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Nullity of Corona of a Path with Smith Graphs

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Abstract. Let G be a graph and $A(G)$ be its adjacency matrix. The nullity of graph is the presence of zero as an eigenvalue in the spectrum of G. In this paper, we have established the results on nullity of $(P_n \odot S_m)$ where S_m is smith graph and $\odot$ is corona product. Moreover we have shown that nullity of $(P_n \odot S_m)$ depends upon the nullity of S_m, which comes out to be a multiple of nullity of S_m.

2010 Mathematics Subject Classifications: 05C50

Key Words and Phrases: smith graphs, eigenvalue, nullity, corona product

1. Introduction and Preliminaries

For all terminology and notations in graph theory and spectral graph theory not especially defined in this paper, we refer the reader to the standard text books [4] and [1] respectively. By a graph we mean finite, simple, connected and undirected graph. The eigenvalues of a graph G is the eigenvalues of its adjacency matrix. The nullity of a graph G is the presence of zero as an eigenvalue in the spectrum of a graph G. It is denoted by $\eta(G)$. Firstly we recall some basic definitions and existing results from [5].

A function $f : V(G) \to \mathbb{R}$ where $\mathbb{R}$ is the set of real numbers, which assigns a weight (real number) to each vertex of G is called a vertex weighting of graph G. If there exist at least one vertex $v \in V(G)$ for which $f(v) \neq 0$, then it is called non trivial weighting.

A non-trivial vertex weighting of a graph G is called a zero-sum weighting of a graph G if for each $v \in V(G)$, $\sum f(u) = 0$, where $\sum$ is taken to all neighbor v.
For any two non zero real number a and b the zero-sum weighting of a graph G is shown in Figure 1.

The maximum number of non-zero independent variables used in a zero-sum weighting is called a high zero-sum weighting of the graph.

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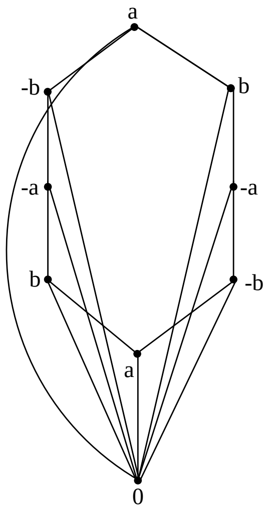

Figure 1: G.

In a high zero-sum weighting of G, the maximum number of non-zero independent variables is equal to the nullity of G.
For a graph G, shown in Figure 1, we have used two non-zero independent variables a and b for high zero-sum weighting of G. In view of the above definition, we conclude that $\eta(G) = 2$. For a connected graph, two non-adjacent vertices are said to be co-neighbor vertices if they have same set of neighbors.

Let G_1 and G_2 be two graphs with vertex set $V(G_1) = \{v_1, v_2, \ldots, v_{p_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{p_2}\}$, respectively. Then, the corona of G_1 and G_2, denoted by $G_1 \odot G_2$ is defined as take one copy of G_1 and p_1 copies of G_2 by adjoining i^{th} vertex of G_1 to each vertex of G_2 in i^{th} copy. The following Lemma is important in the study of nullity of a graph G and is known as co-neighbor lemma.

Lemma 1. Let G be a connected graph and v_i and v_j be two co-neighbor vertices of G. Then, $\eta(G) = \eta(G - v_i) + 1 = \eta(G - v_j) + 1$.

A graph is called smith if one of its eigenvalue is 2 and a smith graph on m vertices is denoted by S_m. Upto isomorphic, there are precisely 6 kinds of smith graphs namely $W_m; m \geq 6$ (double head snake graph), $C_m; m \geq 3$ (cycle graph), H_7, H_8, H_9 and $K_{1,4}$. From [2] except $K_{1,4}$ other smith graphs ($W_m; m \geq 6$, $C_m; m \geq 3$, H_7, H_9) are extended form of Dynkin graphs ($\tilde{D_m}, \tilde{A_m}, \tilde{E_6}, \tilde{E_7}$) and H_8 is Dynkin graph E_8.

The concept of nullity is very much applicable for the stability of unsaturated conjugate hydrocarbons molecules by Huckel molecular orbital theory (HMO) [3]. According to HMO theory, following two cases occurs:

(i) If $\eta(G) > 0$, then the isomorphic chemical molecule is more reactive and unstable.

(ii) If $\eta(G) = 0$, then the isomorphic chemical molecule is stable and less reactive.

Motivated by the earlier study on the nullity of a graph. Here, we have determined the nullity of corona of a path with smith graphs.

2. Main Results

In this section, we study the nullity of corona of a path with smith graphs.

Lemma 2. For a smith graphs S_m, nullity is given by

(i)

$$\eta(C_m) = \begin{cases} 2, & m \equiv 0 \ (mod\ 4) \\ 0, & otherwise \end{cases}$$

(ii)

$$\eta(W_m) = \begin{cases} 3, & if\ m\ is\ odd \\ 2, & otherwise \end{cases}$$

(iii)

$$\eta(K_{1,4}) = 3$$

(iv)

$$\eta(H_7) = 1$$

(v)

$$\eta(H_8) = 0$$

(vi)

$$\eta(H_9) = 1.$$

Proof. We will prove the entire result by weighting technique.

(i) Firstly we assume that S_m be C_m; $m \geq 3$. There are two cases *viz.* $m \equiv 0 \ (mod\ 4)$ and $m \not\equiv 0 \ (mod\ 4)$.
For $m \equiv 0 \ (mod\ 4)$, let x_i be the weights of vertices of C_m. We have the following conditions

$$\sum_{w \in N_{C_m}(v)} f(w) = 0, \ \forall\ v \in V(C_m).$$

On solving the equation we get

$$x_1 = x_5 = \dots x_{m-3} = a_1(say)$$

$$x_3 = x_7 = \dots = x_{m-1} = -a_1$$

and

$$x_2 = x_6 = \dots x_{m-2} = a_2(say)$$

$$x_4 = x_8 = \dots x_m = -a_2.$$

We have used two non-zero independent variables a_1 and a_2 in a zero-sum weighting of C_m. Therefore, $\eta(C_m) = 2$.
For $m \not\equiv 0 \ (mod\ 4)$, we have used same procedure as above. After solving we get solution

$$x_1 = x_2 = \dots x_m = 0.$$

Therefore, $\eta(C_m) = 0$.
Hence, by the above two cases

$$\eta(C_m) = \begin{cases} 2, & m \equiv 0 \ (mod\ 4) \\ 0, & otherwise \end{cases}$$

(ii) Let S_m to be W_m; $m \geq 6$. We tackle following cases:
Case (i) If m is even, then we have used two independent variables $a_1 \neq 0, a_2 \neq 0$ in a zero-sum weighting of W_m. Therefore, $\eta(W_m) = 2$.
Case (ii) If m is odd, then we have used three non-zero independent variables in a zero-sum weighting of W_m. Therefore, $\eta(W_m) = 3$. Hence,

$$\eta(W_m) = \begin{cases} 3, & if\ m\ is\ odd \\ 2, & otherwise \end{cases}$$

(iii) Let S_m to be isomorphic to $K_{1,4}$. Let x_i, $\forall\ i = 1, 2, \dots, 5$ be the weights of vertices respectively. Then, we have following conditions

$$\sum_{w \in N_{K_{1,4}}\ (v)} f(w) = 0, \ \forall\ v \in V(K_{1,4}).$$

$$\sum_{i=2}^{5} x_i = 0$$

and

$$x_1 = 0, \ \forall\ x_i; i = 2, 3, 4, 5.$$

After solving these equations, we get $x_2 = a_1$, $x_3 = a_2$, $x_4 = a_3$ and $x_5 = -(a_1 + a_2 + a_3)$. Here, we have used three non-zero independent variables in a zero-sum weighting of $K_{1,4}$. Therefore, $\eta(K_{1,4}) = 3$.

(iv) Let us take S_m to be H_7. In zero-sum weighting of H_7, we have used only one non-zero independent variables. Therefore, $\eta(H_7) = 1$.

(v) We suppose that S_m be H_8. We have not found the non-zero independent variables for zero-sum weighting of H_8. Hence, $\eta(H_8) = 0$.

(vi) If we take S_m be H_9, then we have used one non-zero independent variable in zero-sum weighting of H_9. Hence, $\eta(H_9) = 1$. ■

Theorem 1. Let S_m be any smith graph with m vertices and let $\eta(S_m)$ denote the nullity of S_m. Then the nullity of smith graph S_m belongs to the set $\{0, 1, 2, 3\}$.
Proof. The proof of the result can be given by Lemma 2. ■

Corollary 1. The converse of Theorem 2. does not hold in general, i.e. $\eta(G) \in \{0, 1, 2, 3\}$ then G need not be a smith graph.
As for instance, the nullity of both P_n and $K_n \in (0, 1)$ however none of them is smith.

Remark. It is interesting to note here that we can not have a graph as from [3] $\eta(G) = n$ if and only if G is a null graph.
Thus the following problem arises.
Problem: For a given n, does there exist a graph of order $p > n$, such that $\eta(G) = n$.

We answer to this problem in affirmative due to the following theorem:

Theorem 2. Let $(P_n \odot S_m)$ be the corona of a path with S_m, where S_m is H_9. Then $\eta(P_n \odot S_m) = n$.
Proof. Let us consider smith graph S_m to be H_9 and let the vertices of P_n are $v_1, v_2, v_3, \ldots v_n$ and vertices of H_9 are $u_1, u_2, u_3, \ldots u_9$ in a usual manner as shown in Figure 2.
The corona of P_n with H_9 has vertex set $V^i(G) = \{u_{ij}, v_i : \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, 9\}$.

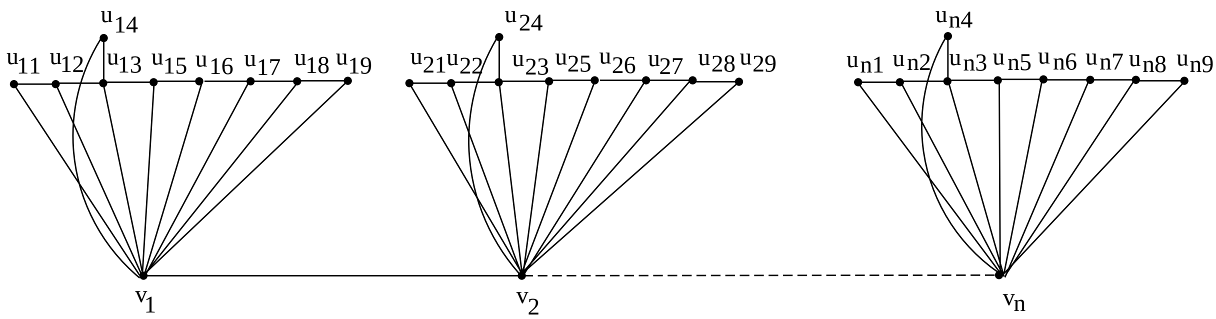

Figure 2: $(P_n \odot H_9)$

Let x_{ij} and y_i be weights of the vertices of $(P_n \odot H_9)$ as indicated in Figure 3.
Then,

$$\sum_{w \in N_{(P_n \odot H_9)}(v)} f(w) = 0, \ \forall \ v \in V(P_n \odot H_9).$$

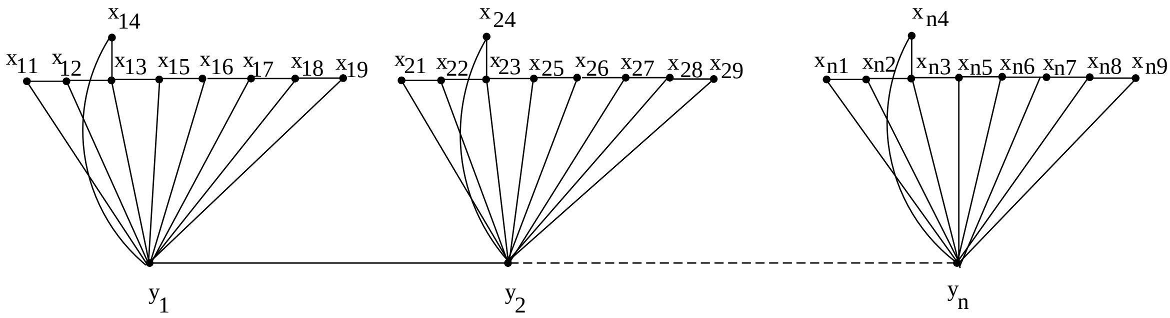

Figure 3: $(P_n \odot H_9)$

These equations possess solution if and only if

$$x_{nj} = 0; j = 1, 2, 3, 6, 8$$

and

$$x_{n4} = x_{n7} = a_1; x_{n5} = x_{n9} = -a_1.$$

Clearly, we have used n independent variable in a zero-sum weighting of $(P_n \odot H_9)$. Therefore, $\eta(P_n \odot H_9) = n$.
Hence from the above discussion it is clear that $\eta(P_n \odot S_m) = n$, where S_m is H_9. ■

Theorem 3. Let $(P_n \odot S_m)$ denotes the corona of a path with S_m. Then $\eta(P_n \odot S_m) \in \{0, 2n, 3n\}$, where S_m is either W_m or $K_{1,4}$ or C_m or H_7 or H_8.
Proof. We will prove the entire result for each of the smith graph separately. First consider S_m to be $W_m; m \geq 6$ to the vertices of $(P_n \odot S_m)$. We need to tackle two cases for m, *viz.*, $m = 4k + 1$ and $m \neq 4k + 1$ where $k = 2, 3, \ldots$.
Let $m = 4k+1$ on applying co-neighbor lemma. In this case, we have 2 pairs of co-neighbor vertices in each copy of W_m, it means that we have to remove 2 vertices in each copy of W_m. It implies that total $2n$ vertices have been removed from $(P_n \odot W_m)$. Thus we get $\eta(P_n \odot W_m) = \eta(P_n \odot W'_m) + 2n$, where $W'_m = P_{m-2}; m - 2 = 4k - 1$ and $k = 2, 3, \ldots$. Therefore, we conclude that $\eta(P_n \odot W_m) = 3n$.
Next, let $m \neq 4k + 1$, using the same procedure as above, we conclude that $\eta(P_n \odot W_m) = 2n$. Hence. We get

$$\eta(P_n \odot W_m) = \begin{cases} 3n, & m = 4k + 1, \ where \ k = 2, 3, \ldots \\ 2n, & otherwise \end{cases}$$

Consider S_m to be $K_{1,4}$. The co-neighbor vertices of $(P_n \odot K_{1,4})$ are (u_2, u_3), (u_3, u_4), (u_4, u_5) in each copy. On applying co-neighbor lemma, we remove three vertices from each copy. Then the nullity of $(P_n \odot K_{1,4}) = \eta(P_n \odot K_2) + 3n$. Hence, we conclude that $\eta(P_n \odot K_{1,4}) = 3n$. Let S_m to be C_m. Here we need to tackle two cases for m, *viz.* $m \equiv 0 \ (mod\ 4)$ and $m \not\equiv 0 \ (mod\ 4)$.
Case (i). For $m \equiv 0 \ (mod\ 4)$ we will find the nullity of $(P_n \odot C_m)$. We assume that

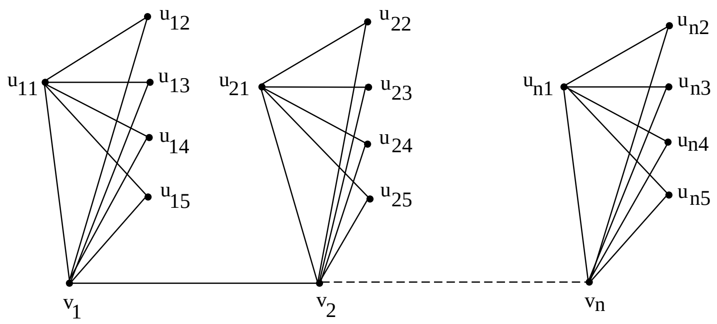

Figure 4: $(P_n \odot K_{1,4})$

$u_{ij} = x_{ij}$ and $v_i = y_i$ be weighting of graph $(P_n \odot C_m)$, where ; $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.
Then from

$$\sum_{w \in N_{(P_n \odot C_m)(v)}} f(w) = 0, \ \forall \ v \in V(P_n \odot C_m).$$

We get equations, and after solving these equations, we have used 2 non-zero variables in each copy of C_m. It means that we have to use $2n$ independent variables, for a zero- sum weighting of $(P_n \odot C_m)$.
Case (ii). For $m \not\equiv 0$ $(mod\ 4)$. We found that no non-zero independent variables in a zero-sum weighting of $(P_n \odot C_m)$.
From the above cases, we conclude that

$$\eta(P_n \odot C_m) = \begin{cases} 2n, & m \equiv 0 \ (mod\ 4) \\ 0, & m \not\equiv 0 \ (mod\ 4) \end{cases}$$

Finally, let S_m to be either H_7 or H_8 respectively. Using the procedure analogues as done in Theorem 2, we have used no independent variables in zero-sum weighting of $(P_n \odot H_7)$ and $(P_n \odot H_8)$. Therefore, nullity of both the graphs is zero.
Therefore,

$$\eta(P_n \odot H_7) = 0$$

or

$$\eta(P_n \odot H_8) = 0.$$

From the above analysis, it is clear that, $\eta(P_n \odot S_m) \in \{0, 2n, 3n\}$. ■

Theorem 4. Let $(P_n \odot S_m)$ denotes the corona of a path with any smith graph S_m. Then $\eta(P_n \odot S_m) \in \{0, n, 2n, 3n\}$.
Proof. The proof of the result can be given by Theorem 2 and Theorem 3. ■

Now we give the following result which established the connection between nullity of corona of P_n with S_m and nullity of S_m.

Theorem 5. Let $(P_n \odot S_m)$ denotes the corona of a path with smith graph S_m, where S_m is either $K_{1,4}$ or $C_m; m \geq 3$ or H_8 or H_9 or W_m, $m = 4k+5$ or $m = 2k+4$, $k = 1, 2, 3, \dots$. Then $\eta(P_n \odot S_m) = n.\eta(S_m)$, where n is the order of path.

Theorem 6. Let $(P_n \odot S_m)$ denotes the corona of a path with smith graph S_m, where S_m is either H_7 or W_m, $m = 2k+5$, $k = 1, 3, 5, \dots$. Then $\eta(P_n \odot S_m) = n.(\eta(S_m) - 1)$, where n is the order of path.

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