



On the Semi-bounded Solution of Cauchy Type Singular Integral Equations of the First Kind

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Abstract. This paper presents an efficient approximate method to obtain a numerical solution, which is bounded at the end point $x = -1$, for Cauchy type singular integral equations of the first kind on the interval $[-1, 1]$. The solution is derived by approximating the unknown density function using the weighted Chebyshev polynomials of the third kind, and then computing the Cauchy singular integral which is obtained analytically. The known force function is interpolated using the Chebyshev polynomials of the fourth kind. The exactness of this approximate method is shown for characteristic equation when the force function is a cubic. Particular result is also given to show the exactness of this method.

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1. Introduction

Let us consider the Cauchy type singular integral equations of the first kind

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 K(x,t)\varphi(t) dt = f(x), \quad -1 < x < 1, \quad (1.1)$$

where K and f are assumed to be real-valued functions belong to the class of Hölder continues functions on the sets $[-1, 1] \times [-1, 1]$ and $[-1, 1]$, respectively. φ is unknown function to be determined. The singular integral equations have been widely used [1–4] in solving problems associated with aerodynamic, hydrodynamic and elasticity.

The characteristic singular integral equation of equation (1.1) is of the form

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = f(x), \quad -1 < x < 1. \quad (1.2)$$

Eshkovatov et al. [5] discussed the efficient approximate method to solve characteristic equation (1.2) using Chebyshev polynomial approximations of the first, second, third, and fourth kinds with corresponding weight functions for four cases. The collocation points are chosen to be the zeros of Chebyshev polynomials. They showed that, the approximate method gives exact solution when the force function f is a linear.

Abdulkawi et al. [6] presented a numerical solution of equation (1.1), which is bounded at the end points $x \pm 1$. They used Chebyshev polynomials of the second kind with the corresponding weight function to approximate the density function and the Chebyshev polynomials of the first kind to approximate the force function. They showed that the numerical solution of characteristic equation is identical to the exact solution when the force function is a cubic.

It is well known that the analytical solution of characteristic equation (1.2), which

is bounded at the end point $x = -1$, is given by the following formula

$$\varphi(x) = -\frac{1}{\pi^2} \sqrt{\frac{1+x}{1-x}} \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} \frac{f(t)}{t-x} dt. \tag{1.3}$$

By solving equation (1.1) with respect to its characteristic part, we will find that it is equivalent to the Fredholm equation type of the second kind [7]

$$\left. \begin{aligned} \varphi(t) + \int_{-1}^1 N(t, \tau) \varphi(\tau) d\tau &= F(t), \\ N(t, \tau) &= -\frac{1}{\pi^2} \sqrt{\frac{1+t}{1-t}} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \frac{K(x, \tau)}{x-t} dx, \\ F(t) &= -\frac{1}{\pi^2} \sqrt{\frac{1+t}{1-t}} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \frac{f(x)}{x-t} dx. \end{aligned} \right\} \tag{1.4}$$

in the sense of obtaining the solution which one can apply the Fredholm's theorems.

In this paper, we present an approximate solution for equation (1.1) which is bounded at the end point $x = -1$.

2. Approximate Solution of Equation (1.1)

Guiding by the analytic solutions of characteristic equation given by (1.3), using the Chebyshev interpolation polynomials of third kind V_i and fourth kind W_i with corresponding weight functions ω_1 and ω_2 [8];

$$\left. \begin{aligned} V_i(x) &= \frac{\cos\left(\frac{2i+1}{2} \cos^{-1} x\right)}{\cos\left(\frac{1}{2} \cos^{-1} x\right)}, & \omega_1(x) &= \sqrt{\frac{1+x}{1-x}}, \\ W_i(x) &= \frac{\sin\left(\frac{2i+1}{2} \cos^{-1} x\right)}{\sin\left(\frac{1}{2} \cos^{-1} x\right)}, & \omega_2(x) &= \sqrt{\frac{1-x}{1+x}}. \end{aligned} \right\}, \tag{2.1}$$

and helping of the following important formula for singular integrals with the Cauchy kernel

$$\int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{V_i(t)}{t-x} dt = \pi W_i(x), \quad -1 < x < 1, \quad i = 0, 1, \dots, n, \tag{2.2}$$

the approximate solution, which is bounded at the end point $x = -1$, of equation (1.1) is obtained.

We will interpolate the known function $f(x)$ by using the Chebyshev orthogonal polynomial of the fourth kind $f_n(x)$ of degree n as

$$f(x) \approx f_n(x) = \sum_{k=0}^n f_k W_k(x) \tag{2.3}$$

where

$$f_k = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} f(t) W_k(t) dt. \tag{2.4}$$

Approximating the unknown function φ by φ_n which is defined as

$$\varphi_n(x) = \sqrt{\frac{1+x}{1-x}} \sum_{j=0}^n a_j V_j(x) \tag{2.5}$$

where the unknown coefficients $\{a_j\}_0^n$ are to be determined.

Substituting (2.5) into (1.1) we obtain

$$\sum_{j=0}^n a_j \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{V_j(t)}{t-x} dt + \sum_{j=0}^n a_j \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} K(x,t) V_j(t) dt = f(x). \tag{2.6}$$

Using (2.2) into (2.6) we obtain

$$\pi \sum_{j=0}^n a_j W_j(x) + \sum_{j=0}^n a_j \zeta_j(x) = f(x) \tag{2.7}$$

where

$$\zeta_j(x) = \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} K(x,t) V_j(t) dt. \tag{2.8}$$

Interpolating the function $\zeta_j(x)$ by using the Chebyshev orthogonal polynomial of the fourth kind as follows

$$\zeta_j(x) \approx \sum_{k=0}^n \mu_{j,k} W_k(x) \tag{2.9}$$

where

$$\mu_{j,k} = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} K(x,t) V_j(t) W_k(x) dt dx. \tag{2.10}$$

Due to (2.3-2.4) and (2.9-2.10), equation (2.7) becomes

$$\sum_{j=0}^n a_j W_j(x) + \frac{1}{\pi} \sum_{k=0}^n \sum_{j=0}^n a_j \mu_{j,k} W_k(x) = \frac{1}{\pi} \sum_{k=0}^n f_k W_k(x). \tag{2.11}$$

The unknown coefficients $\{a_j\}_0^n$ are determined by solving the system of linear equations obtained by comparing the coefficients of $W_j, j = 0, 1, 2, \dots, n$ in both sides of equation (2.11) which is

$$\left. \begin{aligned} a_0 + \frac{1}{\pi} \sum_{j=0}^n a_j \mu_{j,0} &= \frac{1}{\pi} f_0, \\ a_1 + \frac{1}{\pi} \sum_{j=0}^n a_j \mu_{j,1} &= \frac{1}{\pi} f_1, \\ \dots &\dots \dots \\ \vdots &\vdots \vdots \\ \dots &\dots \dots \\ a_n + \frac{1}{\pi} \sum_{j=0}^n a_j \mu_{j,n} &= \frac{1}{\pi} f_n. \end{aligned} \right\} \tag{2.12}$$

where the coefficients $\{f_k\}$ and $\{\mu_{j,k}\}$ are given by (2.4) and (2.10), respectively.

3. Approximate Solution of the Characteristic Equation (1.2)

Theorem 3.1. *If $f(x)$ in characteristic equation (1.2) is a cubic function, then the approximate solution (2.5) is identical to the exact solution.*

Proof. Let us consider the characteristic singular integral equation

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = f(x), \quad -1 < x < 1. \tag{3.1}$$

Let $f(x)$ in (3.1) be a cubic function i.e

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3, \quad -1 < x < 1. \tag{3.2}$$

Substituting (3.2) into (2.4) yields

$$f_k = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} (c_0 + c_1 t + c_2 t^2 + c_3 t^3) W_k(t) dt. \tag{3.3}$$

Using the following Chebyshev recurrence relations of the third and fourth kinds, respectively,

$$\left. \begin{aligned} V_0(x) &= 1, \quad V_1(x) = 2x - 1, \\ V_n(x) &= 2x V_{n-1}(x) - V_{n-2}(x), \quad n \geq 2. \end{aligned} \right\}, \tag{3.4}$$

$$\left. \begin{aligned} W_0(x) &= 1, \quad W_1(x) = 2x + 1, \\ W_n(x) &= 2x W_{n-1}(x) - W_{n-2}(x), \quad n \geq 2. \end{aligned} \right\}. \tag{3.5}$$

we have

$$\left. \begin{aligned} t^3 &= \frac{1}{8} [V_3(t) + V_2(t) + 3(V_1(t) + V_0(t))] \\ &= \frac{1}{8} [W_3(t) - W_2(t) + 3(W_1(t) - W_0(t))], \\ t^2 &= \frac{1}{4} [V_2(t) + V_1(t) + 2V_0(t)] \\ &= \frac{1}{4} [W_2(t) - W_1(t) + 2W_0(t)], \\ t &= \frac{1}{2} [V_1(t) + V_0(t)] \\ &= \frac{1}{2} [W_1(t) - W_0(t)]. \end{aligned} \right\} \tag{3.6}$$

It is known that [8]

$$\int_{-1}^1 \sqrt{\frac{1+t}{1-t}} V_m(t) V_n(t) dt = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases} \tag{3.7}$$

and

$$\int_{-1}^1 \sqrt{\frac{1-t}{1+t}} W_m(t) W_n(t) dt = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases} \tag{3.8}$$

Due to (3.3), (3.6) and (3.8), we obtain

$$\left. \begin{aligned} f_0 &= c_0 - \frac{c_1 - c_2}{2} - \frac{3c_3}{8}, \\ f_1 &= \frac{2c_1 - c_2}{4} + \frac{3c_3}{8}, \\ f_2 &= \frac{2c_2 - c_3}{8}, \\ f_3 &= \frac{c_3}{8}. \end{aligned} \right\} \quad (3.9)$$

From (2.12) when $K(x, t) = 0$, yields

$$a_j = \frac{1}{\pi} f_j, \quad j = 0, \dots, n. \quad (3.10)$$

The approximate solution (2.5) with $n = 3$ becomes

$$\varphi_n(x) = \frac{1}{\pi} \sqrt{\frac{1+x}{1-x}} [f_0 + f_1 V_1(x) + f_2 V_2(x) + f_3 V_3(x)]. \quad (3.11)$$

Substituting (3.9) and (3.10) into (3.11), we obtain the approximate solutions of characteristic equation (3.1) which is

$$\left. \begin{aligned} \varphi_n(x) &= \frac{1}{\pi} \sqrt{\frac{1+x}{1-x}} p(x), \\ p(x) &= c_0 - c_1 + \frac{1}{2}(c_2 - c_3) + (c_1 - c_2 + \frac{1}{2}c_3)x + (c_2 - c_3)x^2 + c_3x^3. \end{aligned} \right\} \quad (3.12)$$

In order to obtain the exact solution of equation (3.1), we substitute (3.2) into (1.3) which gives

$$\varphi(x) = -\frac{1}{\pi^2} \sqrt{\frac{1+x}{1-x}} \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} \frac{c_0 + c_1 t + c_2 t^2 + c_3 t^3}{t-x} dt. \quad (3.13)$$

It is easy to see that

$$\left. \begin{aligned} \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} \frac{1}{t-x} dt &= -\pi, \\ \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} \frac{t}{t-x} dt &= -\pi(x-1), \\ \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} \frac{t^2}{t-x} dt &= -\pi(x^2-x+0.5), \\ \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} \frac{t^3}{t-x} dt &= -\pi(x^3-x^2+0.5x-0.5). \end{aligned} \right\} \quad (3.14)$$

Using (3.14) into (3.13), we obtain the exact solution of equation (3.1) which is identical to the approximate solutions (3.12).

4. Particular Result

Let us consider the integral equation

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 (x^3 + t^3) \varphi(t) dt = 3x^3 + 2x^2 + x, \quad -1 < x < 1 \quad (4.1)$$

and we seek the solution of this equation which is bounded at $x = -1$.

From (2.4) and (3.6) yields

$$\left. \begin{aligned} f_k &= \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} (3t^3 + 2t^2 + t) W_k(t) dt \\ &= \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \left[\frac{3}{8} W_3(t) + \frac{1}{8} W_2(t) + \frac{9}{8} W_1(t) - \frac{5}{8} W_0(t) \right] W_k(t) dt. \end{aligned} \right\} \quad (4.2)$$

Using (3.8) into (4.2), we have

$$\left\{ f_0 = -\frac{5}{8}, f_1 = \frac{9}{8}, f_2 = \frac{1}{8}, f_3 = \frac{3}{8} \right\}. \quad (4.3)$$

Due to (2.10) we get

$$\mu_{j,k} = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} (x^3 + t^3) V_j(t) W_k(x) dt dx. \quad (4.4)$$

Using orthogonal property (3.7) into (4.4) we obtain

$$\mu_{0,k} = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \left[\pi x^3 + \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} t^3 V_0 dt \right] W_k(x) dx \tag{4.5}$$

and

$$\mu_{j,k} = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \left[\int_{-1}^1 \sqrt{\frac{1+t}{1-t}} t^3 V_j dt \right] W_k(x) dx, \quad j = 1, 2, \dots, n. \tag{4.6}$$

Due to (3.6-3.7), equation (4.5) becomes

$$\mu_{0,k} = \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} \left[x^3 + \frac{3}{8} \right] W_k(x) dx \tag{4.7}$$

which gives

$$\mu_{0,0} = 0, \quad \mu_{0,1} = \frac{3\pi}{8}, \quad \mu_{0,2} = -\frac{\pi}{8}, \quad \mu_{0,3} = \frac{\pi}{8}. \tag{4.8}$$

From (4.6) with help of (3.6-3.7), yields

$$\mu_{1,k} = \frac{3}{8} \int_{-1}^1 \sqrt{\frac{1-x}{1+x}} W_k(x) dx \tag{4.9}$$

which gives

$$\mu_{1,0} = \frac{3\pi}{8}, \quad \mu_{1,k} = 0, \quad k = 1, 2, 3. \tag{4.10}$$

Similarly, we obtain

$$\mu_{2,0} = \mu_{3,0} = \frac{\pi}{8}, \quad \mu_{2,k} = \mu_{3,k} = 0, \quad k = 1, 2, 3. \tag{4.11}$$

Due to (2.12), (4.3) and (4.8, 4.10-4.11) we have the following system of linear equations

$$a_k + \frac{1}{\pi} \sum_{j=0}^3 a_j \mu_{j,k} = \frac{1}{\pi} f_k, \quad k = 0, 1, 2, 3. \tag{4.12}$$

It is not difficult to see that the solution of the system (4.12) is

$$a_0 = -\frac{71}{55\pi}, \quad a_1 = \frac{177}{110\pi}, \quad a_2 = -\frac{2}{55\pi}, \quad a_3 = \frac{59}{110\pi}. \tag{4.13}$$

Substituting the values of the coefficients $\{a_j\}_0^3$ into (2.5) yields the approximate solution of equation (4.1)

$$\varphi_n(x) = \frac{1}{55\pi} \sqrt{\frac{1+x}{1-x}} [236x^3 - 126x^2 + 63x - 128] \quad (4.14)$$

which is identical to the exact solution.

5. Conclusion

The Chebyshev orthogonal polynomials of the third and fourth kinds are used to approximate the unknown density function which is bounded at the end point $x = -1$, and the known force function, respectively, for solving the Cauchy type singular integral equation of the first kind. Theorem 3.1 shows the exactness of the approximate method presented for characteristic equation when the force function is a cubic. Particular result also shows that this approximate method does not only give the exact solution for characteristic equation but also for other Cauchy type singular integral equations of the first kind.

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