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Hypersemigroups and fuzzy hypersemigroups

Niovi Kehayopulu

University of Athens, Department of Mathematics, 15784 Panepistimiopolis, Greece

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Abstract. The aim is to show that the theory of hypersemigroups and the theory of fuzzy hypersemigroups are parallel to each other, in the following sense: An hypersemigroup H is intra-regular, for example, if and only if $A \cap B \subseteq B * A$ for every right ideal A and every left ideal B of H. And an hypersemigroup H is intra-regular if and only if $f \wedge g \preceq g \circ f$ for every fuzzy right ideal f and every fuzzy left ideal g of H. An hypersemigroup H is left quasi-regular if and only if $A \cap B \subseteq A * B$ for every ideal A and every nonempty subset B of H. And an hypersemigroup H is left quasi-regular if and only if $f \wedge g \preceq f \circ g$ for every fuzzy ideal f and every fuzzy subset g of H.

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1. Introduction and prerequisites

A semigroup $(S, \cdot)$ is called *regular* (*von Neumann regular*) if for every $a \in S$ there exists $x \in S$ such that $a = axa$. This is equivalent to saying that $a \in aSa$ for every $a \in S$ or $A \subseteq ASA$ for every $A \subseteq S$. A nonempty subset A of a groupoid $(S, \cdot)$ is called a *right* (resp. *left*) *ideal* of S if $AS \subseteq A$ (resp. $SA \subseteq A$). If A is both a right and a left ideal of S, then it is called an *ideal* of S. A nonempty subset B of a semigroup $(S, \cdot)$ is called a *bi-ideal* of S if $BSB \subseteq B$. It is well known that a semigroup S is regular if and only if for every right ideal A and every left ideal B if S, we have $A \cap B = AB$ (Iséki [2]). A semigroup $(S, \cdot)$ is called *intra-regular* [1] if for every $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$. This is equivalent to saying that $a \in Sa^2S$ for every $a \in S$ or $A \subseteq SA^2S$ for every $A \subseteq S$. It is also well known that a semigroup S is intra-regular if and only if for any right ideal A and any left ideal B of S, we have $A \cap B \subseteq BA$ (Lajos and Szász [9]). For the concepts of left (right) quasi-regular and semisimple semigroups we refer to [11]: A semigroup S is called *left* (resp. *right*) *quasi-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a = xaya$ (resp. $a = axay$). We remark that a semigroup S is left (resp. right) quasi-regular if and only if $a \in SaSa$ (resp. $a \in aSaS$) for every $a \in S$, equivalently if $A \subseteq SASA$ (resp. $A \subseteq ASAS$) for every $A \subseteq S$. A semigroup S is called *semisimple* if for every $a \in S$ there exist $x, y, z \in S$ such that $a = xayaz$. We note that a semigroup S is

Email address: nkehayop@math.uoa.gr (N. Kehayopulu)

semisimple if and only if $a \in SaSaS$ for every $a \in S$, equivalently if $A \subseteq SASAS$ for every $A \subseteq S$. In the present paper we show that the theories of hypersemigroups and fuzzy hypersemigroups are parallel to each other, in the following sense: An hypersemigroup H is regular if and only if $A \cap B = A * B$, equivalently if $A \cap B \subseteq A * B$ for every right ideal A and every left ideal B of H. On the other hand, an hypersemigroup H is regular if and only if $f \wedge g = f \circ g$, equivalently if $f \wedge g \preceq f \circ g$ for every fuzzy right ideal f and every fuzzy left ideal g of H. An hypersemigroup H is intra-regular if and only if for every right ideal A and every left ideal B of H we have $A \cap B \subseteq B * A$. An hypersemigroup H is intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of H we have $f \wedge g \preceq g \circ f$. An hypersemigroup H is left quasi-regular if and only if $A \cap B \subseteq A * B$ for every ideal A and every bi-ideal B of H, equivalently if $A \cap B \subseteq A * B$ for every ideal A and every left ideal B of H. An hypersemigroup H is left quasi-regular if and only if $f \wedge g \preceq f \circ g$ for every fuzzy ideal f and every fuzzy bi-ideal g of H, equivalently if $f \wedge g \preceq f \circ g$ for every fuzzy ideal f and every fuzzy left ideal g of H. An hypersemigroup H is left quasi-regular if and only if the left ideals of H are idempotent. And an hypersemigroup H is left quasi-regular if and only if the fuzzy left ideals of H are idempotent. An hypersemigroup H is semisimple if and only if the ideals of H are idempotent, equivalently if $A \cap B = A * B$ for all ideals A, B of H. An hypersemigroup H is semisimple if and only if the fuzzy ideals of H are idempotent, equivalently if for each fuzzy ideals f and g of H we have $f \wedge g = f \circ g$. Our aim being to show that the theories of hypersemigroups and of fuzzy hypersemigroups are parallel to each other, we restrict ourselves to an hypersemigroup (without order) and further interesting information related to this parallelism will be given in a forthcoming paper. However, analogous results with the results given in this paper for ordered hypersemigroups also hold. For related results on ordered semigroups on which the present article is based we refer to [3, 7]. The left (right) quasi-regular fuzzy ordered semigroups under the name left (right) weakly regular and the semisimple ordered semigroups have been studied by Shabir and Khan in [12, 13], and by Kehayopulu in [4]. Hypersemigroups under the name semihypergroups have been first systematically studied in 2011 by Mahmood in his PhD thesis [10]. However a revision in the notation in this thesis is necessary. For the sake of completeness, let us mention some definitions and results already given in [5, 6].

An *hypergroupoid* is a nonempty set H with an hyperoperation

$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$ on H and an operation

$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$ on $\mathcal{P}^*(H)$ (induced by the operation of H) such that $A * B = \bigcup\limits_{(a,b) \in A \times B} (a \circ b)$ for every $A, B \in \mathcal{P}^*(H)$, $\mathcal{P}^*(H)$ being the set of nonempty subsets of H. As the operation "$*$" depends on the hyperoperation "$\circ$", an hypergroupoid can be denoted by $(H, \circ)$ (instead of $(H, \circ, *)$). If $(H, \circ)$ is an hypergroupoid then, for every $x, y \in H$, we have $\{x\} * \{y\} = x \circ y$. If $(H, \circ)$ is an hypergroupoid and $A, B, C \in \mathcal{P}^*(H)$, then $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$. We also have $H * H \subseteq H$. If $(H, \circ)$ is an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$, then the following two properties, though clear, are essential to the investigation:

(1) If $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$.

(2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$ [5, 6].

Lemma 1.1 [5] *If* $(H, \circ)$ *is an hypergroupoid and* $A_i, B \in \mathcal{P}^*(H)$, $i \in I$, *then we have the following:*

(1) $(\bigcup\limits_{i \in I} A_i) * B = \bigcup\limits_{i \in I} (A_i * B)$.

(2) $B * (\bigcup\limits_{i \in I} A_i) = \bigcup\limits_{i \in I} (B * A_i)$.

An hypergroupoid $(H, \circ)$ is called *hypersemigroup* if $(x \circ y) * \{z\} = \{x\} * (y \circ z)$ for every $x, y, z \in H$. For short, we can identify the singleton $\{x\}$ by the element x and the $\{z\}$ by z and define the hypersemigroup as $(x \circ y) * z = x * (y \circ z)$.

If $(H, \circ)$ is an hypersemigroup, then the operation "$*$" on $\mathcal{P}^*(H)$ is associative so, for any subsets $A, B, C \in \mathcal{P}^*(H)$ we can write $(A * B) * C = A * (B * C) = A * B * C$ and for any product $A_1 * A_2 * ..... * A_n$ of elements of $\mathcal{P}^*(H)$ we can put parentheses in any place beginning with some A_i and ending in some A_j $(1 \le i, j \le n)$. Following the concepts of right and left ideals of groupoids, a nonempty subset A of an hypergroupoid $(H, \circ)$ is called a *right* (resp. *left*) *ideal* of H if $A * H \subseteq A$ (resp. $H * A \subseteq A$). It is called an *ideal* of H if it is both a right and left ideal of H. For a subset A of H, we denote by $R(A)$ (resp. $L(A)$) the right (resp. left) ideal of H generated by A and by $I(A)$ the ideal of H generated by A. If $(H, \circ)$ is an hypersemigroup, then $R(A) = A \cup (A * H)$, $L(A) = A \cup (H * A)$ and $I(A) = A \cup (A * H) \cup (H * A) \cup (H * A * H)$. For $A = \{a\}$ $(a \in H)$, we write $R(a)$, $L(a)$, $I(a)$ instead of $L(\{a\})$, $R(\{a\})$, $I(\{a\})$. Following the concepts of bi-ideals of semigroups, a nonempty subset B of an hypersemigroup $(H, \circ)$ is called a *bi-ideal* of H if $B * H * B \subseteq B$. Following the concepts of regular and intra-regular semigroups, an hypersemigroup $(H, \circ)$ is called *regular* if for every $a \in H$ there exists $x \in H$ such that $a \in (a \circ x) * \{a\}$; it is called *intra-regular* if for every $a \in H$ there exist $x, y \in H$ such that $a \in (x \circ a) * (a \circ y)$.

Lemma 1.2. *Let* $(H, \circ)$ *be an hypersemigroup. The following are equivalent:*

(1) H *is regular.*

(2) $a \in \{a\} * H * \{a\}$ *for every* $a \in H$.

(3) $A \subseteq A * H * A$ *for every* $A \in \mathcal{P}^*(H)$.

Indeed: If H is regular and $a \in H$, then there exists $x \in H$ such that

$$a \in (a \circ x) * \{a\} = \{a\} * \{x\} * \{a\} \subseteq \{a\} * H * \{a\}.$$

"Conversely", if $A \subseteq A * H * A$ for every $A \in \mathcal{P}^*(H)$ and $a \in A$, then $\{a\} \subseteq \Big(\{a\} * H\Big) * \{a\}$, then there exists $u \in \{a\} * H$ such that $a \in u \circ a$ and $h \in H$ such that $u \in a \circ h$. Then we have $a \in u \circ a = \{u\} * \{a\} \subseteq (a \circ h) * \{a\}$, where $h \in H$, so H is regular. In a similar way, an hypersemigroup H is intra-regular if and only if, for every $a \in H$, we have $a \in H * \{a\} * \{a\} * H$, equivalently if for any nonempty subset A of H we have $A \subseteq H * A * A * H$.

Following Zadeh, any mapping $f : H \to [0, 1]$ of an hypergroupoid H into the closed interval $[0, 1]$ of real numbers is called a *fuzzy subset* of H (or a *fuzzy set* in H) and the mapping f_A (the so called characteristic function of A) is the fuzzy subset of H defined as follows:

$$f_A : H \to \{0,1\} \mid x \to f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For an element a of H, we denote by A_a the subset of $H \times H$ defined by

$$A_a := \{(y,z) \in H \times H \mid a \in y \circ z\}.$$

For two fuzzy subsets f and g of H, we denote by $f \circ g$ the fuzzy subset of H defined as follows:

$$f \circ g : H \to [0,1] \;\; a \to \begin{cases} \bigvee\limits_{(y,z)\in A_a} \min\{f(y), g(z)\} & \text{if } \;\; A_a \neq \emptyset \\ 0 \quad \text{if } \;\; A_a = \emptyset. \end{cases}$$

As no confusion is possible, we denote the operation between fuzzy subsets of H and the hyperoperation on H by the same symbol. Denote by $F(H)$ the set of all fuzzy subsets of H and by "$\preceq$" the order relation on $F(H)$ defined by

$$f \preceq g \iff f(x) \leq g(x) \text{ for every } x \in H.$$

For two fuzzy subsets f and g of an hypergroupoid H we denote by $f \wedge g$ the fuzzy subset of H defined as follows:

$$f \wedge g : H \to [0,1] \mid x \to (f \wedge g)(x) := \min\{f(x), g(x)\}.$$

One can easily see that the fuzzy subset $f \wedge g$ is the infimum of the fuzzy subsets f and g, and this is why we write $f \wedge g = \inf\{f, g\}$. If f is a fuzzy subset of H, then $f \wedge f = f$. Indeed, if $x \in H$, then $(f \wedge f)(x) := \min\{f(x), f(x)\} = f(x)$.

The concepts of fuzzy right and fuzzy left ideal of a groupoid due to Kuroki [8] can be naturally transferred to hypergroupoids as follows: A fuzzy subset f of an hypergroupoid H is called a *fuzzy right ideal* of H if

$$f(x \circ y) \geq f(x) \text{ for every } x, y \in H,$$

in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$.
A fuzzy subset f of an hypergroupoid H is called a *fuzzy left ideal* of H if

$$f(x \circ y) \geq f(y) \text{ for every } x, y \in H,$$

meaning that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$.
If f is both a fuzzy right and a fuzzy left ideal of H, then it is called a *fuzzy ideal* of H.
A fuzzy subset f of an hypersemigroup H is called a *fuzzy bi-ideal* of H if

$$f\Big((x \circ y) * \{z\}\Big) \geq \min\{f(x), f(z)\} \text{ for every } x, y, z \in H,$$

in the sense that if $x, y, z \in H$ and $u \in (x \circ y) * \{z\}$, then $f(u) \geq \min\{f(x), f(z)\}$.
Exactly as in groupoids–semigroups, the following hold:

(1) If H is an hypergroupoid, then A is a right (resp. left) ideal of H if and only if the characteristic function f_A is a fuzzy right (resp. fuzzy left) ideal of H.

(2) If H is an hypersemigroup, then A is a bi-ideal of H if and only if f_A is a fuzzy bi-ideal of H.

2. Main results

Theorem 2.1. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is regular.*

(2) *$A \cap B = A * B$ for every right ideal A and every left ideal B of H.*

(3) *$A \cap B \subseteq A * B$ for every right ideal A and every left ideal B of H.*

(4) *$R(A) \cap L(A) \subseteq R(A) * L(A)$ for every $A \in \mathcal{P}^*(H)$.*

(5) *$R(a) \cap L(a) \subseteq R(a) * L(a)$ for every $a \in H$.*

Proof. (1) $\Longrightarrow$ (2). Let A be a right ideal and B a left ideal of H. The set $A \cap B$ is a nonempty subset of H. Indeed: Take an element $a \in A$ and an element $b \in B$ $(A, B \neq \emptyset)$. Then $a \circ b \subseteq A * H \subseteq A$ and $a \circ b \subseteq H * B \subseteq B$, so $a \circ b \subseteq A \cap B$. Since $a \circ b \neq \emptyset$, we have $A \cap B \neq \emptyset$. Since H is regular and $A \cap B \in \mathcal{P}^*(H)$, we have

$$\begin{aligned} A \cap B &\subseteq (A \cap B) * H * (A \cap B) \subseteq A * H * B = (A * H) * B \\ &\subseteq A * B \subseteq (A * H) \cap (H * B) \subseteq A \cap B. \end{aligned}$$

Thus we have $A \cap B = A * B$.
The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.
(5) $\Longrightarrow$ (1). Let $a \in H$. Since $R(a)$ is a right ideal of H and $L(a)$ is a left ideal of H, by hypothesis, we have

$$\begin{aligned} a &\in R(a) \cap L(a) \subseteq R(a) * L(a) = \Big(\{a\} \cup \Big(\{a\} * H\Big)\Big) * \Big(\{a\} \cup \Big(H * \{a\}\Big)\Big) \\ &= (a \circ a) \cup \Big(\{a\} * H * \{a\}\Big) \cup \Big(\{a\} * H * H * \{a\}\Big) \text{ (by Lemma 1.1)} \\ &= (a \circ a) \cup \Big(\{a\} * H * \{a\}\Big). \end{aligned}$$

We have $a \in a \circ a$, so $a \in (a \circ a) * \{a\}$ or $a \in \{a\} * H * \{a\}$. In each case, H is regular. $\square$

Proposition 2.2. *Let $(H, \circ)$ be an hypergroupoid, f a fuzzy right ideal and g a fuzzy left ideal of H. Then we have $f \circ g \preceq f \wedge g$.*

Proof. Let $a \in H$. Then $(f \circ g)(a) \leq (f \wedge g)(a)$. In fact: If $A_a = \emptyset$, then $(f \circ g)(a) := 0$. Since $a \in H$ and $f \wedge g$ is a fuzzy subset of H, we have $(f \wedge g)(a) \geq 0$, thus we have $(f \circ g)(a) \leq (f \wedge g)(a)$. Let now $A_a \neq \emptyset$. Then

$$(f \circ g)(a) := \bigvee_{(x,y) \in A_a} \min\{f(x), g(y)\} \qquad (*)$$

We have

$$\min\{f(x), g(y)\} \le (f \wedge g)(a) \text{ for every } (x, y) \in A_a \qquad (**)$$

Indeed: Let $(x, y) \in A_a$. Then $a \in x \circ y$. Since f is a fuzzy right ideal of H, we have $f(x \circ y) \ge f(x)$, then we have $f(a) \ge f(x)$. Since g is a fuzzy left ideal of H, we have $g(x \circ y) \ge g(y)$, then $g(a) \ge g(y)$, so

$$(f \wedge g)(a) := \min\{f(a), g(a)\} \ge \min\{f(x), g(y)\},$$

and condition $(**)$ is satisfied. By $(**)$, we have

$$\bigvee_{(x,y)\in A_a} \min\{f(x), g(y)\} \le (f \wedge g)(a).$$

Then, by $(*)$, $(f \circ g)(a) \le (f \wedge g)(a)$. □

Proposition 2.3. *Let $(H, \circ)$ be a regular hypersemigroup, f a fuzzy right ideal of H and g a fuzzy subset of H. Then we have $f \wedge g \preceq f \circ g$.*

Proof. Let $a \in H$. Since H is regular, there exists $x \in H$ such that $a \in (a \circ x) * \{a\}$. Then $a \in u \circ a$ for some $u \in a \circ x$. Since $a \in u \circ a$, we have $(u, a) \in A_a$, then

$$(f \circ g)(a) := \bigvee_{(y,z)\in A_a} \min\{f(y), g(z)\} \ge \min\{f(u), g(a)\}.$$

Since f is a fuzzy right ideal of H, we have $f(a \circ x) \ge f(a)$. Since $u \in a \circ x$, we have $f(u) \ge f(a)$. Then we have

$$(f \circ g)(a) \ge \min\{f(u), g(a)\} \ge \min\{f(a), g(a)\} := (f \wedge g)(a),$$

so $f \wedge g \preceq f \circ g$ and the proof is complete. □

Theorem 2.4. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is regular.*

(2) *$f \wedge g = f \circ g$ for every fuzzy right ideal f and every fuzzy left ideal g of H.*

(3) *$f \wedge g \preceq f \circ g$ for every fuzzy right ideal f and every fuzzy left ideal g of H.*

Proof. (1) $\Longrightarrow$ (2). Let f be a fuzzy right ideal and g a fuzzy left ideal of H. By Proposition 2.2, we have $f \circ g \preceq f \wedge g$. By Proposition 2.3, we have $f \wedge g \preceq f \circ g$, and (2) is satisfied.
The implication (2) $\Rightarrow$ (3) is obvious.
(3) $\Longrightarrow$ (1). By Theorem 2.1, it is enough to prove that $R(a) \cap L(a) \subseteq R(a) * L(a)$ for every $a \in H$. So, let $a \in H$ and $b \in R(a) \cap L(a)$. Since $R(a)$ is a right ideal of H, the characteristic function $f_{R(a)}$ is a fuzzy right ideal of H and, since $L(a)$ is a left ideal of H, $f_{L(a)}$ is a fuzzy left ideal of H. By hypothesis, we have $f_{R(a)} \wedge f_{L(a)} \preceq f_{L(a)} \circ f_{R(a)}$, then

$$\Big(f_{R(a)} \wedge f_{L(a)}\Big)(b) \le \Big(f_{R(a)} \circ f_{L(a)}\Big)(b).$$

Thus

$$\min\{f_{R(a)}(b), f_{L(a)}(b)\} \leq \Big(f_{R(a)} \circ f_{L(a)}\Big)(b).$$

Since $b \in R(a)$ and $b \in L(a)$, we have $f_{R(a)}(b) = f_{L(a)}(b) = 1$, and so

$$1 \leq \Big(f_{R(a)} \circ f_{L(a)}\Big)(b).$$

If $A_b = \emptyset$, then $\Big(f_{R(a)} \circ f_{L(a)}\Big)(b) = 0$ which is impossible. Thus we have $A_b \neq \emptyset$ and

$$\Big(f_{R(a)} \circ f_{L(a)}\Big)(b) = \bigvee_{(y,z)\in A_b} \min\{f_{R(a)}(y), f_{L(a)}(z)\}.$$

Then there exists $(y, z) \in A_b$ such that $y \in R(a)$ and $z \in L(a)$ $(*)$
Indeed: Suppose there is no $(y, z) \in A_b$ such that $y \in R(a)$ and $z \in L(a)$. Then, for each $(y, z) \in A_b$, we have $y \notin R(a)$ or $z \notin L(a)$. Then, for each $(y, z) \in A_b$, we have $f_{R(a)}(y) = 0$ or $f_{L(a)}(z) = 0$. Then $\min\{f_{R(a)}(y), f_{R(a)}(z)\} = 0$ for every $(y, z) \in A_b$. Then $\bigvee\limits_{(y,z)\in A_b} \min\{f_{R(a)}(y), f_{L(a)}(z)\} = 0$, so $\Big(f_{R(a)} \circ f_{L(a)}\Big)(b) = 0$ which is no possible.
By $(*)$, we have $b \in y \circ z \subseteq R(a) * L(a)$, and the proof is complete. □

Theorem 2.5. (cf. also [10]) *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is intra-regular.*

(2) *$A \cap B \subseteq B * A$ for every right ideal A and every left ideal B of H.*

(3) *$R(A) \cap L(A) \subseteq L(A) * R(A)$ for every $A \in \mathcal{P}^*(H)$.*

(4) *$R(a) \cap L(a) \subseteq L(a) * R(a)$ for every $a \in H$.*

Theorem 2.6. *An hypersemigroup $(H, \circ)$ is intra-regular if and only if, for every fuzzy right ideal f and every fuzzy left ideal g of H, we have $f \wedge g \preceq g \circ f$.*

Proof. $\Longrightarrow$. Let $a \in H$. Since H is intra-regular, there exist $x, y \in H$ such that $a \in (x \circ a) * (a \circ y)$. Then $a \in u \circ v$ for some $u \in x \circ a$, $v \in a \circ y$. Since $a \in u \circ v$, we have $(u, v) \in A_a$, then we have

$$(g \circ f)(a) := \bigvee_{(h,k)\in A_a} \min\{g(h), f(k)\} \geq \min\{g(u), f(v)\}.$$

Since g is a fuzzy left ideal of H, we have $g(x \circ a) \geq g(a)$. Since $u \in x \circ a$, we get $g(u) \geq g(a)$. Since f is a fuzzy right ideal of H, we have $f(a \circ y) \geq f(a)$. Since $v \in a \circ y$, we have $f(v) \geq f(a)$. Thus we have

$$(g \circ f)(a) \geq \min\{g(u), f(v)\} \geq \min\{g(a), f(a)\} = (f \wedge g)(a),$$

thus $f \wedge g \preceq g \circ f$.

$\Longleftarrow$. By Theorem 2.5, it is enough to prove that $R(a) \cap L(a) \subseteq L(a) * R(a)$ for every $a \in H$. Let now $a \in H$ and $b \in R(a) \cap L(a)$. As $f_{R(a)}$ is a fuzzy right ideal and $f_{L(a)}$ is a fuzzy left ideal of H, by hypothesis, we have

$$\Big(f_{R(a)} \wedge f_{L(a)}\Big)(b) \leq \Big(f_{L(a)} \circ f_{R(a)}\Big)(b).$$

Thus

$$\min\{f_{R(a)}(b), f_{L(a)}(b)\} \leq \Big(f_{L(a)} \circ f_{R(a)}\Big)(b).$$

Since $b \in R(a)$ and $b \in L(a)$, we have $f_{R(a)}(b) = f_{L(a)}(b) = 1$, and so

$$1 \leq \Big(f_{L(a)} \circ f_{R(a)}\Big)(b).$$

If $A_b = \emptyset$, then $\Big(f_{L(a)} \circ f_{R(a)}\Big)(b) = 0$ which is impossible. Then we have $A_b \neq \emptyset$ and then

$$\Big(f_{L(a)} \circ f_{R(a)}\Big)(b) = \bigvee_{(y,z) \in A_b} \min\{f_{L(a)}(y), f_{R(a)}(z)\}.$$

Then there exists $(y, z) \in A_b$ such that $y \in L(a)$ and $z \in R(a)$. Then we get $b \in y \circ z \subseteq L(a) * R(a)$. $\square$

The concept of left quasi-regular semigroups can be naturally transferred to hypersemigroups in the definition below.

Definition 2.7. *An hypersemigroup $(H, \circ)$ is called left quasi-regular if for every $a \in H$ there exist $x, y \in H$ such that $a \in (x \circ a) * (y \circ a)$.*

Proposition 2.8. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is left quasi-regular.*

(2) *$a \in H * \{a\} * H * \{a\}$ for every $a \in H$.*

(3) *$A \subseteq H * A * H * A$ for every $A \in \mathcal{P}^*(H)$.*

Theorem 2.9. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is left quasi-regular.*

(2) *$A \cap B \subseteq A * B$ for every ideal A and every nonempty subset B of H.*

(3) *$A \cap B \subseteq A * B$ for every ideal A and every bi-ideal B of H.*

(4) *$A \cap B \subseteq A * B$ for every ideal A and every left ideal B of H.*

(5) *$I(A) \cap L(A) \subseteq I(A) * L(A)$ for every $A \in \mathcal{P}^*(H)$.*

(6) *$I(a) \cap L(a) \subseteq I(a) * L(a)$ for every $a \in H$.*

Proof. (1) $\Longrightarrow$ (2). Let A be an ideal, B a nonempty subset of H and $a \in A \cap B$. Since $a \in H$ and H is left quasi-regular, by Proposition 2.8 (1) $\Rightarrow$ (2), we have

$$a \in H * \{a\} * H * \{a\} = \Big(H * \{a\} * H\Big) * \{a\} \subseteq (H * A * H) * B.$$

We have $H * A * H = (H * A) * H \subseteq A * H$ since A is a left ideal of H and $A * H \subseteq A$ since A is a right ideal of H. Thus we have $H * A * H \subseteq A$. Then $a \in (H * A * H) * B \subseteq A * B$. The implications (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious, and (3) $\Rightarrow$ (4) since the left ideals of H are bi-ideals of H as well.
(6) $\Longrightarrow$ (1). Let $a \in H$. By hypothesis, we have

$$\begin{aligned} a &\in I(a) \cap L(a) \subseteq I(a) * L(a) \\ &= \Big(\{a\} \cup \Big(H * \{a\}\Big) \cup \Big(\{a\} * H\Big) \cup \Big(H * \{a\} * H\Big)\Big) * \Big(\{a\} \cup \Big(H * \{a\}\Big)\Big) \\ &= \Big(\{a\} * \{a\}\Big) \cup \Big(H * \{a\} * \{a\}\Big) \cup \Big(\{a\} * H * \{a\}\Big) \cup \Big(H * \{a\} * H * \{a\}\Big). \end{aligned}$$

If $a \in \{a\} * \{a\}$, then

$$\begin{aligned} a \in \{a\} &\subseteq \{a\} * \{a\} \subseteq \Big(\{a\} * \{a\}\Big) * \Big(\{a\} * \{a\}\Big) \\ &\subseteq H * \{a\} * H * \{a\}. \end{aligned}$$

If $a \in H * \{a\} * \{a\}$, then

$$\begin{aligned} a \in \{a\} &\subseteq H * \{a\} * \Big(H * \{a\} * \{a\}\Big) \subseteq H * \{a\} * (H * H) * \{a\} \\ &\subseteq H * \{a\} * H * \{a\}. \end{aligned}$$

If $a \in \{a\} * H * \{a\}$, then

$$\begin{aligned} \{a\} &\subseteq \{a\} * H * \Big(\{a\} * H * \{a\}\Big) \subseteq (H * H) * \Big(\{a\} * H * \{a\}\Big) \\ &\subseteq H * \{a\} * H * \{a\}. \end{aligned}$$

In each case, $a \in H * \{a\} * H * \{a\}$, so H is left quasi-regular. $\square$

Theorem 2.10. *An hypersemigroup $(H, \circ)$ is left quasi-regular if and only if, for any left ideals A and B of H, we have $A \cap B \subseteq A * B$.*

Proof. $\Longrightarrow$. Let A, B be left ideals of H and $a \in A \cap B$. Since H is left quasi-regular, there exist $x, y \in H$ such that

$$\begin{aligned} a \in (x \circ a) * (y \circ a) &= \{x\} * \{a\} * \{y\} * \{a\} \\ &\subseteq (H * A) * (H * B) \subseteq A * B. \end{aligned}$$

$\Longleftarrow$. Let $A \in \mathcal{P}^*(H)$. Then $A \subseteq H * A * H * A$. In fact, by hypothesis, we have

$$A \subseteq L(A) = L(A) \cap L(A) \subseteq L(A) * L(A) = \Big(A \cup (H * A)\Big) * \Big(A \cup (H * A)\Big)$$

$$= (A * A) \cup (H * A * A) \cup (A * H * A) \cup (H * A * H * A).$$

Then we have

$$\begin{aligned} A * A &\subseteq (A * A * A) \cup (A * H * A * A) \cup (A * A * H * A) \cup (A * H * A * H * A) \\ &\subseteq A * H * A, \end{aligned}$$

from which $H * A * A \subseteq H * A * H * A$. Thus we obtain

$$A \subseteq (A * H * A) \cup (H * A * H * A),$$

then we get

$$\begin{aligned} A * H * A &\subseteq (A * H * A * H * A) \cup (H * A * H * A * H * A) \\ &\subseteq H * A * H * A, \end{aligned}$$

so $A \subseteq H * A * H * A$, and H is left quasi-regular. $\square$

A subset A of an hypergroupoid $(H, \circ)$ is called *idempotent* if $A * A = A$.

Theorem 2.11. *An hypersemigroup $(H, \circ)$ is left quasi-regular if and only if the left ideals of H are idempotent.*

Proof. $\Longrightarrow$. If L is a left ideal of H then, by Theorem 2.10, we have $L \subseteq L * L \subseteq H * L \subseteq L$, so $L * L = L$.
$\Longleftarrow$. By Theorem 2.9, it is enough to prove that for every ideal A and every left ideal B of H, we have $A \cap B \subseteq A * B$. Let now A be an ideal and B a left ideal of H. The set $A \cap B$ is a nonempty subset of H and $H * (A \cap B) \subseteq (H * A) \cap (H * B) \subseteq A \cap B$, so the set $A \cap B$ is a left ideal of H. By hypothesis, we have $A \cap B = (A \cap B) * (A \cap B) \subseteq A * B$. $\square$

Theorem 2.12. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is left quasi-regular.*

(2) *$f \wedge g \preceq f \circ g$ for every fuzzy ideal f and every fuzzy subset g of H.*

(3) *$f \wedge g \preceq f \circ g$ for every fuzzy ideal f and every fuzzy bi-ideal g of H.*

(4) *$f \wedge g \preceq f \circ g$ for every fuzzy ideal f and every fuzzy left ideal g of H.*

Proof. (1) $\Longrightarrow$ (2). Let f be a fuzzy ideal, g a fuzzy subset of H and $a \in H$. Since H is left quasi-regular, there exist $x, y \in H$ such that $a \in \Big((x \circ a) * \{y\}\Big) * \{a\}$. Then $a \in u \circ a$ for some $u \in (x \circ a) * \{y\}$. In addition, $u \in v \circ y$ for some $v \in x \circ a$. On the other hand, since $(u, a) \in A_a$, we have

$$(f \circ g)(a) := \bigvee_{(h,k) \in A_a} \min\{f(h), g(k)\} \geq \min\{f(u), g(a)\}.$$

Since f is a fuzzy right ideal of H, we have $f(v \circ y) \geq f(v)$ and since $u \in v \circ y$, we have $f(u) \geq f(v)$. Since f is a fuzzy left ideal of H, we have $f(x \circ a) \geq f(a)$ and since $v \in x \circ a$, we have $f(v) \geq f(a)$. Thus we have $f(u) \geq f(a)$, and

$$(f \circ g)(a) \geq \min\{f(a), g(a)\} = (f \wedge g)(a),$$

so $f \wedge g \preceq f \circ g$.
The implication (2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (4) since every fuzzy left ideal is a fuzzy bi-ideal of H.
(4) $\Longrightarrow$ (1). By Theorem 2.9, it is enough to prove that $I(a) \cap L(a) \subseteq I(a) * L(a)$ for every $a \in H$. Let now $a \in H$ and $b \in I(a) \cap L(a)$. Since $I(a)$ is an ideal of H, the characteristic function $f_{I(a)}$ is a fuzzy ideal of H and since $L(a)$ is a left ideal of H, $f_{L(a)}$ is a fuzzy left ideal of H. By hypothesis, we have $f_{I(a)} \wedge f_{L(a)} \preceq f_{I(a)} \circ f_{L(a)}$, and so $\Big(f_{I(a)} \wedge f_{L(a)}\Big)(b) \leq \Big(f_{I(a)} \circ f_{L(a)}\Big)(b)$, that is

$$\min\{f_{I(a)}(b), f_{L(a)}(b)\} \leq \Big(f_{I(a)} \circ f_{L(a)}\Big)(b).$$

Since $b \in I(a)$, we have $f_{I(b)}(b) = 1$ and since $b \in L(a)$, we have $f_{L(b)}(b) = 1$, so $\min\{f_{I(a)}(b), f_{L(a)}(b)\} = 1$, and so $1 \leq \Big(f_{I(a)} \circ f_{L(a)}\Big)(b)$. If $A_b = \emptyset$, then $\Big(f_{I(a)} \circ f_{L(a)}\Big)(b) = 0$ which is impossible. Thus we have $A_b \neq \emptyset$. Then

$$\Big(f_{I(a)} \circ f_{L(a)}\Big)(b) = \bigvee_{(y,z) \in A_b} \min\{f_{I(a)}(y), f_{L(a)}(z)\}.$$

Then there exists $(y, z) \in A_b$ such that $y \in I(a)$ and $z \in L(a)$ $(*)$
Indeed: Suppose there is no $(y, z) \in A_b$ such that $y \in I(a)$ and $z \in L(a)$. Then, for every $(y, z) \in A_b$ we have $y \notin I(a)$ or $z \notin L(a)$. Then, for each $(y, z) \in A_b$ we have $f_{I(a)}(y) = 0$ or $f_{L(a)}(z) = 0$, so for each $(y, z) \in A_b$, we have $\min\{f_{I(a)}(y), f_{L(a)}(z)\} = 0$, then $\Big(f_{I(a)} \circ f_{L(a)}\Big)(b) = 0$ which is no possible.
By $(*)$, we have $b \in y \circ z \subseteq I(a) * L(a)$, and the proof is complete. □

Theorem 2.13. *An hypersemigroup $(H, \circ)$ is left quasi-regular if and only if for any fuzzy left ideals f and g of H, we have $f \wedge g \preceq f \circ g$.*

Proof. $\Longrightarrow$. Let f and g be fuzzy left ideals of H and $a \in H$. Then $(f \wedge g)(a) \leq (f \circ g)(a)$. Indeed: By hypothesis, we have $a \in (x \circ a) * (y \circ a)$, so we have $a \in u \circ v$ for some $u \in x \circ a$, $v \in y \circ a$. Since $(u, v) \in A_a$, we have

$$(f \circ g)(a) = \bigvee_{(h,k) \in A_a} \min\{f(h), g(k)\} \geq \min\{f(u), g(v)\}.$$

Since f is a fuzzy left ideal of H, we have $f(x \circ a) \geq f(a)$ and since $u \in x \circ a$, we have $f(u) \geq f(a)$. Since g is a fuzzy left ideal of H, we have $g(y \circ a) \geq g(a)$ and since $v \in y \circ a$, we have $g(v) \geq g(a)$. Hence we obtain

$$(f \circ g)(a) \geq \min\{f(a), g(a)\} = (f \wedge g)(a).$$

$\Longleftarrow$. By Theorem 2.11, it is enough to prove that the left ideals of H are idempotent. Let now A be a left ideal of H and $a \in A$. Since f_A is a fuzzy left ideal of H, by hypothesis, we have $f_A = f_A \wedge f_A \preceq f_A \circ f_A$. Then $1 = f_A(a) \le f_A \circ f_A)(a)$. If $A_a = \emptyset$, then $(f_A \circ f_A)(a) = 0$ which is impossible. Thus we have $A_a \neq \emptyset$ and

$$(f_A \circ f_A)(a) := \bigvee_{(h,k)\in A_a} \min\{f_A(h), f_A(k)\}.$$

Then there exists $(y, z) \in A_a$ such that $y \in A$ and $z \in A$. Then we have $a \in y \circ z \subseteq A * A$, so $A \subseteq A * A \subseteq H * A \subseteq A$, thus $A * A = A$, and A is idempotent. $\square$

Theorem 2.14. *An hypersemigroup $(H, \circ)$ is left quasi-regular if and only if the fuzzy left ideals of H are idempotent.*

Proof. $\Longrightarrow$. Let f be a fuzzy left ideal of H and $a \in H$. Then $(f \circ f)(a) \le f(a)$. In fact, if $A_a = \emptyset$, then $(f \circ f)(a) = 0 \le f(a)$. Let $A_a \neq \emptyset$. Then

$$(f \circ f)(a) = \bigvee_{(x,y)\in A_a} \min\{f(x), f(y)\}.$$

On the other hand,

$$\min\{f(x), f(y)\} \le f(a) \text{ for every } (x, y) \in A_a.$$

Indeed: Let $(x, y) \in A_a$. Since f is a fuzzy left ideal of H we have $f(x \circ y) \ge f(y)$, and since $a \in x \circ y$, we have $f(a) \ge f(y) \ge \min\{f(x), f(y)\}$. Thus we get $(f \circ f)(a) \le f(a)$. Moreover $f \preceq f \circ f$. Indeed: Let $a \in H$. Since H is left quasi-regular, there exist $x, y \in H$ such that $a \in (x \circ a) * (y \circ a)$. Then $a \in u \circ v$ for some $u \in x \circ a$, $v \in y \circ a$. Since $(u, v) \in A_a$, we have

$$(f \circ f)(a) := \bigvee_{(h,k)\in A_a} \min\{f(h), f(k)\} \ge \min\{f(u), f(v)\}.$$

Since f is a fuzzy left ideal of H, we have $f(x \circ a) \ge f(a)$ and since $u \in x \circ a$, we get $f(u) \ge f(a)$. Again since f is a fuzzy left ideal of H, we have $f(y \circ a) \ge f(a)$ and since $v \in y \circ a$, we get $f(v) \ge f(a)$. Thus we have

$$(f \circ f)(a) \ge \min\{f(a), f(a)\} = f(a),$$

so $f \preceq f \circ f$, and f is idempotent.
$\Longleftarrow$. By Theorem 2.11, it is enough to prove that the left ideals of H are idempotent. Let now A be a left ideal of H and $a \in A$. Since f_A is a fuzzy left ideal of H, by hypothesis, we have $f_A = f_A \circ f_A$, thus $1 = f_A(a) = (f_A \circ f_A)(a)$. Then $1 \le (f_A \circ f_A)(a)$ and for the rest of the proof we refer to the proof of the $\Leftarrow$-part of the previous theorem. $\square$

The concept of right quasi-regular semigroups is naturally transferred to hypersemigroups in the following definition.

Definition 2.15. *An hypersemigroup $(H, \circ)$ is called right quasi-regular if for every $a \in H$ there exist $x, y \in H$ such that $a \in (a \circ x) * (a \circ y)$.*

Proposition 2.16. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is right quasi-regular.*

(2) $a \in \{a\} * H * \{a\} * H$ *for every* $a \in H$.

(3) $A \subseteq A * H * A * H$ *for every* $A \in \mathcal{P}^*(H)$.

The right analogues of Theorems 2.9–2.14 also hold and we have the following:

Theorem 2.17. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is right quasi-regular.*

(2) $A \cap B \subseteq A * B$ *for every nonempty subset A and every ideal B of H.*

(3) $A \cap B \subseteq A * B$ *for every bi-ideal A and every ideal B of H.*

(4) $A \cap B \subseteq A * B$ *for every right ideal A and every ideal B of H.*

(5) $R(A) \cap I(A) \subseteq R(A) * I(A)$ *for every* $A \in \mathcal{P}^*(H)$.

(6) $R(a) \cap I(a) \subseteq R(a) * I(a)$ *for every* $a \in H$.

Let us prove the implication (1) $\Rightarrow$ (2): Let A be a nonempty subset of H, B an ideal of H and $a \in A \cap B$. Since H is right quasi-regular, we have

$$a \in \{a\} * H * \{a\} * H \subseteq A * (H * B * H) \subseteq A * B.$$

□

Theorem 2.18. *An hypersemigroup $(H, \circ)$ is right quasi-regular if and only if, for any right ideals A and B of H, we have $A \cap B \subseteq A * B$.*

Theorem 2.19. *An hypersemigroup $(H, \circ)$ is right quasi-regular if and only if the right ideals of H are idempotent.*

Theorem 2.20. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is right quasi-regular.*

(2) $f \wedge g \preceq f \circ g$ *for every fuzzy subset f and every fuzzy ideal g of H.*

(3) $f \wedge g \preceq f \circ g$ *for every fuzzy bi-ideal f and every fuzzy ideal g of H.*

(4) $f \wedge g \preceq f \circ g$ *for every fuzzy right ideal f and every fuzzy ideal g of H.*

Theorem 2.21. *An hypersemigroup $(H, \circ)$ is right quasi-regular if and only if for any fuzzy right ideals f and g of H, we have $f \wedge g \preceq f \circ g$.*

A fuzzy subset f of an hypergroupoid is called *idempotent* if $f \circ f = f$.

Theorem 2.22. *An hypersemigroup $(H, \circ)$ is right quasi-regular if and only if the fuzzy right ideals of H are idempotent.*

The concept of semisimple semigroups can be naturally transferred to hypersemigroups as follows:

Definition 2.23. *An hypersemigroup $(H, \circ)$ is called semisimple if for every $a \in H$ there exist $x, y, z \in H$ such that $a \in (x \circ a) * (y \circ a) * \{z\}$.*

Proposition 2.24. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is semisimple.*

(2) *$a \in H * \{a\} * H * \{a\} * H$ for every $a \in H$.*

(3) *$A \subseteq H * A * H * A * H$ for every $A \in \mathcal{P}^*(H)$.*

Let us prove the implication (3) $\Rightarrow$ (1): Let $a \in H$. By (3), we have $\{a\} \subseteq \Big(\big(H * \{a\}\big) * \big(H * \{a\}\big)\Big) * H$. Then

$$a \in x \circ z \text{ for some } x \in \big(H * \{a\}\big) * \big(H * \{a\}\big),\ z \in H.$$

Then $x \in u \circ v$ for some $u, v \in H * \{a\}$, $u \in x \circ a$ for some $x \in H$ and $v \in y \circ a$ for some $y \in H$. Then

$$a \in x \circ z,\ x \in u \circ v,\ u \in x \circ a,\ v \in y \circ a,\ z \in H.$$

Thus we have

$$\begin{aligned} a \in x \circ z &= \{x\} * \{z\} \subseteq (u \circ v) * \{z\} = \{u\} * \{v\} * \{z\} \\ &\subseteq (x \circ a) * (y \circ a) * \{z\}. \end{aligned}$$

Since $x, y, z \in H$ and $a \in (x \circ a) * (y \circ a) * \{z\}$, H is semisimple and condition (1) is satisfied. $\square$

Theorem 2.25. *Let $(H, \circ)$ be an hypersemigroup. The following are equivalent:*

(1) *H is semisimple.*

(2) *The ideals of H are idempotent.*

(3) *$A \cap B = A * B$ for all ideals A, B of H.*

(4) *$I(A) = I(A) * I(A)$ for every $A \in \mathcal{P}^*(H)$.*

(5) *$I(a) = I(a) * I(a)$ for every $a \in H$.*

Proof. (1) $\Longrightarrow$ (2). Let A be an ideal of H. Since $A \in \mathcal{P}^*(H)$ and H is semisimple, by Proposition 2.24, we have

$$\begin{aligned} A &\subseteq (H * A) * H * (A * H) \subseteq A * H * A = (A * H) * A \subseteq A * A \\ &\subseteq A * H \subseteq A, \end{aligned}$$

so $A * A = A$, and A is idempotent.
(2) $\Longrightarrow$ (3). Let A, B be ideals of H. Then $A * B \subseteq A * H \subseteq A$ and $A * B \subseteq H * B \subseteq B$, so $A * B \subseteq A \cap B$. On the other hand, $A \cap B$ is an ideal of H and, by hypothesis, we have $A \cap B = (A \cap B) * (A \cap B) \subseteq A * B$. Thus we have $A \cap B = A * B$.
The implications (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious.
(5) $\Longrightarrow$ (1). Exactly as in the Lemma 2 in [3], we prove that

$$I(a) = I(a) * I(a) * I(a) * I(a) * I(a)$$

and that

$$I(a) * I(a) * I(a) * I(a) * I(a) \subseteq H * \{a\} * H * \{a\} * H.$$

Then we get $a \in H * \{a\} * H * \{a\} * H$, and H is semisimple. □

Proposition 2.26. *Let $(H, \circ)$ be an hypersemigroup. Then we have the following:*

(1) *If H is regular, then it is left and right quasi-regular.*

(2) *If H is left (or right) quasi-regular, then it is semisimple.*

(3) *If H is intra-regular, then it is semisimple.*

Proof. (1) Let H be regular and $A \in \mathcal{P}^*(H)$. Then we have

$$\begin{aligned} A &\subseteq A * H * A \subseteq A * H * (A * H * A) \subseteq (H * H) * (A * H * A) \\ &\subseteq H * A * H * A, \end{aligned}$$

so H is left quasi-regular. Similarly H is right quasi-regular.
(2) Let H be left quasi-regular and $A \in \mathcal{P}^*(H)$. Then we have

$$\begin{aligned} A &\subseteq H * A * H * A \subseteq H * (H * A * H * A) * (H * A) \\ &= (H * H) * (A * H * A) * (H * A) \\ &\subseteq (H * H) * (A * H * A) * (H * H) \\ &\subseteq H * A * H * A * H, \end{aligned}$$

thus H is semisimple. If H is right quasi-regular, the proof is analogous.
(3) Let H be intra-regular and A a nonempty subset of H. Then we have

$$\begin{aligned} A &\subseteq H * A * A * H \subseteq H * (H * A * A * H) * A * H \\ &= (H * H) * A * (A * H) * A * H \\ &\subseteq (H * H) * A * (H * H) * A * H \\ &\subseteq H * A * H * A * H, \end{aligned}$$

and H is semisimple. □

Theorem 2.27. *Let H be an hypersemigroup. The following are equivalent:*

(1) *H is semisimple.*

(2) *For every fuzzy ideals f and g of H, we have $f \wedge g = f \circ g$.*

(3) *For every fuzzy ideal f of H, we have $f = f \circ f$.*

Proof. (1) $\Longrightarrow$ (2). Let f and g be fuzzy ideals of H. Since f is a fuzzy right ideal and g is a fuzzy left ideal of H, by Proposition 2.2, we have $f \circ g \preceq f \wedge g$. Let now $a \in H$. Then $(f \wedge g)(a) \leq (f \circ g)(a)$. In fact: Since H is semisimple, there exist $x, y, z \in H$ such that $a \in (x \circ a) * (y \circ a) * \{z\}$. Then there exist $u \in x \circ a$ and $v \in (y \circ a) * \{z\}$ such that $a \in u \circ v$. Since $v \in (y \circ a) * \{z\}$, there exists $w \in y \circ a$ such that $v \in w \circ z$. Thus we have

$$u \in x \circ a, \ a \in u \circ v, \ w \in y \circ a \text{ and } v \in w \circ z.$$

Since $a \in u \circ v$, we have $(u, v) \in A_a$. Since $(u, v) \in A_a$, we have

$$(f \circ g)(a) := \bigvee_{(h,k) \in A_a} \min\{f(h), g(k)\} \geq \min\{f(u), g(v)\}.$$

Since f is a fuzzy left ideal of H, we have $f(x \circ a) \geq f(a)$ and since $u \in x \circ a$, we have $f(u) \geq f(a)$. Since g is a fuzzy right ideal of H, we have $g(w \circ z) \geq g(w)$ and since $v \in w \circ z$, we have $g(v) \geq g(w)$. Since g is a fuzzy left ideal of H, we have $g(y \circ a) \geq g(a)$ and since $w \in y \circ a$, we have $g(w) \geq g(a)$. Thus we get $g(v) \geq g(a)$. Hence we obtain

$$(f \circ g)(a) \geq \min\{f(a), g(a)\} = (f \wedge g)(a),$$

so $f \wedge g \preceq f \circ g$.
The implication (2) $\Rightarrow$ (3) is obvious.
(3) $\Longrightarrow$ (1). Let $a \in H$. We prove that $I(a) \subseteq I(a) * I(a)$. Then, since $I(a)$ is an ideal of H, we have $I(a) = I(a) * I(a)$ and, by Theorem 2.25, H is semisimple. Let now $b \in I(a)$. Then $b \in I(a) * I(a)$. In fact: Since $I(a)$ is an ideal of H, the characteristic function $f_{I(a)}$ is a fuzzy ideal of H. By hypothesis, we have $f_{I(a)} = f_{I(a)} \circ f_{I(a)}$, then $f_{I(a)}(b) = \Big(f_{I(a)} \circ f_{I(a)}\Big)(b)$. Since $b \in I(a)$, we have $f_{I(a)}(b) = 1$, then $1 = \Big(f_{I(a)} \circ f_{I(a)}\Big)(b)$. If $A_b = \emptyset$, then $\Big(f_{I(a)} \circ f_{I(a)}\Big)(b) = 0$ which is impossible. Thus we have $A_b \neq \emptyset$ and

$$\Big(f_{I(a)} \circ f_{I(a)}\Big)(b) = \bigvee_{(y,z) \in A_b} \min\{f_{I(a)}(y), f_{I(a)}(z)\}.$$

Then there exists $(y, z) \in A_b$ such that $y \in I(a)$ and $z \in I(a)$ (otherwise, $\Big(f_{I(a)} \circ f_{I(a)}\Big)(b) = 0$ which is impossible). Therefore, we have

$$b \in y \circ z \subseteq I(a) * I(a), \text{ and then } b \in I(a) * I(a).$$

□

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