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Coupled Fixed Point Theorems on Bipolar Metric Spaces

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Abstract. In this article, certain coupled fixed point theorems, which can be considered as generalizations of Banach fixed point theorem, are extended to bipolar metric spaces. Also, some results which are related to these theorems are obtained. Finally, it is given an example which presents the applicability of obtained results.

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1. Introduction

In literature, the notion of coupled fixed point has been introduced by Guo and Lakshmikantham [6] in 1987. Afterward, Bhaskar and Lakshmikantham [3] introduced certain coupled fixed point theorems in partially ordered metric spaces. Since then, when many authors saw that these fixed point theorems can be utilized to investigate existence and uniqueness of solutions of periodic boundary value problems, differential equations and nonlinear integral equations, these theorems attracted their attention. And, they extended these theorems to various generalizations of metric spaces as cone, partial and modular, e.g. [1, 2, 4, 5, 7–12, 14–20].

The notion of metric space has many generalizations in literature. One of the most recent of them is bipolar metric space which is introduced by Mutlu and Gürdal [13] in 2016. Also, they established some fixed point theorems as Banach's and Kannan's on this space.

In this paper, we extend certain coupled fixed point theorems, which can be considered as generalization of Banach fixed point theorem, to bipolar metric spaces. Also, we obtain some results which are related to these theorems. Finally, we give an example which presents the applicability of our obtained results.

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2. Bipolar Metric Spaces

We express a series of definitions of some fundamental notions related to bipolar metric spaces.

Definition 1. *[13] A bipolar metric space is a triple (X, Y, d) such that $X, Y \neq \emptyset$ and $d : X \times Y \to \mathbb{R}^+$ is a function satisfying the properties*

(B0) *if $d(x, y) = 0$, then $x = y$,*

(B1) *if $x = y$, then $d(x, y) = 0$,*

(B2) *if $x, y \in X \cap Y$, then $d(x, y) = d(y, x)$,*

(B3) *$d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$,*

for all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$, where $\mathbb{R}^+$ symbolises the set of all non-negative real numbers. Then d is called a bipolar metric on the pair (X, Y).

Definition 2. *[13] Let (X_1, Y_1) and (X_2, Y_2) be pairs of sets and given a function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$. If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$, we call f a covariant map from (X_1, Y_1) to (X_2, Y_2) and denote this with $f : (X_1, Y_1) \rightrightarrows (X_2, Y_2)$. If $f(X_1) \subseteq Y_2$ and $f(Y_1) \subseteq X_2$, then we call f a contravariant map from (X_1, Y_1) to (X_2, Y_2) and write $f : (X_1, Y_1) \nearrow\!\!\!\!\swarrow (X_2, Y_2)$. In particular, if d_1 and d_2 are bipolar metrics on (X_1, Y_1) and (X_2, Y_2), respectively, we sometimes use the notations $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \nearrow\!\!\!\!\swarrow (X_2, Y_2, d_2)$.*

Definition 3. *[13] Let (X, Y, d) be a bipolar metric space. A point $u \in X \cup Y$ is called a left point if $u \in X$, a right point if $u \in Y$ and a central point if it is both left and right point. Similarly a sequence (x_n) on the set X is called a left sequence and a sequence (y_n) on Y is called a right sequence. In a bipolar metric space, a left or a right sequence is called simply a sequence. A sequence (u_n) is said to be convergent to a point u, iff (u_n) is a left sequence, u is a right point and $\lim_{n\to\infty} d(u_n, u) = 0$; or (u_n) is a right sequence, u is a left point and $\lim_{n\to\infty} d(u, u_n) = 0$. A bisequence (x_n, y_n) on (X, Y, d) is a sequence on the set $X \times Y$. If the sequences (x_n) and (y_n) are convergent, then the bisequence (x_n, y_n) is said to be convergent, and if (x_n) and (y_n) converge to a common point, then (x_n, y_n) is called biconvergent. (x_n, y_n) is a Cauchy bisequence, if $\lim_{n,m\to\infty} d(x_n, y_m) = 0$. In a bipolar metric space, every convergent Cauchy bisequence is biconvergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.*

Definition 4. *[13] Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces.*

(1) *A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at a point $x_0 \in X_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x_0, y) < \delta$ implies $d_2(f(x_0), f(y)) < \varepsilon$ all $y \in Y_1$.*

(2) *A map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called right-continuous at a point $y_0 \in Y_1$, if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_1(x, y_0) < \delta$ implies $d_2(f(x), f(y_0)) < \varepsilon$ for all $x \in X_1$.*

(3) *A map f is called continuous, if it is left-continuous at each point $x \in X_1$ and right-continuous at each point $y \in Y_1$.*

(4) A contravariant map $f : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ is continuous if and only if it is continuous as a covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (Y_2, X_2, \bar{d_2})$

It can be seen from the Definition 4 that a covariant or a contravariant map f from (X_1, Y_1, d_1) to (X_2, Y_2, d_2) is continuous if and only if $(u_n) \to v$ on (X_1, Y_1, d_1) implies $(f(u_n)) \to f(v)$ on (X_2, Y_2, d_2).

3. Main Results

Definition 5. *Let (X, Y, d) be a bipolar metric space, $F : (X^2, Y^2) \rightrightarrows (X, Y)$ be a covariant mapping. $(a, b) \in X^2 \cup Y^2$ is said to be a coupled fixed point of F if*

$$F(a, b) = a \text{ and } F(b, a) = b.$$

Theorem 1. *Let (X, Y, d) be a complete bipolar metric space, $F : (X^2, Y^2) \rightrightarrows (X, Y)$ be a covariant mapping and k, l be non-negative constants. If F satisfies the condition*

$$d(F(a, b), F(p, q)) \leq kd(a, p) + ld(b, q), \ \ k + l < 1 \tag{1}$$

for all $a, b \in X$, $p, q \in Y$, then $F : X^2 \cup Y^2 \to X \cup Y$ has a unique coupled fixed point.

Proof. Let $a_0, b_0 \in X$ and $p_0, q_0 \in Y$. We take $a_1, b_1 \in X$ and $p_1, q_1 \in Y$ with $a_1 = F(a_0, b_0)$, $b_1 = F(b_0, a_0)$, $p_1 = F(p_0, q_0)$, $q_1 = F(q_0, p_0)$. And similarly, we take $a_2, b_2 \in X$ and $p_2, q_2 \in Y$ with $a_2 = F(a_1, b_1)$, $b_2 = F(b_1, a_1)$, $p_2 = F(p_1, q_1)$, $q_2 = F(q_1, p_1)$. In this way, we obtain bisequences (a_n, b_n) and (p_n, q_n) with

$$a_{n+1} = F(a_n, b_n), \ b_{n+1} = F(b_n, a_n), \ p_{n+1} = F(p_n, q_n) \text{ and } q_{n+1} = F(q_n, p_n)$$

for all $n \in \mathbb{N}^+$. Let $k + l = \lambda$. From (1), we get

$$\begin{aligned} d(a_n, p_{n+1}) &= d(F(a_{n-1}, b_{n-1}), F(p_n, q_n)), \\ &\leq kd(a_{n-1}, p_n) + ld(b_{n-1}, q_n) \end{aligned} \tag{2}$$

and

$$\begin{aligned} d(b_n, q_{n+1}) &= d(F(b_{n-1}, a_{n-1}), F(q_n, p_n)), \\ &\leq kd(b_{n-1}, q_n) + ld(a_{n-1}, p_n) \end{aligned} \tag{3}$$

for all $n \in \mathbb{N}^+$ and $\lambda < 1$. Let

$$e_n = d(a_n, p_{n+1}) + d(b_n, q_{n+1})$$

for all $n \in \mathbb{N}^+$. Combining (2) and (3), we observe that

$$\begin{aligned} e_n &= d(a_n, p_{n+1}) + d(b_n, q_{n+1}) \\ &\leq kd(a_{n-1}, p_n) + ld(b_{n-1}, q_n) + kd(b_{n-1}, q_n) + ld(a_{n-1}, p_n) \end{aligned}$$

$$\begin{aligned} &= (k+l)(d(a_{n-1},p_n)+d(b_{n-1},q_n)) \\ &= \lambda e_{n-1}. \end{aligned}$$

Then we get

$$0 \le e_n \le \lambda e_{n-1} \le \lambda^2 e_{n-2} \le \cdots \le \lambda^n e_0. \tag{4}$$

On the other hand,

$$\begin{aligned} d(a_{n+1},p_n) &= d(F(a_n,b_n),F(p_{n-1},q_{n-1})), \\ &\le kd(a_n,p_{n-1})+ld(b_n,q_{n-1}) \end{aligned} \tag{5}$$

and

$$\begin{aligned} d(b_{n+1},q_n) &= d(F(b_n,a_n),F(q_{n-1},p_{n-1})), \\ &\le kd(b_n,q_{n-1})+ld(a_n,p_{n-1}) \end{aligned} \tag{6}$$

for all $n \in \mathbb{N}^+$ and $\lambda < 1$. Let

$$s_n = d(a_{n+1},p_n)+d(b_{n+1},q_n)$$

for all $n \in \mathbb{N}^+$. Combining (5) and (6), we observe that

$$\begin{aligned} s_n &= d(a_{n+1},p_n)+d(b_{n+1},q_n) \\ &\le kd(a_n,p_{n-1})+ld(b_n,q_{n-1})+kd(b_n,q_{n-1})+ld(a_n,p_{n-1}) \\ &= (k+l)(d(a_n,p_{n-1})+d(b_n,q_{n-1})) \\ &= \lambda s_{n-1}. \end{aligned}$$

Then similar to Equation (4), we obtain that

$$0 \le s_n \le \lambda s_{n-1} \le \lambda^2 s_{n-2} \le \cdots \le \lambda^n s_0. \tag{7}$$

Moreover,

$$\begin{aligned} d(a_n,p_n) &= d(F(a_{n-1},b_{n-1}),F(p_{n-1},q_{n-1})), \\ &\le kd(a_{n-1},p_{n-1})+ld(b_{n-1},q_{n-1}) \end{aligned} \tag{8}$$

and

$$\begin{aligned} d(b_n,q_n) &= d(F(b_{n-1},a_{n-1}),F(q_{n-1},p_{n-1})), \\ &\le kd(b_{n-1},q_{n-1})+ld(a_{n-1},p_{n-1}) \end{aligned} \tag{9}$$

for all $n \in \mathbb{N}^+$ and $\lambda < 1$. Therefore, let

$$t_n = d(a_n,p_n)+d(b_n,q_n)$$

for all $n \in \mathbb{N}^+$. Combining (8) and (9), we observe that

$$\begin{aligned} t_n &= d(a_n, p_n) + d(b_n, q_n) \\ &\leq kd(a_{n-1}, p_{n-1}) + ld(b_{n-1}, q_{n-1}) + kd(b_{n-1}, q_{n-1}) + ld(a_{n-1}, p_{n-1}) \\ &= (k+l)(d(a_{n-1}, p_{n-1}) + d(b_{n-1}, q_{n-1})) \\ &= \lambda t_{n-1}. \end{aligned}$$

Thus, we obtain that

$$0 \leq t_n \leq \lambda t_{n-1} \leq \lambda^2 t_{n-2} \leq \cdots \leq \lambda^n t_0. \tag{10}$$

Using the property (B3), we get

$$\begin{aligned} d(a_n, p_m) &\leq d(a_n, p_{n+1}) + d(a_{n+1}, p_{n+1}) + \cdots + d(a_{m-1}, p_m), \\ d(b_n, q_m) &\leq d(b_n, q_{n+1}) + d(b_{n+1}, q_{n+1}) + \cdots + d(b_{m-1}, q_m) \end{aligned} \tag{11}$$

and

$$\begin{aligned} d(a_m, p_n) &\leq d(a_m, p_{m-1}) + d(a_{m-1}, p_{m-1}) + \cdots + d(a_{n+1}, p_n), \\ d(b_m, q_n) &\leq d(b_m, q_{m-1}) + d(b_{m-1}, q_{m-1}) + \cdots + d(b_{n+1}, q_n) \end{aligned} \tag{12}$$

for each $n, m \in \mathbb{N}$, $n < m$. Then, from (4), (7) (10), (11) and (12), we have

$$\begin{aligned} d(a_n, p_m) + d(b_n, q_m) &\leq (d(a_n, p_{n+1}) + d(b_n, q_{n+1})) \\ &\quad +(d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})) + \cdots \\ &\quad +(d(a_{m-1}, p_{m-1}) + d(b_{m-1}, q_{m-1})) \\ &\quad +(d(a_{m-1}, p_m) + d(b_{m-1}, q_m)), \\ &= e_n + t_{n+1} + e_{n+1} + \cdots + t_{m-1} + e_{m-1}, \\ &\leq \lambda^n e_0 + \lambda^{n+1} t_0 + \lambda^{n+1} e_0 + \cdots + \lambda^{m-1} t_0 + \lambda^{m-1} e_0, \\ &= (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) e_0 + (\lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}) t_0, \\ &\leq \frac{\lambda^n}{1-\lambda} e_0 + \frac{\lambda^{n+1}}{1-\lambda} t_0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} d(a_m, p_n) + d(b_m, q_n) &\leq (d(a_m, p_{m-1}) + d(b_m, q_{m-1})) \\ &\quad +(d(a_{m-1}, p_{m-1}) + d(b_{m-1}, q_{m-1})) + \cdots \\ &\quad +(d(a_{n+1}, p_{n+1}) + d(b_{n+1}, q_{n+1})) \\ &\quad +(d(a_{n+1}, p_n) + d(b_{n+1}, q_n)), \\ &= s_{m-1} + t_{m-1} + \cdots + s_{n+1} + t_{n+1} + s_n, \\ &\leq \lambda^{m-1} s_0 + \lambda^{m-1} t_0 + \cdots + \lambda^{n+1} s_0 + \lambda^{n+1} t_0 + \lambda^n s_0, \\ &= (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) s_0 + (\lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}) t_0, \\ &\leq \frac{\lambda^n}{1-\lambda} s_0 + \frac{\lambda^{n+1}}{1-\lambda} t_0 \end{aligned} \tag{14}$$

for $n < m$. Since, for an arbitrary $\epsilon > 0$, there exists n_0 such that $\frac{\lambda^{n_0}}{1-\lambda}e_0 + \frac{\lambda^{n_0+1}}{1-\lambda}t_0 < \frac{\epsilon}{3}$ and $\frac{\lambda^{n_0}}{1-\lambda}s_0 + \frac{\lambda^{n_0+1}}{1-\lambda}t_0 < \frac{\epsilon}{3}$, from (13) and (14), we have

$$d(a_n, p_m) + d(b_n, q_m) < \frac{\epsilon}{3}$$

for each $n, m \geq n_0$. Then (a_n, p_n) and (b_n, q_n) are Cauchy bisequences. Because of completeness of (X, Y, d), there exist $a, b \in X$ and $p, q \in Y$ with

$$\lim_{n\to\infty} a_n = p,\ \lim_{n\to\infty} b_n = q,\ \lim_{n\to\infty} p_n = a \text{ and } \lim_{n\to\infty} q_n = b. \tag{15}$$

Then there exists $n_1 \in \mathbb{N}$ with $d(a_n, p) < \frac{\epsilon}{3}$, $d(b_n, q) < \frac{\epsilon}{3}$, $d(a, p_n) < \frac{\epsilon}{3}$ and $d(b, q_n) < \frac{\epsilon}{3}$ for all $n \geq n_1$ and every $\epsilon > 0$. Since (a_n, p_n) and (b_n, q_n) are Cauchy bisequences, we get $d(a_n, p_n) < \frac{\epsilon}{3}$ and $d(b_n, q_n) < \frac{\epsilon}{3}$. So, from (1), we have

$$\begin{aligned} d(F(a,b),p) &\leq d(F(a,b),p_{n+1}) + d(a_{n+1},p_{n+1}) + d(a_{n+1},p) \\ &= d(F(a,b),F(p_n,q_n)) + d(a_{n+1},p_{n+1}) + d(a_{n+1},p) \\ &\leq kd(a,p_n) + ld(b,q_n) + d(a_{n+1},p_{n+1}) + d(a_{n+1},p) \\ &< k\frac{\epsilon}{3} + l\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \lambda\frac{\epsilon}{3} + 2\frac{\epsilon}{3} < \epsilon \end{aligned}$$

for each $n \in \mathbb{N}$ and $\lambda < 1$. Then $d(F(a, b), p) = 0$. Hence, $F(a, b) = p$. Similarly, we get $F(b, a) = q$, $F(p, q) = a$ and $F(q, p) = b$. On the other hand, from (15) we get

$$d(a,p) = d(\lim_{n\to\infty} p_n, \lim_{n\to\infty} a_n) = \lim_{n\to\infty} d(a_n, p_n) = 0$$

and

$$d(b,q) = d(\lim_{n\to\infty} q_n, \lim_{n\to\infty} b_n) = \lim_{n\to\infty} d(b_n, q_n) = 0.$$

So, $a = p$ and $b = q$. Therefore, $(a, b) \in X^2 \cap Y^2$ is a coupled fixed point of F.

Now, to show the uniqueness, we begin by taking another coupled fixed point $(a^*, b^*) \in X^2 \cup Y^2$. If $(a^*, b^*) \in X^2$, then we get

$$d(a^*, a) = d(F(a^*, b^*), F(a, b)) \leq kd(a^*, a) + ld(b^*, b)$$

and

$$d(b^*, b) = d(F(b^*, a^*), F(b, a)) \leq kd(b^*, b) + ld(a^*, a).$$

Therefore, we have

$$d(a^*, a) + d(b^*, b) \leq \lambda(d(a^*, a) + d(b^*, b)). \tag{16}$$

Since $\lambda < 1$, by (16) this means that $d(a^*, a) + d(b^*, b) = 0$. So, we obtain that $a^* = a$ and $b^* = b$. Similarly, if $(a^*, b^*) \in Y^2$, we have $a^* = a$ and $b^* = b$. Then (a, b) is a unique coupled fixed point of F.

The following corollary is obtained, if we take equal the constants k, l in Theorem 1.

Corollary 1. *Let (X, Y, d) be a complete bipolar metric space, $F : \left(X^2, Y^2\right) \rightrightarrows (X, Y)$ be a covariant mapping and k, l be non-negative constants. If the condition*

$$d(F(a,b),F(p,q)) \leq \frac{k}{2}(d(a,p)+d(b,q)),\ k<1 \tag{17}$$

holds for all $a, b \in X$, $p, q \in Y$, then $F : X^2 \cup Y^2 \to X \cup Y$ has a unique coupled fixed point.

Now, we express another generalization of coupled fixed point theorem in bipolar metric spaces.

Definition 6. *Let (X, Y, d) be a bipolar metric space, $a \in X$, $p \in Y$ and $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping. (a, p) is said to be a coupled fixed point of F if*

$$F(a,p) = a \text{ and } F(p,a) = p.$$

Theorem 2. *Let (X, Y, d) be a complete bipolar metric space, $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping and k, l be non-negative constants. If the condition*

$$d(F(a,p),F(q,b)) \leq kd(a,q)+ld(b,p),\ k+l<1 \tag{18}$$

holds for all $a, b \in X$, $p, q \in Y$, then $F : (X \times Y) \cup (Y \times X) \to X \cup Y$ has a unique coupled fixed point.

Proof. Similar to the proof of Theorem 1, we define bisequences (a_n, p_n) and (b_n, q_n) as follows:

$$a_{n+1} = F(a_n,p_n),\ p_{n+1} = F(p_n,a_n),\ b_{n+1} = F(b_n,q_n) \text{ and } q_{n+1} = F(q_n,b_n)$$

for all $n \in \mathbb{N}^+$. Let $k + l = \lambda$. Then, from (18), we get

$$\begin{aligned} d(a_n,q_{n+1}) &= d(F(a_{n-1},p_{n-1}),F(q_n,b_n)), \\ &\leq kd(a_{n-1},q_n)+ld(b_n,p_{n-1}) \end{aligned} \tag{19}$$

$$\begin{aligned} d(a_{n+1},q_n) &= d(F(a_n,p_n),F(q_{n-1},b_{n-1})), \\ &\leq kd(a_n,q_{n-1})+ld(b_{n-1},p_n) \end{aligned} \tag{20}$$

$$\begin{aligned} d(b_n,p_{n+1}) &= d(F(b_{n-1},q_{n-1}),F(p_n,a_n)), \\ &\leq kd(b_{n-1},p_n)+ld(a_n,q_{n-1}) \end{aligned} \tag{21}$$

$$\begin{aligned} d(b_{n+1},p_n) &= d(F(b_n,q_n),F(p_{n-1},a_{n-1})), \\ &\leq kd(b_n,p_{n-1})+ld(a_{n-1},q_n) \end{aligned} \tag{22}$$

for all $n \in \mathbb{N}^+$ and $\lambda < 1$. Let

$$e_n = d(a_n, q_{n+1}) + d(b_{n+1}, p_n)$$

and

$$s_n = d(a_{n+1}, q_n) + d(b_n, p_{n+1})$$

for all $n \in \mathbb{N}^+$. Using equations (19), (20), (21) and (22), we get

$$\begin{aligned} e_n &= d(a_n, q_{n+1}) + d(b_{n+1}, p_n) \\ &\leq kd(a_{n-1}, q_n) + ld(b_n, p_{n-1}) + kd(b_n, p_{n-1}) + ld(a_{n-1}, q_n) \\ &= (k+l)(d(a_{n-1}, q_n) + d(b_n, p_{n-1})) \\ &= \lambda e_{n-1} \end{aligned}$$

and

$$\begin{aligned} s_n &= d(a_{n+1}, q_n) + d(b_n, p_{n+1}) \\ &\leq kd(a_n, q_{n-1}) + ld(b_{n-1}, p_n) + kd(b_{n-1}, p_n) + ld(a_n, q_{n-1}) \\ &= (k+l)(d(a_n, q_{n-1}) + d(b_{n-1}, p_n)) \\ &= \lambda s_{n-1}. \end{aligned}$$

Then we obtain that

$$0 \leq e_n \leq \lambda e_{n-1} \leq \lambda^2 e_{n-2} \leq \cdots \leq \lambda^n e_0 \tag{23}$$

and

$$0 \leq s_n \leq \lambda s_{n-1} \leq \lambda^2 s_{n-2} \leq \cdots \leq \lambda^n s_0. \tag{24}$$

On the other hand,

$$\begin{aligned} d(a_n, q_n) &= d(F(a_{n-1}, p_{n-1}), F(q_{n-1}, b_{n-1})), \\ &\leq kd(a_{n-1}, q_{n-1}) + ld(b_{n-1}, p_{n-1}) \end{aligned} \tag{25}$$

and

$$\begin{aligned} d(b_n, p_n) &= d(F(b_{n-1}, p_{n-1}), F(p_{n-1}, a_{n-1})), \\ &\leq kd(b_{n-1}, p_{n-1}) + ld(a_{n-1}, q_{n-1}) \end{aligned} \tag{26}$$

for all $n \in \mathbb{N}^+$ and $\lambda < 1$. If we take

$$t_n = d(a_n, q_n) + d(b_n, p_n)$$

for all $n \in \mathbb{N}^+$ and combine (25) and (26), we have

$$\begin{aligned} t_n &= d(a_n, q_n) + d(b_n, p_n) \\ &\leq kd(a_{n-1}, q_{n-1}) + ld(b_{n-1}, p_{n-1}) + kd(b_{n-1}, p_{n-1}) + ld(a_{n-1}, q_{n-1}) \\ &= (k+l)(d(a_{n-1}, q_{n-1}) + d(b_{n-1}, p_{n-1})) \end{aligned}$$

$$= \lambda t_{n-1}.$$

So, we get

$$0 \leq t_n \leq \lambda t_{n-1} \leq \lambda^2 t_{n-2} \leq \cdots \leq \lambda^n t_0. \tag{27}$$

We obtain that

$$\begin{array}{rcl} d(a_n, q_m) & \leq & d(a_n, q_{n+1}) + d(a_{n+1}, q_{n+1}) + \cdots + d(a_{m-1} q_m), \\ d(b_n, p_m) & \leq & d(b_n, p_{n+1}) + d(b_{n+1}, p_{n+1}) + \cdots + d(b_{m-1}, p_m), \\ d(a_m, q_n) & \leq & d(a_m, q_{m-1}) + d(a_{m-1}, q_{m-1}) + \cdots + d(a_{n+1}, q_n), \\ d(b_m, p_n) & \leq & d(b_m, p_{m-1}) + d(b_{m-1}, p_{m-1}) + \cdots + d(b_{n+1}, p_n) \end{array} \tag{28}$$

for each $n, m \in \mathbb{N}$, $n < m$. Thus, from (23), (24) (27) and (28), we have

$$\begin{array}{rcl} d(a_n, q_m) + d(b_m, p_n) & \leq & (d(a_n, q_{n+1}) + d(b_{n+1}, p_n)) \\ & & +(d(a_{n+1}, q_{n+1}) + d(b_{n+1}, p_{n+1})) + \cdots \\ & & +(d(a_{m-1}, q_{m-1}) + d(b_{m-1}, p_{m-1})) \\ & & +(d(a_{m-1}, q_m) + d(b_m, p_{m-1})), \\ & = & e_n + t_{n+1} + e_{n+1} + \cdots + t_{m-1} + e_{m-1}, \\ & \leq & \lambda^n e_0 + \lambda^{n+1} t_0 + \lambda^{n+1} e_0 + \cdots + \lambda^{m-1} t_0 + \lambda^{m-1} e_0, \\ & = & (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) e_0 + (\lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}) t_0, \\ & \leq & \dfrac{\lambda^n}{1-\lambda} e_0 + \dfrac{\lambda^{n+1}}{1-\lambda} t_0 \end{array} \tag{29}$$

and

$$\begin{array}{rcl} d(a_m, q_n) + d(b_n, p_m) & \leq & (d(a_m, q_{m-1}) + d(b_{m-1}, p_m)) \\ & & +(d(a_{m-1}, q_{m-1}) + d(b_{m-1}, p_{m-1})) + \cdots \\ & & +(d(a_{n+1}, q_{n+1}) + d(b_{n+1}, p_{n+1})) \\ & & +(d(a_{n+1}, q_n) + d(b_n, p_{n+1})), \\ & = & s_{m-1} + t_{m-1} + \cdots + s_{n+1} + t_{n+1} + s_n, \\ & \leq & \lambda^{m-1} s_0 + \lambda^{m-1} t_0 + \cdots + \lambda^{n+1} s_0 + \lambda^{n+1} t_0 + \lambda^n s_0, \\ & = & (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) s_0 + (\lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}) t_0, \\ & \leq & \dfrac{\lambda^n}{1-\lambda} s_0 + \dfrac{\lambda^{n+1}}{1-\lambda} t_0 \end{array} \tag{30}$$

for $n < m$. Since, for an arbitrary $\epsilon > 0$, there exists n_0 such that $\frac{\lambda^{n_0}}{1-\lambda} e_0 + \frac{\lambda^{n_0+1}}{1-\lambda} t_0 < \frac{\epsilon}{3}$ and $\frac{\lambda^{n_0}}{1-\lambda} s_0 + \frac{\lambda^{n_0+1}}{1-\lambda} t_0 < \frac{\epsilon}{3}$, from (29) and (30), we have for each $n, m \geq n_0$ that

$$d(a_n, q_m) + d(b_m, p_n) < \frac{\epsilon}{3}.$$

Then (a_n, q_n) and (b_n, p_n) are Cauchy bisequences. Using completeness of (X, Y, d), we say that there exist $a, b \in X$ and $p, q \in Y$ with

$$\lim_{n\to\infty} a_n = q, \ \lim_{n\to\infty} b_n = p, \ \lim_{n\to\infty} p_n = b \text{ and } \lim_{n\to\infty} q_n = a. \tag{31}$$

Then there exists $n_1 \in \mathbb{N}$ with $d(a_n, q) < \frac{\epsilon}{3}$, $d(b_n, p) < \frac{\epsilon}{3}$, $d(b, p_n) < \frac{\epsilon}{3}$ and $d(a, q_n) < \frac{\epsilon}{3}$ for all $n \geq n_1$ and every $\epsilon > 0$. Since (a_n, q_n) and (b_n, p_n) are Cauchy bisequences, we get $d(a_n, q_n) < \frac{\epsilon}{3}$ and $d(b_n, p_n) < \frac{\epsilon}{3}$. Thus, from (18), we have

$$\begin{aligned} d(F(a,p),q) &\leq d(F(a,p),q_{n+1}) + d(a_{n+1},q_{n+1}) + d(a_{n+1},q) \\ &= d(F(a,p),F(p_n,b_n)) + d(a_{n+1},q_{n+1}) + d(a_{n+1},q) \\ &\leq kd(a,q_n) + ld(b_n,p) + d(a_{n+1},q_{n+1}) + d(a_{n+1},q) \\ &< k\frac{\epsilon}{3} + l\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \lambda\frac{\epsilon}{3} + 2\frac{\epsilon}{3} < \epsilon \end{aligned}$$

for each $n \in \mathbb{N}$ and $\lambda < 1$. Then $d(F(a,p),q) = 0 \Rightarrow F(a,p) = q$. In a similar manner, we get $F(p,a) = b$, $F(b,q) = p$ and $F(q,b) = a$. And, from (31) we have

$$d(a,q) = d(\lim_{n\to\infty} q_n, \lim_{n\to\infty} a_n) = \lim_{n\to\infty} d(a_n, q_n) = 0$$

and

$$d(b,p) = d(\lim_{n\to\infty} p_n, \lim_{n\to\infty} b_n) = \lim_{n\to\infty} d(b_n, p_n) = 0.$$

Therefore, $a = q$ and $b = p$. Then $(a,p) \in (X \times Y) \cap (Y \times X)$ is a coupled fixed point of F. As in the proof of the Theorem 1, uniqueness of the coupled fixed point of F can be shown easily.

Corollary 2. *Let (X, Y, d) be a complete bipolar metric space. $F : (X \times Y, Y \times X) \rightrightarrows (X, Y)$ be a covariant mapping and k, l be non-negative constants. If the condition*

$$d(F(a,p),F(q,b)) \leq \frac{k}{2}(d(a,q) + d(b,p)), \ k < 1 \tag{32}$$

holds for all $a, b \in X$, $p, q \in Y$, then $F : (X \times Y) \cup (Y \times X) \to X \cup Y$ has a unique coupled fixed point.

Example 1. *Let $U_n(\mathbb{R})$ and $L_n(\mathbb{R})$ be the sets of all $n \times n$ upper and lower triangular matrices over $\mathbb{R}$, respectively. A function $d : U_n(\mathbb{R}) \times L_n(\mathbb{R}) \to \mathbb{R}^+$ be defined as*

$$d(A,B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|$$

for all $A = (a_{ij})_{n\times n} \in U_n(\mathbb{R})$ and $B = (b_{ij})_{n\times n} \in L_n(\mathbb{R})$. Then it is apparent that $(U_n(\mathbb{R}), L_n(\mathbb{R}), d)$ is a complete bipolar metric space. We take a covariant mapping

$$F : \left(U_n(\mathbb{R})^2, L_n(\mathbb{R})^2\right) \rightrightarrows (U_n(\mathbb{R}), L_n(\mathbb{R}))$$

such as $F(A,B) = \left(\frac{a_{ij}+b_{ij}}{3}\right)_{n\times n}$ where $(A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}) \in U_n(\mathbb{R})^2 \cup L_n(\mathbb{R})^2$. Then we get

$$d(F(A,B),F(C,D)) = d\left(\left(\frac{a_{ij}+b_{ij}}{3}\right)_{n\times n}, \left(\frac{c_{ij}+d_{ij}}{3}\right)_{n\times n}\right)$$

$$= \sum_{i,j=1}^{n} \left| \frac{a_{ij} + b_{ij} - c_{ij} - d_{ij}}{3} \right|$$
$$\leq \sum_{i,j=1}^{n} \left| \frac{a_{ij} - c_{ij}}{3} \right| + \left| \frac{b_{ij} - d_{ij}}{3} \right|$$
$$= \frac{1}{3} \left(d(A, C) + d(B, D) \right)$$

for all $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in U_n(\mathbb{R})$ *and* $C = (c_{ij})_{n\times n}, D = (d_{ij})_{n\times n} \in L_n(\mathbb{R})$. *Therefore, the equation* (17) *is satisfied for* $k = \frac{2}{3}$. *Then from Corollary 1,* F *has a unique coupled fixed point. It is obvious that the coupled fixed point is* $(0_{n\times n}, 0_{n\times n}) \in U_n(\mathbb{R}) \cap L_n(\mathbb{R})$ *where* $0_{n\times n}$ *is the null matrix.*

On the other hand, if

$$F : \left(U_n(\mathbb{R})^2, L_n(\mathbb{R})^2\right) \rightrightarrows \left(U_n(\mathbb{R}), L_n(\mathbb{R})\right)$$

is defined by $F(A, B) = \left(\frac{a_{ij}+b_{ij}}{2}\right)_{n\times n}$ *where* $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in U_n(\mathbb{R})^2 \cup L_n(\mathbb{R})^2$. *Then it can be observed that*

$$d(F(A, B), F(C, D)) \leq \frac{1}{2}(d(A, C) + d(B, D)).$$

Then F *satisfies the equation* (17) *for* $k = 1$. *Therefore, coupled fixed points of* F *are both* $(0_{n\times n}, 0_{n\times n}) \in U_n(\mathbb{R}) \cap L_n(\mathbb{R})$ *and* $(I_n, I_n) \in U_n(\mathbb{R}) \cap L_n(\mathbb{R})$ *where* $0_{n\times n}$ *is the null matrix and* I_n *is the identity matrix. As it can be seen from this expression,* F *has not a unique coupled fixed point. Thus, the conditions* $k < 1$ *in Corollary 1 and* $k + l < 1$ *in Theorem 1 are the most appropriate conditions for satisfing the uniqueness of coupled fixed point.*

References

[1] M. Abbas, M. Ali Khan, S. Radenovic. Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, Appl. Math. Comput. 217, 195-202, 2010.

[2] I. Altun, H. Simsek. Some fixed point theorems on ordered metric spaces and application, *Fixed Point Theory Appl.* 2010, 17 pages, Article ID 621492, 2010.

[3] T.G. Bhaskar, V. Lakshmikantham. Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.* 65, 1379-1393, 2006.

[4] Y.J. Cho, B.E. Rhoades, R. Saadati, B. Samet, W. Shatanawi. Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type, *Fixed Point Theory Appl.* 2012(8), 1–14, 2012.

[5] L.j. Círíc, V. Lakshmikantham. Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Stoch. Anal. Appl.* 27, 1246-1259, 2009.

[6] D. Guo, V. Lakshmikantham. Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal.* 11, 623-632, 1987.

[7] E. Karapinar. Coupled fixed point theorems for nonlinear contractions in cone metric spaces, *Comput. Math. Appl.* 59, 3656-3668, 2010.

[8] V. Lakshmikantham, L.j. Círíc. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.* 70, 4341-4349, 2009.

[9] N.V. Luong, N.X. Thuan. Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal. 74, 983-992, 2011.

[10] A. Mutlu, N. Yolcu, B. Mutlu, N. Bildik. On common coupled fixed point theorems for comparable mappings in ordered partially metric spaces, *Abstract and Applied Analysis* 2014, 6 pages Article ID 486384, 2014.

[11] A. Mutlu, N. Yolcu, B. Mutlu. Coupled fixed point theorem for mixed monotone mappings on partially ordered dislocated quasi metric spaces, *Global Journal of Mathematics* 1(1), 12-17, 2015.

[12] A. Mutlu, N. Yolcu, B. Mutlu. Fixed point theorems in partially ordered rectangular metric spaces, *British Journal of Mathematics & Computer Science* 15(2), 1-9, 2016.

[13] A. Mutlu, U. Gürdal. Bipolar metric spaces and some fixed point theorems, *J. Nonlinear Sci. Appl.* 9(9), 5362–5373, 2016.

[14] A. Mutlu, , K. Özkan, U. Gürdal. Coupled fixed point theorem in partially ordered modular metric spaces and its an application, *J. Comput. Anal. Appl.* 25(2), 207–216, 2018.

[15] A. Petruşel, G. Petruşel, B. Samet, J.C. Yao. Coupled fixed point theorems for symmetric contractions in b-metric spaces with applications to operator equation systems, *Fixed Point Theory* 17(2), 457–476, 2016.

[16] F. Sabetghadam, H.P. Masiha, A.H. Sanatpour. Some coupled fixed point theorems in cone metric spaces, *Fixed Point Theory Appl.* 8 pages Article ID 125426 (2009) doi:10.1155/2009/125426.

[17] B. Samet. Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, *Nonlinear Anal.* 72, 4508-4517, 2010.

[18] W. Shatanawi, E. Karapinar, H. Aydi. Coupled coincidence points in partially ordered cone metric spaces with a c-distance, *Journal of Applied Mathematics* 2012, Article ID 312078, 2012.

[19] W. Shatanawi , A. Pitea. Some coupled fixed point theorems in quasi-partial metric spaces, *Fixed Point Theory Appl.* 2013(153), 1-15, 2013.

[20] N. Tahat, H. Aydi, E. Karapinar, W. Shatanawi. Common fixed points for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces, *Fixed Point Theory Appl.* 2012(48), 2012, doi:10.1186/1687-1812-2012-48.