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Monomorphism and Epimorphism Properties of Soft Categories

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Abstract. In this paper, firstly we recall some definitions and basic properties of soft set theory, category theory and soft category theory. We study on soft monomorphism, soft epimorphism, equalizer and coequalizer for soft categories. We gave some properties of these soft morphisms. We proved that the soft Category $SFun$ has coequalizers, equalizer of a morphism pair in $SFun$ category is soft monic and coequalizers of a morphism pair in $SFun$ category is soft epic.

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1. Introduction

The concept of soft sets was introduced by D. Molodtsov [6] in 1999 and Soft set theory became an alternative and useful tool for computer science, modeling problems in engineering, economics, medical and social science. Theorical properties of soft set theory has also been studied some mathematicians. Maji et all [5] defined some operations on soft sets. On the other hand Aras, Sönmez, Çakallı [1] and Zorlutuna, Çakır [11] worked on continuity of soft mappings. Also Probabilistic Soft Set Theory has been studied by Aras and Poşul in [2] and soft topological spaces have been studied by Shabir and Naz in [9]. The soft category theory studied by Sardar and Gupta in [8] and Zhou, Li and Akram in [10] and Öztunç [7]. They introduced the basic notions of the theory of soft categories and gave some introductory results of the soft category theory. The purpose of this paper is to study some new properties of soft category theory. Zhou, Li and Akram [10] defined the $SFun$ Category and gave some results such as $SFun$ has equalizers in [10]. Then we present monomorphism and epimorphism properties of $SFun$ category and also prove $SFun$ has coequalizers. We used some fundamental books from Category theory such as [3] and [4].

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2. Preliminaries

We express a series of definitions of some fundamental notions related to soft set theory and category theory.

Definition 1. *[6] A pair (F, A) is said to be a soft set over the universe X, where F is a mapping given by $F : A \to P(X)$ and $A \subseteq E$. Any soft set (F, A) can be extended to a soft set of type (F, E), where $F(e) \neq \varnothing$ for all $e \in A$ and $F(e) = \varnothing$ for all $e \in E \backslash A$. $S(X, E)$ indicates the family of all soft sets over X.*

Definition 2. *[10] Let (F, A) and (G, B) be two soft sets over the set X. Then one says that the mapping $f : (F, A) \to (G, B)$ is a soft function from (F, A) to (G, B) if it satisfies $F(a) \subseteq (G \circ f)(a)$ for each $a \in A$.*

Definition 3. *[3] A category $\mathcal{C}$ consists of the data which is given below:*
- *Objects: $A, B, C, ...$*
- *Arrows: $f, g, h, ...$*
- *For each arrow f, there are given objects*

$$dom(f), \;\; cod(f)$$

which is called the domain and codomain of f. It is written

$$f : A \to B$$

to indicate that $A = dom(f)$ and $B = cod(f)$
- *Given arrows $f : A \to B$ and $g : B \to C$, that is, with*

$$cod(f) = dom(g)$$

there is an arrow given by

$$g \circ f : A \to C$$

called the composite of f and g.
- *For each object A, there is given an arrow*

$$1_A : A \to A$$

called the identity arrow of A.
This property must satisfy the following laws:
Associativity : $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B$,$g : B \to C$ and $h : C \to D$.
Unit : $f \circ 1_A = f = 1_B \circ f$ for all $f : A \to B$.

Let $SFun$ denote the category of all soft sets over X and soft functions. [10]

3. Monomorphism and Epimorphism of the Category SFun

Definition 4. *Let $SFun$ be a soft category and (F, A) and (G, B) be two $SFun-$objects. If a $SFun-$morphism $f_s : (F, A) \to (G, B)$ in $SFun$ Category is left cancellable, then f_s is said to be a soft monomorphism.*

Theorem 1. *Let (F, A), (G, B) and (H, C) be $SFun-$objects over X. Suppose that $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ be two soft functions. If f_s and g_s are soft monic, then $g_s \circ f_s$ is soft monic.*

Proof. If $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ are $SFun-$morphisms, then there is a $y \in B$ such that $f_s(x) = y$ for every $x \in A$ and there is a $z \in C$ such that $g_s(y) = z$ for every $y \in B$.

We have $F(x) \subseteq (G \circ f_s)(x)$ for all $x \in A$, since f_s is a soft function and $G(y) \subseteq (H \circ g_s)(y)$ for all $y \in B$, since g_s is a soft function. We must show that

$$F(x) \subseteq (H \circ g_s \circ f_s)(x)$$

and $g_s \circ f_s$ $SFun-$morphism is left cancellable in order to prove that

$$g_s \circ f_s : (F, A) \to (H, C)$$

is monic. Thus we have the following:

$$F(x) \subseteq (G \circ f_s)(x) = G(f_s(x)) = G(y) \tag{1}$$

$$G(y) \subseteq (H \circ g_s)(y) = H(g_s(y)) = H(z) \tag{2}$$

If $F(x) \subseteq G(y)$ and $G(y) \subseteq H(z)$, then $F(x) \subseteq H(z)$. We obtain that

$$F(x) \subseteq H(z) = H(g_s(y)) = H(g_s(f(x))) = (H \circ g_s \circ f_s)(x)$$

by 1 and 2.

Let now show that the left cancellable property. Suppose that $(g_s \circ f_s) \circ h = (g_s \circ f_s) \circ k_s$ for any $h_s, k_s : (K, D) \to (F, A)$ $SFun-$morphisms. We get $g_s \circ (f_s \circ h_s) = g_s \circ (f_s \circ k_s)$ because of associativity of morphisms and obtain that $f_s \circ h_s = f_s \circ k_s$ since g_s $SFun-$morphism is left cancellable. f_s is left cancellable since it is a $SFun-$monomorphism. Thus we conclude that $h_s = k_s$.

■

Theorem 2. *Let (F, A), (G, B) and (H, C) be $SFun-$objects over X and let $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ be two $SFun-$morhisms. If $g_s \circ f_s$ is soft monic, then f_s is soft monic.*

Proof. If $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ are $SFun-$morphisms, then there is a $b \in B$ such that $f_s(a) = b$ for all $a \in A$ and there is a $c \in C$ such that $g_s(b) = c$ for all $b \in B$.

For every $a \in A$ we have $F(a) \subseteq (G \circ f_s)(a)$, since f_s is a soft function and for every $b \in B$, we have $G(b) \subseteq (H \circ g_s)(b)$, since g_s is a soft function.
At first we must show that $g_s \circ f_s : (F, A) \to (H, C)$ is a $SFun-$ morphism and then f_s is left cancellable. Thus we have the following inclusions:

$$F(a) \subseteq (G \circ f_s)(a) = G(f_s(a)) = G(b) \tag{3}$$

$$G(b) \subseteq (H \circ g_s)(b) = H(g_s(b)) = H(c) \tag{4}$$

By 3 and 4 we obtain that $F(a) \subseteq H(c)$ and

$$F(a) \subseteq H(c) = H(g_s(b)) = H(g_s(f_s(a))) = (H \circ g_s \circ f_s)(a).$$

Therefore $g_s \circ f_s$ is on $SFun-$ morphism.
Let now show that the left cancellable property. Assume that $f_s \circ h_s = f_s \circ k_s$ for any $h_s, k_s : (K, D) \to (F, A)$ $SFun-$ morphisms. Applying the $SFun-$ morphism g_s

$$g_s \circ f_s \circ h_s = g_s \circ f_s \circ k_s.$$

$g_s \circ f_s$ is left cancellable since $g_s \circ f_s$ is a $SFun-$ monomorphism. Therefore we obtain that $h_s = k_s$.

■

Definition 5. *Let $SFun$ be a soft category and (F, A) and (G, B) be two $SFun-$ objects. If $f_s : (F, A) \to (G, B)$ $SFun-$ morphism is right cancellable, then f_s is said to be a soft epimorphism.*

Theorem 3. *Let (F, A), (G, B) and (H, C) be $SFun-$ objects over X. Suppose that $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ be two soft functions. If f_s and g_s are soft epic, then $g_s \circ f_s$ is soft epic.*

Proof. If $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ are $SFun-$ morphisms, then there is a $b \in B$ such that $f_s(a) = b$ for all $a \in A$ and there is a $c \in C$ such that $g_s(b) = c$ for all $b \in B$. Since f_s is a soft function, $F(a) \subseteq (G \circ f_s)(a)$ for every $a \in A$ and since g_s is a soft function $G(b) \subseteq (H \circ g_s)(b)$ for every $b \in B$.
At first we must show that $F(a) \subseteq (H \circ g_s \circ f_s)(a)$ and $SFun-$ morphism $g_s \circ f_s$ is right cancellable in order to show that $g_s \circ f_s : (F, A) \to (H, C)$ is epic. Thus we have the following:

$$F(a) \subseteq (G \circ f_s)(a) = G(f_s(a)) = G(b) \tag{5}$$

$$G(b) \subseteq (H \circ g_s)(b) = H(g_s(b)) = H(c) \tag{6}$$

By (5) and (6) it is obtained that $F(a) \subseteq H(c)$ and then,

$$F(a) \subseteq H(c) = H(g_s(b)) = H(g_s(f_s(a))) = (H \circ g_s \circ f_s)(a).$$

Let now show the right cancellable property. Let $h_s \circ (g_s \circ f_s) = k_s \circ (g_s \circ f_s)$ for any $h_s, k_s : (K, D) \to (F, A)$ $SFun-$ morphisms. We have $(h_s \circ g_s) \circ f_s = (k_s \circ g_s) \circ f_s$, since morphisms are associative and f_s is right cancellable since it is a $SFun-$ epimorphism. Similarly g_s is right cancellable since it is a $SFun-$ epimorphism. Therefore we obtained that $h_s = k_s$.

■

Theorem 4. *Let (F, A), (G, B) and (H, C) be $SFun-$ objects over X. Suppose that $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ be two soft functions. If $g_s \circ f_s$ is epic, then g_s is soft epic.*

Proof. If $f_s : (F, A) \to (G, B)$ and $g_s : (G, B) \to (H, C)$ are $SFun-$ morphisms, then there is a $b \in B$ such that $f_s(a) = b$ for all $a \in A$ and there is a $c \in C$ such that $g_s(b) = c$ for all $b \in B$. Also we have $F(a) \subseteq (G \circ f_s)(a)$ for every $a \in A$ since f_s is a soft function and we have $G(b) \subseteq (H \circ g_s)(b)$ for every $b \in B$ since g_s is a soft function.
Now we must show the map $g_s \circ f_s : (F, A) \to (H, C)$ is a $SFun-$ morphism. Thus we have the following inclusions:

$$F(a) \subseteq (G \circ f_s)(a) = G(f_s(a)) = G(b) \tag{7}$$

$$G(b) \subseteq (H \circ g_s)(b) = H(g_s(b)) = H(c) \tag{8}$$

By (8) and (9) we obtained that $F(a) \subseteq H(c)$ and

$$F(a) \subseteq H(c) = H(g(b)) = H(g_s(f_s(a))) = (H \circ g_s \circ f_s)(a).$$

Hence $g_s \circ f_s$ is a $SFun-$ morphism. Next show that g_s is right cancellable. Let $h_s \circ g_s = k_s \circ g_s$ for any $h_s, k_s : (K, D) \to (F, A)$ soft morphisms. Applying $SFun-$morphism f_s

$$h_s \circ g_s \circ f_s = k_s \circ g_s \circ f_s$$

. Thus we get $h_s = k_s$, since $g_s \circ f_s$ is a soft epimorphism.

■

Theorem 5 (11)**.** *SFun has equalizers.*

$$\begin{tikzcd}
(H', C') \arrow[d, "\bar{e}"'] \arrow[dr, "e'"] & & \\
(H, C) \arrow[r, "e"'] & (F, A) \arrow[r, shift left, "f_s"] \arrow[r, shift right, "g_s"'] & (G, B)
\end{tikzcd}$$

Theorem 6. *In category SFun , if the equalizer of a morphism pair is $((H,C),e)$, then $((H,C),e)$ is monic.*

Proof.

$$\begin{tikzcd}
(H',C') \arrow[d, shift right, "\bar{e}"'] \arrow[d, shift left, "\bar{\bar{e}}"] \arrow[dr, "e'"] & & \\
(H,C) \arrow[r, "e"'] & (F,A) \arrow[r, shift left, "f_s"] \arrow[r, shift right, "g_s"'] & (G,B)
\end{tikzcd}$$

Suppose that $((H,C),e)$, is equalizer of f_s and g_s. Let $\bar{e}$ and $\bar{\bar{e}}$ be two soft morphisms as illustrated above diagram. We have $H'(c') \subseteq (F \circ e')(c')$ since $e' : (H',C') \to (F,A)$ is soft morphism, $H'(c') \subseteq (H \circ \bar{e})(c')$ since $\bar{\bar{e}} : (H',C') \to (H,C)$ is soft morphism and $H'(c') \subseteq (H \circ \bar{\bar{e}})(c')$ since $\bar{\bar{e}} : (H',C') \to (H,C)$ is soft morphism. Thus we have the following inclusion:
$H'(c') \subseteq (F \circ e')(c') = F(e')(c') = F(e \circ \bar{e})(c') = F(e \circ \bar{\bar{e}})(c')$.

Assume that $e\bar{e} = e\bar{\bar{e}}$. Then we want to show that $\bar{e} = \bar{\bar{e}}$. Put $e' = e\bar{e} = e\bar{\bar{e}}$. Then

$$f_s e' = f_s e\bar{e} = f_s e\bar{\bar{e}}$$

$$f_s e' = f_s e\bar{e} = g_s e\bar{\bar{e}} = g_s e'$$

. By using Universal Mapping Property , there is unique $u : (H',C') \to (H,C)$ such that $eu = e'$. Hence we obtain that $\bar{e} = u = \bar{\bar{e}}$, because we have $e\bar{e} = e'$ and $e\bar{\bar{e}} = e'$. Since $e\bar{e} = e\bar{\bar{e}}$ implies that $\bar{e} = \bar{\bar{e}}$, e is left cancellable. Therefore e is monic.

■

Theorem 7. *SFun has coequalizers.*

$$\begin{tikzcd}
(F,A) \arrow[r, shift left, "f_s"] \arrow[r, shift right, "g_s"'] & (G,B) \arrow[r, "e"] \arrow[dr, "e'"'] & (H,C) \arrow[d, "\bar{e}"] \\
& & (H',C')
\end{tikzcd}$$

Proof. Define the set $C = \{b \in B : f_s(b) = g_s(b)\}$, the embedding map $e : B \to C$ and $G = H \circ e$. From the diagram $e \circ f_s = e \circ g_s$ and $G(b) = (H \circ e)(b)$ for every $b \in B$. Thus e is a $SFun-$morphism. Now show that $((G,B),e)$ is coequalizer of f_s and g_s.

Let (H,C) is a $SFun-$object and be a morphism of (G,B) into (H,C) satisfying $e' \circ f_s = e' \circ g_s$. Define the map $\bar{e} : (H,C) \to (H',C')$ such that $\bar{e} = e'$. We obtain that $e' = \bar{e} \circ e$ from the above diagram.

Since $e' \circ f_s = e' \circ g_s$, we have $e'(f_s(c)) = e'(g_s(c))$ for every $c \in C$. Hence $g_s(c) \in C$ and $\bar{e} \circ e'$ is well defined. Since $G = H \circ e$, $\bar{e} = e'$ and e' is a $SFun-$ morphism we have the following inclusion:

$$G(b) \subseteq (H \circ \bar{e})(b) = (H \circ e')(b) = H(\bar{e}(e(b)))$$

$$= (H \circ \bar{e})(e(b)) = (H \circ e)(b) = G(b)$$

. Hence $\bar{e}$ is a $SFun-$ morphism. Since $e' = \bar{e} \circ e$ and $\bar{e}$ is unique, we conclude that $((G,B),e)$ is a coequalizer of f_s and g_s.

■

Theorem 8. *If $((G,B),e)$ is a coequalizer of a morphism pair in $SFun$ category, then $((G,B),e)$ is epic.*

Proof.

$$\begin{tikzcd}
(F,A) \arrow[r, shift left, "f_s"] \arrow[r, shift right, "g_s"'] & (G,B) \arrow[r, "e"] \arrow[dr, "e'"'] & (H,C) \arrow[d, shift right, "\bar{e}"'] \arrow[d, shift left, "\bar{\bar{e}}"] \\
 & & (H',C')
\end{tikzcd}$$

Suppose that $((G,B),e)$ is coequalizer of f_s and g_s. Let $\bar{e}$ and $\bar{\bar{e}}$ be two $SFun-$ morphisms as illustrated above diagram. We have $G(b) \subseteq (H' \circ e')(b)$ since $e' : (G,B) \to (H',C')$ is a $SFun-$ morphism, $H(c) \subseteq (H' \circ \bar{e})(c)$ since $\bar{e} : (H,C) \to (H',C')$ is a $SFun-$morphism and $H(c) \subseteq (H' \circ \bar{\bar{e}})(c)$ since $\bar{\bar{e}} : (H,C) \to (H',C')$ is a $SFun-$ morphism. From the above diagram

$$G(b) \subseteq (H' \circ e')(b) = H'(e')(b) = H'(\bar{e} \circ e)(b) = H'(\bar{\bar{e}} \circ e)(b).$$

Next suppose that $\bar{e}e = \bar{\bar{e}}e$. We want to show that $\bar{e} = e\bar{\bar{e}}$. Put $e' = \bar{e}e = \bar{\bar{e}}e$. Then

$$e'f_s = \bar{e}ef_s = \bar{\bar{e}}ef_s$$

$$e'f_s = \bar{e}ef_s = \bar{e}eg_s = e'g_s.$$

By using universal mapping property, there is unique $u : (H,C) \to (H',C')$ such that $ue = e'$. Hence we obtain $\bar{e} = u = \bar{\bar{e}}$ from $\bar{e}e = e'$ and $\bar{\bar{e}}e = e'$. Since $\bar{e}e = \bar{\bar{e}}e$ implies that $\bar{e} = \bar{\bar{e}}$, e is right cancellable. Therefore e is epic.

■

4. Conclusion

We have worked on some monomorphism and epimorphism properties of $SFun$ category. We conclude that $SFun$ has coequalizers, equalizers in $SFun$ category are monic and coequalizers in $SFun$ category are epic.

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