EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 10, No. 4, 2017, 730-738
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Modules that Have a δ-supplement in Every Extension

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Abstract. Let R be a ring and M be a left R-module. In this paper, we define modules with the properties (δ-E) and (δ-EE), which are generalized version of Zöschinger's modules with the properties (E) and (EE), and provide various properties of these modules. We prove that the class of modules with the property (δ-E) is closed under direct summands and finite direct sums. It is shown that a module M has the property (δ-EE) if and only if every submodule of M has the property (δ-E). It is a known fact that a ring R is perfect if and only if every left R-module has the property (E). As a generalization of this, we prove that if R is a δ-perfect ring then every left R-module has the property (δ-E). Moreover, the converse is also true on δ-semiperfect rings.

2010 Mathematics Subject Classifications: 16D10, 16D90

Key Words and Phrases: Supplement, δ-supplement, δ-perfect ring, δ-semiperfect ring, module extension

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1. Introduction

In this paper R is an associative ring with identity and all modules are unital left R-modules. Let M be a module $X \leq M$ means that X is a submodule of M or M is an extension of X. Recall that a submodule $N \leq M$ is called *small*, denoted by $N \ll M$, if $N + L \neq M$, for all proper submodules L of M. We call T a *supplement* of N in M if $M = T + N$ and $T \cap N$ is small in T. A module M is called *supplemented* if every submodule of M has a supplement in M [14]. $L \leq M$ is said to be *essential* in M, denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for each nonzero submodule $K \leq M$. The *singular submodule* of a module M (denoted by $Z(M)$) is $Z(M) = \{x \in M \mid Ix = 0 \text{ for some ideal } I \trianglelefteq R\}$. A module M is called *singular* if $Z(M) = M$. Every submodule and every factor module of a singular module is singular. We refer to [6] for the further properties of singular modules.

In [15], Zhou introduced the concept of δ-small submodules as a generalization of small submodules. A submodule N of M is said to be *δ-small* in M (denoted by $N \ll_\delta M$) if whenever $M = N + K$ and $\frac{M}{K}$ is singular, we have $M = K$. And we denote the sum of all δ-small submodules of M by $\delta(M)$. A submodule L of M is called a *δ-supplement* of

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N in M if $M = N + L$ and $N \cap L \ll_\delta L$ and M is called *δ-supplemented* in case every submodule of M has a δ-supplement in M [7].

For a module M consider the following conditions:

(E) : M has a supplement in every extension.

(EE) : M has ample supplements in every extension.

The concept of these modules with these properties was first introduced by Zöschinger [16]. Adapting his concept in [4], Çalışıcı and Türkmen introduced modules with the properties (CE) and (CEE) as a generalization of the properties (E) and (EE). In addition, in [9] the authors worked on modules that have a weak supplement in every extension and in [5] Eryılmaz introduced modules that have a δ-supplement in every torsion extension.

In this paper we investigate the structure of modules with the properties (δ-E) and (δ-EE) as a generalization of Zöschinger's modules with the properties (E) and (EE). We prove that a module has the property (δ-EE) if and only if every submodule has the property (δ-E). We show that every direct summand and δ-small cover of M with the property (δ-E) has the property (δ-E). Using the property (δ-E), we present a relation between δ-perfect rings and modules with the property (δ-E), which are a generalization of perfect rings, that is, R is a δ-perfect ring, then every left R-module has the property (δ-E). Moreover we obtain that if every left R-module has the property (δ-E), then R is a δ-semiperfect ring.

2. Preliminaries

In this section, we begin by stating the following lemmas and theorems for the completeness.

2.1. δ-Small Submodues

Lemma 1. *([15, Lemma 1.2]). Let N be a submodule of M. The following are equivalent:*

1. *$N \ll_\delta M$.*
2. *If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$.*
3. *If $X + N = M$ with $\frac{M}{X}$ Goldie torsion, then $X = M$.*

Lemma 2. *([15, Lemma 1.3]). Let M be a module.*

1. *For submodules N, K, L of M with $K \subseteq N$, we have*
   (a) $N \ll_\delta M$ if and only if $K \ll_\delta M$ and $\frac{N}{K} \ll_\delta \frac{M}{K}$.
   (b) $N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.
2. *If $K \ll_\delta M$ and $f : M \longrightarrow N$ is a homomorphism, then $f(K) \ll_\delta N$.*
   In particular, if $K <<_\delta M \subseteq N$, then $K \ll_\delta N$.

3. *Let* $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ *and* $M = M_1 \oplus M_2$. *Then* $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ *if and only if* $K_1 \ll_\delta M_1$ *and* $K_2 \ll_\delta M_2$.

2.2. δ-Supplemented Modules

Lemma 3. ([7, *Prop.*2.7]). *Let* U *and* V *be submodules of a module* M. *Assume that* V *is a* δ*-supplement of* U *in* M. *Then*

1. If $W + V = M$ for some $W \subseteq U$, then V is a δ-supplement of W in M.
2. If $K \ll_\delta M$, then V is a δ-supplement of $U + K$ in M.
3. For $K \ll_\delta M$ we have $K \cap V \ll_\delta V$ and so $\delta(V) = V \cap \delta(M)$.
4. For $L \subseteq U$, $\frac{V+L}{L}$ is a δ-supplement of $\frac{U}{L}$ in $\frac{M}{L}$.
5. If $\delta(M) \ll_\delta M$, or $\delta(M) \subseteq U$ and if $p : M \longrightarrow \frac{M}{\delta(M)}$ is the canonical projection, then $\frac{M}{\delta(M)} = p(U) \oplus p(V)$.

In [7], a projective module P is called a *projective* δ*-cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $Ker(f) \ll_\delta M$, and a ring R is called δ*-perfect* (resp., δ-semiperfect) if every R-module (resp., every simple R-module) has a projective δ-cover. In addition, a module M is called δ-lifting if for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is δ-small in B since B is a direct summand of M.

Theorem 4. [7, *Theorem*3.3]. *The following are equivalent for a ring* R :

1. *R is* δ*-semiperfect.*
2. *Every finitely generated module is* δ*-supplemented.*
3. *Every finitely generated projective module is* δ*-supplemented.*
4. *Every finitely generated projective module is* δ*-lifting.*
5. *Every left ideal of* R *has a* δ*-supplement in* ${}_RR$.

Theorem 5. *[7, Theorem 3.4]. The following statements are equivqlent for a ring* R :

1. *R is* δ*-perfect.*
2. *Every module is* δ*-supplemented.*
3. *Every projective module is* δ*-supplemented.*
4. *Every projective module is* δ*-lifting.*

3. Modules with the Properties (δ-E) and (δ-EE)

In this section, we define the concept of modules with the properties (δ-E) and (δ-EE).

Definition 1. *A module M has the property (δ-E) if it has a δ-supplement in each module in which it is contained as a submodule.*

Definition 2. *A module M has the property (δ-EE) if it has ample δ-supplements in each module in which it is contained as a submodule, where $U \leq M$ has ample δ-supplements in M if for every $V \leq M$ with $U + V = M$, there is a δ-supplement V' of U with $V' \leq V$.*

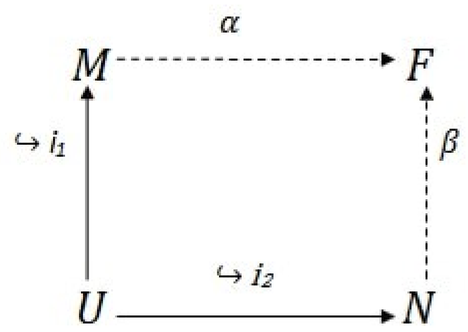
It is clear that every module with the property (E) has the property (δ-E). Also there exists the same relation between modules with the properties (EE) and (δ-EE). At the end of this section, we shall give an example of a module which has the property (δ-E) but not (E).

Zöschinger proved in [16] that a module has the property (EE) if and only if every submodule has the property (E).We give an analogous characterization of our modules with the following proposition.

Proposition 1. *A module M has the property (δ-EE) if and only if every submodule of M has the property (δ-E).*

Proof. Let M be a module and N be any extension of M. Suppose that for a submodule $X \leq N$, $X + M = N$. By hypothesis, the submodule $X \cap M$ of M has a δ-supplement V in X, that is, $(X \cap M) + V = X$ and $(X \cap M) \cap V \ll_\delta V$. Then, $N = M + X = M + [(X \cap M) + V] = M + V$ and $M \cap V = M \cap (V \cap X) = (X \cap M) \cap V \ll_\delta V$. Hence, V is a δ-supplement of M in N such that $V \leq X$.

Conversely, let U be a submodule of M and N be any module containing U. Then we can draw the following pushout:



i_1 and i_2 are inclusion homomorphisms in this diagram. Additionally $\alpha : M \longrightarrow F$ and $\beta : N \longrightarrow F$ are monomorphisms by the properties of push out (see, for example, [11, Exercise 5.10]). Let $\alpha(M) = M' \subseteq F$ and $\beta(N) = N' \subseteq F$. Then it can be easily shown that $F = M' + N'$. So by using hypothesis, $M' \cong M$ has a δ-supplement V in F such that $V \leq N'$, that is, $M' + V = F$ and $M' \cap V \ll_\delta V$. Hence,

$$(M' \cap N') + V = (N' \cap M') + V = N' \cap (M' + V) = N' \cap F = N', \text{ and}$$

$$(M' \cap N') \cap V = M' \cap (N' \cap V) = M' \cap V \ll_\delta V.$$

So V is a δ-supplement of $M' \cap N'$ in N'. Now we will show that $\beta^{-1}(V)$ is a δ-supplement of U in N. We have an isomorphism $\widetilde{\beta} : N \longrightarrow N'$ defined as $\beta(x) = \widetilde{\beta}(x)$ for all $x \in N$, since β is a monomorphism. Using this, we obtain $\widetilde{\beta^{-1}}(V)$ is a δ-supplement of $\beta^{-1}(M' \cap N')$ in $\widetilde{\beta^{-1}}(N')$ since V is a δ-supplement of $M' \cap N'$ in N'. It can be seen that $\widetilde{\beta^{-1}}(V) = \beta^{-1}(V)$, $\widetilde{\beta^{-1}}(N') = N$ and $\widetilde{\beta^{-1}}(M' \cap N') = U$. Thus $\beta^{-1}(V)$ is a δ-supplement of U in N.

Corollary 1. *A module with the property (δ-EE) has the property (δ-E) and it is also δ-supplemented.*

Recall that R is a (right) δ-V ring if for any right R-module M, $\delta(M) = 0$ (see, [13]).

Proposition 2. *Let R be δ-V ring and M be an R-module. Then the following statements are equivalent:*

1. M has the property (δ-E).
2. M is injective.

Proof. $(1) \Longrightarrow (2)$: Suppose that M has the property (δ-E). Let N be any extension of M. So, there exists a δ-supplement V of M in N, that is, $M + V = N$ and $M \cap V \ll_\delta V$ and so $M \cap V \leq \delta(V)$. Since R is a δ-V ring, $\delta(V) = 0$. So, $N = M \oplus V$. Therefore, M is injective.

$(2) \Longrightarrow (1)$: is clear.

Now we show that the property (δ-E) is preserved by direct summands in the following proposition:

Proposition 3. *Every direct summand of any module with the property (δ-E) has the property (δ-E).*

Proof. Let M be a module with the property (δ-E), U be a direct summand of M and N be any extension of U. Then there exists a submodule A of M such that $M = U \oplus A$. By hypothesis, M has a δ-supplement V in $A \oplus N$ such that $(A \oplus U) + V = A \oplus N$ and $(A \oplus U) \cap V << _\delta V$. Let $g : A \oplus N \longrightarrow N$ be the projection onto N. Then

$$N = g(A \oplus N) = g((A \oplus U) + V) = g(A \oplus U) + g(V) = U + g(V), \text{ and}$$

$$g((A \oplus U) \cap V) = U \cap g(V) \ll_\delta g(V).$$

Hence, $g(V)$ is a δ-supplement of U in N.

Proposition 4. *Let $A \leq B$. If A and $\frac{B}{A}$ have the property (δ-E), so does B.*

Proof. Let N be any extension of B. So, there is a δ-supplement $\frac{V}{A}$ of $\frac{B}{A}$ in $\frac{N}{A}$ and a δ-supplement T of A in V. We have δ-small epimorphisms $f : T \longrightarrow \frac{V}{A}$ and $g : \frac{V}{A} \longrightarrow \frac{N}{B}$ that

$Ker\ f = T \cap A \ll_\delta T$ and $Ker\ g = \frac{V}{A} \cap \frac{B}{A} \ll_\delta \frac{V}{A}$. Then, $g \circ f : T \longrightarrow \frac{N}{B}$ is a δ-small epimorphism such that $T \cap B = Ker\ (g \circ f) \ll_\delta T$. Moreover, we have

$$B + T = (B + A) + T = B + (A + T) = B + V = N$$

since $\frac{V}{A}$ is a δ-supplement of $\frac{B}{A}$ in $\frac{N}{A}$. This completes the proof.

Corollary 2. *If M_1 and M_2 have the property (δ-E), so does $M_1 \oplus M_2$.*

Proof. Let $0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence. Result follows by Proposition 4.

Proposition 5. *Let $0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0$ be a short exact sequence. If K and L have the property (δ-E), so does M. If the sequence splits the converse is also true.*

Proof. Let N be any extension of M. So $\frac{N}{K}$ is an extension of $\frac{M}{K}$ and is is a well known fact that $\frac{M}{K} \cong L$. Then there exists a δ-supplement $\frac{V}{K}$ for $\frac{M}{K}$ in $\frac{N}{K}$, that means $\frac{M}{K} + \frac{V}{K} = \frac{N}{K}$ and $\frac{M}{K} \cap \frac{V}{K} \ll_\delta \frac{V}{K}$ for some $\frac{V}{K} \leq \frac{N}{K}$. Since $K \leq V$ and K has the property (δ-E), $K + K' = V$, $K \cap K' \ll_\delta K'$ for some $K' \leq V$. Hence, $N = M + V = M + K + K' = M + K'$. Now we claim that $M \cap K' \ll_\delta K'$. For this let $M \cap K' + T = K'$ with $\frac{K'}{T}$ is singular. $K + M \cap K' + T = K + K'$ and by the modular law $(K + K') \cap M + T = V$. Following this, $V \cap M + T = V$ is obtained. It can be easily seen written that $\frac{V \cap M}{K} + \frac{T+K}{K} = \frac{V}{K}$, additionally, $\frac{V}{T+K}$ is singular since,

$$\frac{V}{T+K} = \frac{K+K'}{T+K} = \frac{K+(K'+T)}{T+K} = \frac{(T+K)+K'}{T+K} \cong \frac{K'}{(T+K) \cap K'} = \frac{K'}{T+(K \cap K')} \leq \frac{K'}{T}$$

and $\frac{M}{K} \cap \frac{V}{K} \ll_\delta \frac{V}{K}$. So $\frac{T+K}{K} = \frac{V}{K}$ and of course $T + K = V$. $(T + K) \cap K' = K'$ can be seen and by the modular lae, $T + (K \cap K') = K'$ is obtained. This provides $T = K'$ since $K \cap K' << _\delta K'$ and $\frac{K'}{T}$ is singular. Moreover, suppose that the sequence splits, Then K and L have the property (δ-E) by *corollary* 2.

Corollary 3. *Let M_i $(i = 1, 2, ..., n)$ be any finite collection of modules and $M = M_1 \oplus M_2 \oplus \ ... \oplus M_n$. Then M has the property (δ-E) if and only if M_i has the property (δ-E) for each $i = 1, 2, ..., n$.*

Proof. It can be proved easily for $n = 2$ by using the previous theorem and can be generalized on n.

We give the following known lemma for the completeness.

Lemma 6. *Every simple submodule S of a module M is either a direct summand of M or small in M (see in [10])*

Proposition 6. *Every simple module has the property (δ-E).*

Proof. Let S be a simple module and N be any extension of S. Then by Lemma 4, $S \ll N$ and so $S \ll_\delta N$ or $S \oplus S' = N$ for a submodule $S' \leq N$. If $S \ll_\delta N$, then N is a δ-supplement of S in N or if S is a direct summand of N then S' is a δ-supplement of S in N . So in each case S has a δ-supplement in N . This means that S has the property (δ-E).

Theorem 7. *Every module with composition series has the property (δ-E).*

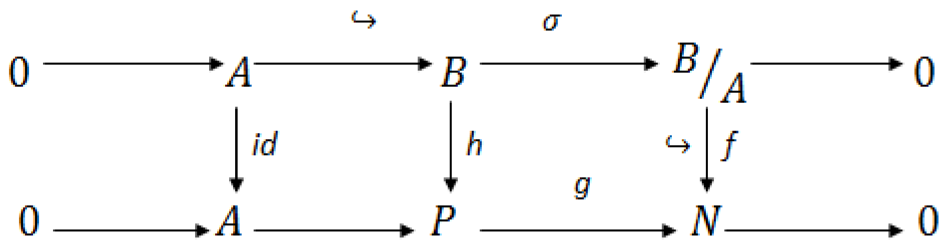
Proof. Let $0 = M_0 \leq M_1 \leq M_2 \leq ... \leq M_{n-1} \leq M_n = M$ be any composition series of a module M. We shall prove the theorem by induction on $n \epsilon \mathbb{N}$. If $n = 1$, then $M = M_1$ is simple, and so M has the property (δ-E) by Proposition 6. Assume that this is true for each $k \leq n-1$.Then M_{n-1} has the property (δ-E). Since $\frac{M_n}{M_{n-1}}$ has the property (δ-E) as a simple module, M has the property (δ-E) by Proposition 4.

Corollary 4. *A finitely generated semisimple module has the property (δ-E).*

In the following proposition we will prove that modules with the property (δ-E) are closed onder factor modules, under a special condition.

Proposition 7. *Let $A \leq B \leq C$ with $\frac{C}{A}$ injective. If B has the property (δ-E), so does $\frac{B}{A}$.*

Proof. Let N be any extension of $\frac{B}{A}$. So we have the following commutative diagram with exact rows since $\frac{C}{A}$ is injective, (see in [10]).



Since h is monic and B has the property (δ-E), $B \cong h(B)$ has a δ-supplement V in P, that is, $h(B) + V = P$ and $h(B) \cap V \ll_\delta V$. We claim that $g(V)$ is a δ-supplement of $\frac{B}{A}$ in N.

$$\frac{B}{A} + g(V) = (f\sigma)(B) + g(V) = g(h(B)) + g(V) = g(P) = N, \text{ and}$$

$$\frac{B}{A} \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap V] \ll_\delta g(V)$$

since $h(B) \cap V \ll_{\delta} V$ and g is a homomorphism.

A ring R is left perfect if and only if every left R-module has the property (E) (see [16]). Now we show only one side of this fact is valid for δ-perfect rings.

Proposition 8. *If R is a δ-perfect ring, then every left R-module has the property (δ-E).*

Proof. Suppose that a ring R is δ-perfect. Let M be an R-module and N be any extension of M. N is δ-supplemented since R is δ-perfect. So M has a δ-supplemented in N as a submodule of N. Hence, M has the property (δ-E).

Proposition 9. *Let R be a ring. If every left R-module has the property (δ-E), then R is a δ-semiperfect ring.*

Proof. Since every left R-module has the property (δ-E), every ideal of R also has the property (δ-E) as a submodule of ${}_RR$. So every ideal of R has a δ-supplement in ${}_RR$. Hence R is δ-semiperfect by [6, Theorem 3.3].

Example 1. *Let F be a field,*

$$I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix},\ R = \{(x_1,\ ...,\ x_n,\ x,\ x, ...) \mid n \in \mathbb{N},\ x_i \in M_2(F),\ x \in I\}$$

with component-wise operations, R is a ring. By Example 4.3 in [15], R is a δ-perfect ring that is not perfect. And so ${}_RR$ is an example of a module that has the property (δ-E) but not have the property (E).

Acknowledgements

The authors sincerely thank the reviewers for the valuable suggestions and comments.

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