{r}}$ *and* $I_{\max} = \frac{E}{2}\sqrt{\frac{N}{Rr}}$. *If* n *is supposed to be a natural number, then the geometric program should be solved in steps.*

Dictionary *To pass to the reliability domain, we use*

$$\mathcal{E} = e^E,\ \mathcal{I} = e^I,\ r \to -\ln \mathcal{R},\ R \to -\ln \mathcal{R}_1.$$

It follows

$$-\ln \mathcal{I} = \frac{\ln \mathcal{E}^{Nn}}{\ln \mathcal{R}_1^N \mathcal{R}^{n^2}} = log_{\mathcal{R}_1^N \mathcal{R}^{n^2}} \mathcal{E}^{Nn},$$

i.e.,

$$\mathcal{I} = exp\left(-\frac{\ln \mathcal{E}^{Nn}}{\ln \mathcal{R}_1^N \mathcal{R}^{n^2}} \right).$$

Dual example of reliability batteries with minimum intensity

Consider a reliability subsystem with N *components, with the same reliability* r. *The components are grouped many n in series, and the series in parallel. To the total system we attach a subsystem, consisting of a single element with reliability* R, *connected in series.*

The total reliability is

$$\mathcal{R}_s = [1 - (1 - r^n)^{N/n}]R.$$

We use a sequence of arrows based on previous commutative diagrams,

$$\mathcal{R}_s = [1 - (1 - r^n)^{N/n}]R \ \to \ -\ln[1 - (1 - r^n)^{N/n}] - \ln R$$

$$\to \ -\ln \frac{1}{(1 - r^n)^{N/n}} - \ln R \ \to \ \frac{n}{N} \ln(1 - r^n) - \ln R \ \to \ \frac{n}{N} \ln \frac{1}{r^n} - \ln R$$

$$\to \ -\frac{n^2}{N} \ln r - \ln R \ \to \ \frac{n^2}{N} r + R = \frac{nE}{I}.$$

The analog "reliability emf" of the battery is $\mathcal{E}$. *To close the loop, we consider that the analog "reliability intensity" is*

$$\mathcal{I} = c(r;n)(1 - \mathcal{R}_s) = exp\left(-\frac{\ln \mathcal{E}^{Nn}}{\ln \mathcal{R}_1^N \mathcal{R}^{n^2}} \right).$$

The function $n \to \mathcal{I}(n)$ *has minimum for*

$$n = \sqrt{N \ \frac{\ln \mathcal{R}_1}{\ln \mathcal{R}}}.$$

Open problems (i) Clarify and extend the ideas in this Section.

(ii) Extend the previous similarity to magnetic reluctance or general circuits.

5. Generated algebraic structures

The commutative monoid $([0,1], S) : aSb = ab$ is isomorphic to the commutative monoid $([0,1], P) : aPb = 1-(1-a)(1-b)$ via an isomorphism f. For example, $f(x) = 1-x$ (involution). The general isomorphism is of the form $f(x) = h(1-h^{-1}(x))$, where $h : [0,1] \to [0,1]$ is an arbitrary bijection.

The commutative monoid $([0,+\infty], s) : AsB = A+B$ is isomorphic to the commutative monoid $([0,+\infty], p) : ApB = \frac{1}{\frac{1}{A}+\frac{1}{B}}$ via an isomorphism g. As example $g(y) = \frac{1}{y}$ (involution). Generally, the isomorphism is of the form $g(y) = \ell(1/\ell^{-1}(y))$, where $\ell : [0,+\infty] \to [0,+\infty]$ is an arbitrary bijection.

The commutative monoid $([0,1], S)$ is isomorphic to the commutative monoid $([0,+\infty], s)$ by the isomorphism $\varphi(x) = -\ln x$. Then $([0,1], P)$ is isomorphic to $([0,+\infty], p)$ by the same function $\varphi(x) = -\ln x$ if and only if $g(-\ln f^{-1}(x)) = -\ln x$.

Open question

Does exist a bijective and decreasing morphism $\varphi : [0,1] \to [0,+\infty]$ such that
$\varphi(aSb) = \varphi(a)s\varphi(b)$, i.e., $\varphi(ab) = \varphi(a) + \varphi(b)$;
$\varphi(aPb) = \varphi(a)p\varphi(b)$, i.e., $\varphi(a+b-ab) = \frac{\varphi(a)\varphi(b)}{\varphi(a)+\varphi(b)}$?

6. Conclusions

In Sections 2-3 were examined some reliability optimization problems: (1) optimization of series-parallel systems, (2) optimization of series-parallel systems, with additional assumptions, (3) designing series-parallel systems by similarities, (4) minimizing the geometric mean with significative constraints. Section 4 rises and solves the problem of mathematical relationship between total resistance of an electrical circuit and total reliability of a reliability system. As consequence, there are introduced "reliability batteries" with minimum intensity. The fundamental characteristic of our techniques is that similar techniques can be applied for simple and complex reliability systems.

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