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Fractional orders of the generalized Bessel matrix polynomials

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Abstract. This paper presents and investigates generalized Bessel matrix polynomials (GBMPs) with order $\alpha \in \Re$ (the set of real numbers). The given result is supposed to be an enhanced and a generalized form of the scalar form to the fractional analysis setting. By using the Liouville-Caputo operator of fractional analysis and Rodrigues type representation form of fractional order, the generalized Bessel matrix functions (GBMFs) $\mathcal{Y}_\alpha(t;\mathbb{A},\mathbb{B})$, $t \in \mathbb{C}$, for matrices $\mathbb{A}$ and $\mathbb{B}$ in the complex space $\mathbb{C}^{N\times N}$ are derived and supplied with a matrix hypergeometric representation that are satisfied by these functions. Subsequently, a fractional matrix recurrence relationship, a fractional matrix of second-order differential equation and an orthogonal system are then developed for GBMFs.

2010 Mathematics Subject Classifications: 33C05, 33C45, 34A05.
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1. Introduction

The generalized Bessel polynomials (GBPs) formula, a class of orthogonal polynomials which is intimately related with the Bessel functions. They emerged in the solution of differential equation of spherical waves. These polynomials have been studied first by Bochner [4] who pointed out their connection with Bessel functions. A comprehensive study on these polynomials was given by Krall and Frink [17]. Several other authors (see, e.g., [2, 5, 12]) have contributed to the study of the Bessel polynomials. Special matrix functions latterly show in several fields (see, for example [15, 24, 25]). A new extension of hypergeomatric, Humbert and Appel matrix functions were introduced and studied in [19, 20, 21]. In [1, 22] the scalar case of the generalized Bessel and reverse Bessel polynomials have already been expanded into matrix setting.

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Several articles and books have been written recently in fractional calculus area, of which we recommend (for instance, [6, 7, 10, 11, 29, 14, 18, 26, 27, 32]).

In recent years, many researchers have studied various special functions associated with fractional calculus. Laguerre polynomials, Bell polynomials, Legendre polynomials and generalized ultraspherical or Gegenbauer functions of arbitrary (fractional) orders have been defined in [8, 9, 30, 31]. In addition, fractional derivatives of various multivariable functions have been derived (for examples, [3, 23]).

The major purpose of this work is to obtain generalizations of the (GBMPs) by making use of fractional calculus and Rodrigues type exemplification form of fractional order. Therefore, the (GBMPs) with fractional order are obtained and some of their properties such as a fractional matrix recurrence relations, the fractional matrix differential equation and an orthogonality property are given. Starting, we mention some the fundamental definitions of the fractional calculus and some properties of the matrix functions used in the present work.

Definition 1. *The fractional integral of order $\beta \in \Re^+$, being the set of positive real numbers, of the function $f(\tau),\ \tau \geq b$ is defined by (see [13, 28] and [26])*

$$I_b^\beta f(\tau) \ = \ \int_b^\tau \frac{(\tau-u)^{\beta-1}}{\Gamma(\beta)} f(u)\ du. \tag{1}$$

The Liouville-Caputo fractional derivative of order $\alpha \in (n-1,n)$ $(n \in \mathbb{N} := \{1,2,...\})$ of $f(\tau),\ \tau \geq a$ is defined by

$$D_b^\alpha\ f(\tau) \ = \ I_b^{n-\alpha}\ D^n\ f(\tau), \quad D = \frac{d}{d\tau}. \tag{2}$$

The fractional derivative of the product $g(v)f(v)$ by [26], the Leibniz rule for fractional differentiation takes the form

$$D^\alpha[g(v)f(v)] = \sum_{s=0}^{\infty} \binom{\alpha}{s} g^{(s)}(v) D^{\alpha-s} f(v). \tag{3}$$

Definition 2. *(cf. [1, 16]) For all $\mathbb{A}$ in the complex space of matrices $\mathbb{C}^{N\times N}$, and*

$$\mathbb{A} + nI \quad is\ \ invertible\ \ for\ \ all\, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{4}$$

then the Pochhammer symbol (the shifted factorial) is defined by

$$(\mathbb{A})_n = \mathbb{A}(\mathbb{A}+I)...(\mathbb{A}+(n-1)I) = \Gamma(\mathbb{A}+nI)\Gamma^{-1}(\mathbb{A}); \quad (\mathbb{A})_0 \equiv I. \tag{5}$$

where I is unite matrix in $\mathbb{C}^{N\times N}$.

Definition 3. *[1, 16] Suppose that $\mathbb{A}, \mathbb{B}$ and $\mathbb{D}$ are matrices in $\mathbb{C}^{N\times N}$, and $\mathbb{D}$ satisfy condition (4), then, the matrix power series of the hypergeometric matrix function is defined in the form*

$$F(\mathbb{A}, \mathbb{B}; \mathbb{D}; z) = \sum_{m=0}^{\infty} \frac{(\mathbb{A})_m (\mathbb{B})_m [(\mathbb{D})_m]^{-1}}{m!} z^m. \tag{6}$$

2. Generalized Bessel matrix functions of fractional order

The classical (GBMPs) $Y_n(z, \mathbb{A}, \mathbb{B})$ are defined by Rodrigues' type formula (see [1, 22])

$$Y_n(z, \mathbb{A}, \mathbb{B}) = \mathbb{B}^{-n} z^{2I-\mathbb{A}} e^{\frac{\mathbb{B}}{z}} D^n\Big(z^{2(n-1)I+\mathbb{A}} e^{\frac{-\mathbb{B}}{z}}\Big), \tag{7}$$

where $n \geq 0$, $\mathbb{A}$ and $\mathbb{B}$ are parameter matrices. When $\mathbb{A} = \mathbb{B} = 2I$, the analogue Rodrigues' type formula for the (GBMPs) (7) reduces to the analogue Rodrigues' type formula Bessel polynomials proper:

$$y_n(z) = 2^{-n} e^{\frac{2}{z}} D^n\Big(z^{2n} e^{\frac{-2}{z}}\Big). \tag{8}$$

By taking the the Liouville-Caputo fractional derivative D^α in (7), we introduce functions which are naturally refereed to as **generalized Bessel matrix functions (GBMFs)**.

Definition 4. *Suppose that $\alpha \in (n-1, n)$ $(n \in \mathbb{N})$ and $\mathbb{A}$ and $\mathbb{B}$ are commuting matrices in $\mathbb{C}^{N\times N}$ satisfying the spectral condition (4). Then the GBMFs are defined by the formula*

$$\mathcal{Y}_\alpha(t; \mathbb{A}, \mathbb{B}) = \mathbb{B}^{-\alpha} t^{2I-\mathbb{A}} e^{\frac{\mathbb{B}}{t}} \mathcal{L}_\alpha(t); \quad \mathcal{L}_\alpha(t) = D^\alpha(t^{\mathbb{A}+(2\alpha-2)I} e^{-\frac{\mathbb{B}}{t}}). \tag{9}$$

Using (9), the GBMFs would be represented by the hypergeometric matrix function ${}_1F_1(\mathbb{A}, \mathbb{B}; t)$ in the following result:

Theorem 1. *The GBMFs can be written as*

$$\begin{aligned}\mathcal{Y}_\alpha(t; \mathbb{A}, \mathbb{B}) =& (t\mathbb{B}^{-1})^\alpha \Gamma^{-1}(\mathbb{A} + (\alpha-1)I)\Gamma(\mathbb{A} + (2\alpha-1)I)\\ &\times {}_1F_1(-\alpha I; -\mathbb{A} + 2(1-\alpha)I; \frac{\mathbb{B}}{t}).\end{aligned} \tag{10}$$

Proof. From (9) and the relation (3), we find that

$$\begin{aligned}\mathcal{Y}_\alpha(t; \mathbb{A}, \mathbb{B}) =& \mathbb{B}^{-\alpha} t^{2I-\mathbb{A}} e^{\frac{B}{t}} D^\alpha \sum_{s=0}^{\infty} \frac{(-\mathbb{B})^s}{s!} t^{\mathbb{A}+(2\alpha-2-s)I}\\ =& \mathbb{B}^{-\alpha} t^{2I-\mathbb{A}} e^{\frac{\mathbb{B}}{t}} \sum_{s=0}^{\infty} \frac{(-\mathbb{B})^s}{s!} t^{\mathbb{A}+(\alpha-2-s)I}\\ &\times \Gamma^{-1}(\mathbb{A} + (\alpha-1-s)I)\Gamma(\mathbb{A} + (2\alpha-1-s)I)\\ =& (t\mathbb{B}^{-1})^\alpha e^{\frac{\mathbb{B}}{t}} \sum_{s=0}^{\infty} \frac{(-\mathbb{B})^s}{s!} t^{-s}\\ &\times \Gamma(\mathbb{A} + (2\alpha-1-s)I)\, \Gamma^{-1}(\mathbb{A} + (\alpha-1-s)I)\\ =& (t\mathbb{B}^{-1})^\alpha\, \Gamma(\mathbb{A} + (2\alpha-1)I)\, \Gamma^{-1}(\mathbb{A} + (\alpha-1)I)\\ &\times {}_1F_1(-\alpha I; -\mathbb{A} + 2(1-\alpha)I; \frac{\mathbb{B}}{t}),\end{aligned}$$

which yields the desired result.

3. Recurrence relations and the differential equation

In this section, we shall show some recurrence relations for the matrix functions $\mathcal{Y}_\alpha(t;\mathbb{A},\mathbb{B})$ which generalize (interpolate) those of the GBMPs $Y_n(z,\mathbb{A},\mathbb{B})$ (see[1, 22]). In addition, we generalize the GBMFs (9) by solving the following linear homogeneous fractional matrix differential equation:

$$\begin{aligned}&t^2\,\mathcal{Y}''_\alpha(t;\mathbb{A},\mathbb{B})+(t\mathbb{A}+\mathbb{B})\,\mathcal{Y}'_\alpha(t;\mathbb{A},\mathbb{B})\\&=\alpha(\mathbb{A}+(\alpha-1)I)\,\mathcal{Y}_\alpha(t;\mathbb{A},\mathbb{B}).\end{aligned}$$

The following lemma enables us to establish Theorem 2.

Lemma 1. *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are commuting matrices in $\mathbb{C}^{N\times N}$ satisfying the condition (4). For any $\alpha\in(n-1,n)$ $(n\in\mathbb{N})$,*

(i) $\mathcal{L}_{\alpha+1}(t)\ (\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)=\mathcal{L}_\alpha(t)\ \big[(\mathbb{A}+2\alpha I)(\mathbb{A}+2(\alpha-1)I)t+\mathbb{B}(\mathbb{A}-2I)\big](\mathbb{A}+(2\alpha-1)I)+\mathcal{L}_{\alpha-1}(t)\ \alpha\ \mathbb{B}^2(\mathbb{A}+2\alpha I)$.

(ii) $\mathcal{L}_{\alpha+1}(t)\ (\mathbb{A}+(\alpha-1)I)=\mathcal{L}'_\alpha(t)\ (\mathbb{A}+2\alpha I)t^2+\mathcal{L}_\alpha(t)\big[(\mathbb{A}+2\alpha I)(\alpha+1)t-\mathbb{B}(\alpha+1)\big]$.

(iii) $\mathcal{L}_{\alpha+1}(t)\ (\mathbb{A}+2(\alpha-1)I)t^2=\big[(\mathbb{A}+2(\alpha-1)I)(\mathbb{A}+(\alpha-2)I)t+\mathbb{B}(\mathbb{A}+(\alpha-2)I)\big]\mathcal{L}_\alpha(t)+\mathcal{L}_{\alpha-1}(t)\mathbb{B}^2\alpha$.

Proof. (i) Using the Leibniz rule for fractional derivative [26], the fractional derivative in (9) yields

$$\begin{aligned}&(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\ \mathcal{L}_{\alpha+1}(t)\\
=&(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)(\mathbb{A}+2\alpha I)\ D^\alpha t^{\mathbb{A}+(2\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}\\
&+\mathbb{B}(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\ \mathcal{L}_\alpha(t)\\
=&(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)(\mathbb{A}+2\alpha I)t\ \mathcal{L}_\alpha(t)\\
&+\alpha(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)(\mathbb{A}+2\alpha I)D^{\alpha-1}t^{\mathbb{A}+2(\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}\\
&+\mathbb{B}(\mathbb{A}+(2\alpha-1)I)(\mathbb{A}-2I)\ \mathcal{L}_\alpha(t)\\
&+\alpha\ \mathbb{B}(\mathbb{A}+2\alpha I)\big[(\mathbb{A}+2(\alpha-1)I)\ D^{\alpha-1}t^{\mathbb{A}+(2\alpha-3)I}\ e^{\frac{-\mathbb{B}}{t}}+\mathbb{B}\ \mathcal{L}_{\alpha-1}(t)\big]\\
=&\big[(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)(\mathbb{A}+2\alpha I)t+\mathbb{B}(\mathbb{A}+(2\alpha-1)I)(\mathbb{A}-2I)\\
&+\alpha(\mathbb{A}+2(\alpha-1)I)(\mathbb{A}+2\alpha I)t\big]\ \mathcal{L}_\alpha(t)+\alpha\ \mathbb{B}^2\ (A+2\alpha I)\ \mathcal{L}_{\alpha-1}(t).\end{aligned}$$

Hence,

$$\begin{aligned}&(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\mathcal{L}_{\alpha+1}(t)\\
=&\big[(\mathbb{A}+2\alpha I)(\mathbb{A}+2(\alpha-1)I)t+\mathbb{B}(\mathbb{A}-2I)\big]\\
&(\mathbb{A}+(2\alpha-1)I)\ \mathcal{L}_\alpha(t)+\alpha\mathbb{B}^2(\mathbb{A}+2\alpha I)\ \mathcal{L}_{\alpha-1}(t).\end{aligned}\tag{11}$$

(ii) We have

$$\begin{aligned}\mathcal{L}_{\alpha+1}(t) &= t^2\ D^{\alpha+1} t^{\mathbb{A}+2(\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}\\ &+ 2(\alpha+1)t\ D^{\alpha} t^{\mathbb{A}+2(\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}\\ &+\alpha(\alpha+1)\ D^{\alpha-1} t^{\mathbb{A}+2(\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}\\ &= t^2\ \mathcal{L}'_{\alpha}(t) + 2(\alpha+1)t\ \mathcal{L}_{\alpha}(t)\\ &+\alpha(\alpha+1)\ D^{\alpha-1} t^{\mathbb{A}+2(\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{L}_{\alpha+1}(t) =& (\mathbb{A}+2\alpha I)\ D^{\alpha} t^{\mathbb{A}+(2\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}} + \mathbb{B}\ \mathcal{L}_{\alpha}(t)\\ =& \big[(\mathbb{A}+2\alpha I)t + \mathbb{B}\big]\ \mathcal{L}_{\alpha}(t)\\ &+\alpha(\mathbb{A}+2\alpha I)\ D^{\alpha-1} t^{\mathbb{A}+2(\alpha-1)I}\ e^{\frac{-\mathbb{B}}{t}}.\end{aligned} \tag{13}$$

If we multiply (12) by $(\mathbb{A}+2\alpha I)$ *and (13) by* $(\alpha+1)$*, then subtract we obtain the required result.*

(iii) Multiply both sides of the equation (ii) above by $(\mathbb{A}+2(\alpha-1))$ *and substitute for* $(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\mathcal{L}_{\alpha+1}(t)$ *from (i) in (ii) and on rearrangement, we obtain (iii).*

To prove the following result:

Theorem 2. *Suppose that* $\mathbb{A}$ *and* $\mathbb{B}$ *are commuting matrices in* $\mathbb{C}^{N\times N}$ *satisfying the condition (4). Then the GBMFs satisfy the following recurrence relations:*

$$\begin{aligned}&(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}_{\alpha+1}(t;\mathbb{A},\mathbb{B})\\ &= \big[(\mathbb{A}+2\alpha I)(\mathbb{A}+2(\alpha-1)I)t\mathbb{B}^{-1} + (\mathbb{A}-2I)\big]\\ &(\mathbb{A}+(2\alpha-1)I)\ \mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}) + \alpha(\mathbb{A}+2\alpha I)\ \mathcal{Y}_{\alpha-1}(t;\mathbb{A},\mathbb{B}).\end{aligned} \tag{14}$$

$$\begin{aligned}&(\mathbb{A}+2\alpha I)t^2\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B}) = \mathbb{B}(\mathbb{A}+(\alpha-1)I)\ \mathcal{Y}_{\alpha+1}(t;\mathbb{A},\mathbb{B})\\ &-(\mathbb{A}+(\alpha-1)I)[(\mathbb{A}+2\alpha I)t+\mathbb{B}]\ \mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}).\end{aligned} \tag{15}$$

$$\begin{aligned}&(\mathbb{A}+2(\alpha-1)I)t^2\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B}) = \alpha\ B\ \mathcal{Y}_{\alpha-1}(t;\mathbb{A},\mathbb{B})\\ &+[\alpha\ (\mathbb{A}+2(\alpha-1)I)t - \alpha\ \mathbb{B}]\ \mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}).\end{aligned} \tag{16}$$

$$\begin{aligned}&\mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B})\ [(\mathbb{A}+2(\alpha-1)I)t+\mathbb{B}]\ (\mathbb{A}+(\alpha-2)I)\\ &+\alpha\ \mathbb{B}\ \mathcal{Y}_{\alpha-1}(t;\mathbb{A},\mathbb{B}) = \alpha\ (\mathbb{A}+(\alpha-2)I)(\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}).\end{aligned} \tag{17}$$

Proof. Using Lemma 1, substitute for

$$\mathcal{L}_{\alpha+1}(t) = \mathbb{B}^{\alpha+1}t^{\mathbb{A}-2I}e^{\frac{-\mathbb{B}}{t}}\mathcal{Y}_{\alpha+1}(t;\mathbb{A},\mathbb{B}),$$
$$\mathcal{L}_{\alpha}(t) = \mathbb{B}^{\alpha}t^{\mathbb{A}-2I}e^{\frac{-\mathbb{B}}{t}}\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})$$

and

$$\mathcal{L}_{\alpha-1}(t) = \mathbb{B}^{\alpha-1}t^{\mathbb{A}-2I}e^{\frac{-\mathbb{B}}{t}}\mathcal{Y}_{\alpha-1}(t;\mathbb{A},\mathbb{B}),$$

in $(i), (ii)$ and (iii) respectively, we get (14), (15) and (16). If we multiply (16) by

$$\frac{1}{t^2}\left[(\mathbb{A}+(\alpha-1)I)[(\mathbb{A}+2(\alpha-1)I)t+\mathbb{B}](\mathbb{A}+2(\alpha-1)I)^{-1}\right]$$

and multiply (15) by

$$\frac{1}{t^2}\left[\alpha\mathbb{B}(\mathbb{A}+2(\alpha-1)I)^{-1}\right]$$

after replace α by $\alpha-1$ and add, we obtain Eq.(17).

Other recurrence relations for the GBMFs $\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})$ may be derived from the relations in Theorem 2.

Now, the major property developed here is the differential equation for the GBMFs $\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})$ which is derived from their recurrence relation established by Theorem 2. By differentiating equation (16) we find

$$\begin{aligned}
&t^2\ (\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}''_{\alpha}(t;\mathbb{A},\mathbb{B})\\
&+2t\ (\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B})\\
&=\alpha[t(\mathbb{A}+2(\alpha-1)I)-\mathbb{B}]\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B})\\
&+\alpha(\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})+\alpha\ B\ \mathcal{Y}'_{\alpha-1}(t;\mathbb{A},\mathbb{B}).
\end{aligned} \tag{18}$$

From (17) and (18), a straightforward computation shows that

$$\begin{aligned}
&t^2\ (\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}''_{\alpha}(t;\mathbb{A},\mathbb{B})+2t\ (\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B})\\
&=\alpha(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}),\\
&t^2\ (\mathbb{A}+2(\alpha-1)I)\ \mathcal{Y}''_{\alpha}(t;\mathbb{A},\mathbb{B})+[t(\mathbb{A}+2(\alpha-1)I)\mathbb{A}\\
&+\mathbb{B}(\mathbb{A}+2(\alpha-1)I)]\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B})\\
&=\alpha(\mathbb{A}+(\alpha-1)I)(\mathbb{A}+2(\alpha-1)I)\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&t^2\ \mathcal{Y}''_{\alpha}(t;\mathbb{A},\mathbb{B})+(t\mathbb{A}+\mathbb{B})\ \mathcal{Y}'_{\alpha}(t;\mathbb{A},\mathbb{B})\\
&=\alpha(\mathbb{A}+(\alpha-1)I)\ \mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B}).
\end{aligned} \tag{19}$$

Therefore the following theorem is proved.

Theorem 3. *Let $\mathbb{A}$ and $\mathbb{B}$ be commuting matrices in $\mathbb{C}^{N\times N}$, satisfying the spectral condition (4). Then the GBMFs satisfies fractional matrix differential equation in (19).*

4. Orthogonality property

The research subject of an orthogonal system for the GBMFs is discussed in this section with the weight function $\varrho(t)$ which is defined by (see, [22])

$$\varrho(t) = \frac{1}{2\pi i}\sum_{s=0}^{\infty}\Gamma^{-1}(\mathbb{A}+(s-1)I)\;\Gamma(\mathbb{A})\;(\frac{-\mathbb{B}}{t})^{s}, \tag{20}$$

which satisfies the related matrix nonhomogeneous equation

$$\varrho(t)^{'}(t^2 = \varrho(t)(\mathbb{A}t+\mathbb{B}) - \frac{[(\mathbb{A}-2I)(\mathbb{A}-I)]t}{2\pi i}. \tag{21}$$

When the relation (19) is multiplied by $\varrho(t)$, we get

$$\begin{aligned}&\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B}))^{'}\;(t^2\varrho(t) - \mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B})\;(t^2\varrho(t))^{'} + \mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B})\;(At+B)\varrho(t)\\ =&\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})\;\alpha I(\mathbb{A}+(\alpha-1)I)\varrho(t),\end{aligned}$$

and using (21), we have

$$\begin{aligned}&(zt^2\varrho(t)\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B}))^{'} + \frac{[(\mathbb{A}-I)(\mathbb{A}-\mathbb{B})]t}{2\pi i}\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B})\\ =&\alpha I(\mathbb{A}+(\alpha-1)I)\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})\varrho(t).\end{aligned} \tag{22}$$

Multiplying $\mathcal{Y}_{\gamma}(t;\mathbb{A},\mathbb{B})$ in (22) and and integrating the result around the unit circle, one gets

$$\begin{aligned}&\int_C (t^2\varrho(t)\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B}))^{'}\mathcal{Y}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt\\ &+\int_C \frac{[(\mathbb{A}-I)(\mathbb{A}-2I)]t}{2\pi i}\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B})\;\mathcal{Y}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt\\ &= \alpha I(\mathbb{A}+(\alpha-1)I)\int_C \varrho(t)\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})\mathcal{Y}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt.\end{aligned} \tag{23}$$

Consider the straightforward computation integrating, we see that

$$\begin{aligned}&\alpha I(\mathbb{A}+(\alpha-1)I)\int_C \varrho(t)\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})\mathcal{Y}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt\\ =&-\int_C t^2\varrho(t)\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B})\mathcal{Y}^{'}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt.\end{aligned} \tag{24}$$

Interchanging α and γ, that is

$$\begin{aligned}&\gamma I(\mathbb{A}+(\gamma-1)I)\int_C \varrho(t)\mathcal{Y}_{\alpha}(t;\mathbb{A},\mathbb{B})\mathcal{Y}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt\\ =&-\int_C t^2\varrho(t)\mathcal{Y}^{'}_{\alpha}(t;\mathbb{A},\mathbb{B})\mathcal{Y}^{'}_{\gamma}(t;\mathbb{A},\mathbb{B})\;dt\end{aligned}$$

and subtracting gives

$$[\alpha I(\mathbb{A}+I(\alpha-1))-\gamma I(\mathbb{A}+(\gamma-1)I)]\int_C \varrho(t)\mathcal{Y}_\alpha(t;\mathbb{A},\mathbb{B})\mathcal{Y}_\gamma(t;\mathbb{A},\mathbb{B})\,dt=0.$$

Finally, for $\alpha\neq\gamma$, we get

$$\int_C \varrho(t)\mathcal{Y}_\alpha(t;\mathbb{A},\mathbb{B})\mathcal{Y}_\gamma(t;\mathbb{A},\mathbb{B})\,dt=0. \tag{25}$$

This result can be expressed as follows:

Theorem 4. *For any real numbers $\alpha\neq\gamma$ and let $\mathbb{A}$ and $\mathbb{B}$ be commutative matrices in $\mathbb{C}^{N\times N}$, satisfying the condition (4), then expression (25) hold true.*

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