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Inverse Problem for a Parabolic Equation in a Rectangle Domain with Integral Conditions

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Abstract. This paper is devoted to study of the nonlocal inverse boundary-value problem for a second-order parabolic equation. The problem is considered in the rectangular domain. First, we introduce a definition of a classical solution of the stated problem. Then, the initial problem is reduced to an equivalent problem, for which using the method of contraction mappings principle the theorem of the existence and uniqueness of solutions is proved. Moreover, using the equivalency, we prove the existence and uniqueness of classical solution of the original problem.

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1. Introduction

Inverse problems for differential equations are called the problem of finding the unknown coefficients of differential equations, right-hand side, boundary or initial conditions, the border of domain. The unknown elements of the initial-boundary value problems defined for some additional information about solving equations. Such information are different kinds of overdetermination condition [3], [9], [14], [16]. Inverse problems for differential equations of mathematical physics are now playing an important role in the field of natural sciences and their applications [1], [6], [7],[8], [18]. Coefficient inverse problems are the problems in which, together with the solutions of differential equations is unknown and is one (or more) of its coefficients. Many important applied problems relating to diffusion processes, electromagnetic oscillations, elastic deformations, geophysics, seismology, and computed tomography, scattering theory, acoustics, optics, theory of molecular oscillations, radiolocation, gravity, and others, lead to the like inverse problems [2], [5], [11], [4], [15],[13], [12], [17], [19]. In the submitted article the inverse boundary value problem with nonlocal conditions for second order parabolic equation is studied.

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2. Problem Statement

Denote by Q_T the domain $\{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$ and consider the equation

$$c(t)u_t(x,t) = u_{xx}(x,t) + a(t)u(x,t) + b(t)g(x,t) + f(x,t), \quad (x,t) \in Q_T \tag{1}$$

with nonlocal initial condition

$$u(x,0) + \delta u(x,T) = \varphi(x) \quad (0 \le x \le 1), \tag{2}$$

Neumann boundary condition

$$u_x(0,t) = 0 \quad (0 \le t \le T), \tag{3}$$

nonlocal integral condition

$$\int_0^1 u(x,t)dx = 0 \quad (0 \le t \le T), \tag{4}$$

and the additional conditions

$$u(0,t) = h_1(t) \quad (0 \le t \le T), \tag{5}$$

$$u(1,t) = h_2(t) \quad (0 \le t \le T), \tag{6}$$

where $\delta \ge 0$ is a fixed number, $0 < c(t), \quad g(x,t), \quad f(x,t), \quad 0 \le p(t), \quad h_i(t) \quad (i = 1,2)$ are given functions, $u(x,t), a(t)$ and $b(t)$ are unknown functions.

Definition 1. *The triple $\{u(x,t), a(t), b(t)\}$ is said to be a classical solution of problem (1)-(6), if for the functions $u(x,t), a(t)$ and $b(t)$ satisfy the following conditions:*

(i) The function $u(x,t)$ and its derivatives $u_t(x,t), u_x(x,t), u_{xx}(x,t)$ are continuous in the domain Q_T;

(ii) the functions $a(t)$ and $b(t)$ are continuous on the interval $[0,T]$;

(iii) equation (1) and conditions (2)-(6) are satisfied in the usual sense.

To investigate the problem (1)-(6), first consider the following problem:

$$c(t)y'(t) = a(t)y(t) \quad (0 \le t \le T), \tag{7}$$

$$y(0) + \delta y(T) = 0 \tag{8}$$

where $\delta \ge 0$ is a given number $c(t), a(t) \in C[0,T]$ are given functions, and $y = y(t)$ is unknown function. Under the classic solution of problem (7),(8) we understand the function $y(t)$, continuous on the interval $[0,T]$ together with all its derivatives contained in equation (7) satisfying both (7) and (8) in the classical sense.

Lemma 1. *Let $\delta \geq 0, 0 < c(t) \in C[0,T]$, and $a(t) \in C[0,T]$. Then the problem (7), (8) has a unique trivial solution.*

Proof. It's obvious that the general solution of equation (7) has the form:

$$y(t) = ce^{\int\limits_0^t \frac{a(\tau)}{c(\tau)} d\tau}. \tag{9}$$

Using (7) we obtain

$$c\left(1 + \delta e^{\int\limits_0^T \frac{a(t)}{c(t)} dt}\right) = 0.$$

By $\delta \geq 0$, from the latter relation we have $c = 0$. Putting the value of $c = 0$ in (9), we get that the problem (7), (8) has only the trivial solution. The proof is complete.

Theorem 1. *Suppose that*

$$\delta \geq 0, \quad 0 < c(t) \in C[0,T], \quad f(x,t) \in C(Q_T), \quad \int\limits_0^1 f(x,t)dx = 0 \quad (0 \leq t \leq T),$$

$$g(x,t) \in C(Q_T), \quad \int\limits_0^1 g(x,t)dx = 0 \quad (0 \leq t \leq T), \quad h_i(t) \in C^1[0,T], \quad (i = 1,2),$$

$$h(t) \equiv h_1(t)g(1,t) - h_2(t)g(0,t) \neq 0 \quad (0 \leq t \leq T)$$

and the compatibility conditions

$$\int\limits_0^1 \varphi(x)dx = 0, \tag{10}$$

$$h_1(0) + \delta h_1(T) = \varphi(0), \quad h_2(0) + \delta h_2(T) = \varphi(1) \tag{11}$$

hold. Then the problem of finding a classical solution of (1)-(6) is equivalent to the problem of determining functions $u(x,t) \in C^{2,1}(Q_T)$, $a(t) \in C[0,T]$, and $b(t) \in C[0,T]$, satisfying equation (1), conditions (2) and (3), and the conditions

$$u_x(1,t) = 0 \quad (0 \leq t \leq T), \tag{12}$$

$$c(t)h_1'(t) = u_{xx}(0,t) + a(t)h_1(t) + b(t)g(0,t) + f(0,t) \quad (0 \leq t \leq T), \tag{13}$$

$$c(t)h_2'(t) = u_{xx}(1,t) + a(t)h_2(t) + b(t)g(1,t) + f(1,t) \quad (0 \leq t \leq T). \tag{14}$$

Proof. Let $\{u(x,t), a(t), b(t)\}$ be a classical solution of (1)-(6). Integrating both sides of (1) with respect to x from 0 to 1 yields

$$c(t)\frac{d}{dt}\int_0^1 u(x,t)dx = u_x(1,t) - u_x(0,t)$$

$$+a(t)\int_0^1 u(x,t)dx + b(t)\int_0^1 g(x,t)dx + \int_0^1 f(x,t)dx \quad (0 \le t \le T). \tag{15}$$

Under the assumptions $\int_0^1 f(x,t)dx = 0, \quad \int_0^1 g(x,t)dx = 0 \quad (0 \le t \le T)$, by virtue of (3) we conclude that (12) is satisfied.

Setting $x = 0$, in (1) we obtain

$$c(t)u_t(0,t) = u_{xx}(0,t) + a(t)u(0,t) + b(t)g(0,t) + f(0,t) \quad (0 \le t \le T). \tag{16}$$

Similarly, from the equation (1), we get

$$c(t)u_t(1,t) = u_{xx}(1,t) + a(t)u(1,t) + b(t)g(1,t) + f(1,t) \quad (0 \le t \le T). \tag{17}$$

Further, assuming $h_i(t) \in C^1[0,T] \quad (i = 1,2)$ and differentiating (5) and (6), we have

$$u_t(0,t) = h_1(t) \quad (0 \le t \le T), \tag{18}$$

$$u_t(1,t) = h_2(t) \quad (0 \le t \le T) \tag{19}$$

respectively.

From (16), by (5) and (18), we conclude that the relation (13) is fulfilled.

Analogously, from (17), by using the relations (6) and (19), we arrive at the satisfying of (14).

Now, suppose that $\{u(x,t), a(t), b(t)\}$ is the solution of (1)-(3), (12)-(14). Then from (15), by means of (3) and (12), we find

$$c(t)\frac{d}{dt}\int_0^1 u(x,t)dx = a(t)\int_0^1 u(x,t)dx \quad (0 \le t \le T). \tag{20}$$

By virtue of (2) and (10), it is not hard to see that

$$\int_0^1 u(x,0)dx + \delta\int_0^1 u(x,T)dx = \int_0^1 \left(u(x,0) + \delta u(x,T)\right)dx = \int_0^1 \varphi(x)dx = 0. \tag{21}$$

Since, by Lemma 1, the problems (20) and (21) has only a trivial solution, it follows that

$$\int_0^1 u(x,t)dx = 0 \quad (0 \le t \le T),$$

i.e. the condition (4) holds.

Moreover, from (13), (14), (16) and (17) we find

$$c(t)\frac{d}{dt}(u(0,t)-h_1(t)) = a(t)(u(0,t)-h_1(t)) \quad (0 \le t \le T), \tag{22}$$

$$c(t)\frac{d}{dt}(u(1,t)-h_2(t)) = a(t)(u(1,t)-h_2(t)) \quad (0 \le t \le T). \tag{23}$$

Using (2) and the compatibility conditions (11) we have

$$u(0,0)-h_1(0)+\delta(u(0,T)-h_1(0)) = (u(0,0)+\delta u(0,T)) - (h_1(0)+\delta h_1(T))$$
$$= \varphi(0) - (h_1(0)+\delta h_1(T)) = 0, \tag{24}$$
$$u(1,0)-h_1(0)+\delta(u(1,T)-h_1(0)) = (u(1,0)+\delta u(1,T)) - (h_2(0)+\delta h_2(T))$$
$$= \varphi(1) - (h_2(0)+\delta h_2(T)) = 0. \tag{25}$$

From (22), (24), and (23),(25), by Lemma 1, we conclude that conditions (5) and (6) are satisfied. The lemma is thus proved.

3. Solvability of inverse boundary-value problem

Since for $\lambda_k = k\pi \;\; (k = 0, 1, \ldots)$, the system $\{\cos \lambda_k x\}_{k=0}^{\infty}$ form an orthogonal system in $L_2(0,1)$. We'll seek the first component $u(x,t)$ of classical solution $u(x,t), a(t), b(t)$ of the problem (1)-(3), (12)-(14) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi), \tag{26}$$

where

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx \quad (k = 0, 1, 2, ...),$$

and

$$m_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1, 2, ... \end{cases}$$

Then applying the formal scheme of the Fourier method, from (1) and (2) we obtain

$$c(t)u_0'(t) = F_0(t; u, a, b) \quad (0 \le t \le T), \tag{27}$$

$$c(t)u_k'(t) + \lambda_k^2 u_k(t) = F_k(t; u, a, b) \quad (k = 1, 2 \ldots; 0 \le t \le T), \tag{28}$$

$$u_k(0) + \delta u_k(T) = \varphi_k \quad (k = 0, 1, 2 \ldots) \tag{29}$$

where

$$F_k(t; u, a, b) = f_k(t) + b(t)g_k(t) + a(t)u_k(t) \quad (k = 0, 1, 2 \ldots),$$

$$g_k(t) = m_k \int_0^1 g(x,t)\cos\lambda_k x dx \quad (k = 0, 1, 2, \ldots),$$

$$f_k(t) = m_k \int_0^1 f(x,t)\cos\lambda_k x dx \quad (k = 0, 1, 2, \ldots),$$

$$\varphi_k = m_k \int_0^1 \varphi(x)\cos\lambda_k x dx \quad (k = 0, 1, 2, \ldots).$$

Solving problem (27)-(29) we get

$$u_0(t) = (1+\delta)^{-1}\left(\varphi_0 - \delta\int_0^T \frac{1}{c(t)}F_0(t; u, a, b)dt\right) + \int_0^t \frac{1}{c(\tau)}F_0(\tau; u, a, b)d\tau, \tag{30}$$

$$u_k(t) = \frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\varphi_k - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_k(\tau; u, a, b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau$$
$$+\int_0^t \frac{1}{c(\tau)}F_k(\tau; u, a, b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau \quad (k = 1, 2, \ldots). \tag{31}$$

After substituting expressions $u_k(t) \ \ (k = 0, 1, ...)$ in (26), we have

$$u(x,t) = (1+\delta)^{-1}\left(\varphi_0 - \delta\int_0^T \frac{1}{c(t)}F_0(t; u, a, b)dt\right) + \int_0^t \frac{1}{c(\tau)}F_0(\tau; u, a, b)d\tau$$
$$+\sum_{k=1}^{\infty}\left\{\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\varphi_k - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}{1+\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)}ds}}\int_0^T \frac{1}{c(\tau)}F_k(\tau; u, a, b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau\right.$$
$$\left.+\int_0^t \frac{1}{c(\tau)}F_k(\tau; u, a, b)e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)}ds}d\tau\right\}\cos\lambda_k x. \tag{32}$$

Now, using (26), from (13) and (14) we find

$$a(t) = [h(t)]^{-1}\{(c(t)h_1'(t) - f(0,t))g(1,t) - (c(t)h_2'(t) - f(1,t))g(0,t)$$
$$\left.+\sum_{k=1}^{\infty}\lambda_k^2 u_k(t)(g(1,t) - (-1)^k g(0,t))\right\}, \tag{33}$$

$$b(t) = [h(t)]^{-1}\{h_1(t)(c(t)h_2'(t) - f(1,t)) - h_2(t)(c(t)h_1'(t) - f(0,t)) \\ \left. + \sum_{k=1}^{\infty} \lambda_k^2 u_k(t)((-1)^k h_1(t) - h_2(t)) \right\}, \tag{34}$$

where

$$h(t) \equiv h_1(t)g(1,t) - h_2(t)g(0,t) \neq 0 \quad (0 \leq t \leq T). \tag{35}$$

Putting the expression of (31) in (33) and (34) we obtain

$$a(t) = [h(t)]^{-1}\{(c(t)h_1'(t) - f(0,t))g(1,t) - (c(t)h_2'(t) - f(1,t))g(0,t)$$

$$+ \sum_{k=1}^{\infty} \lambda_k^2 \left[\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)} ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)} ds}} \varphi_k - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)} ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)} ds}} \int_0^T \frac{1}{c(\tau)} F_k(\tau; u, a, b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)} ds} d\tau \right.$$

$$\left. + \int_0^t \frac{1}{c(\tau)} F_k(\tau; u, a, b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)} ds} d\tau \right] (g(1,t) - (-1)^k g(0,t)), \tag{36}$$

$$b(t) = [h(t)]^{-1}\{h_1(t)(c(t)h_2'(t) - f(1,t)) - h_2(t)(c(t)h_1'(t) - f(0,t))$$

$$+ \sum_{k=1}^{\infty} \lambda_k^2 \left[\frac{e^{-\int_0^t \frac{\lambda_k^2}{c(s)} ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)} ds}} \varphi_k - \frac{\delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)} ds}}{1 + \delta e^{-\int_0^T \frac{\lambda_k^2}{c(s)} ds}} \int_0^T \frac{1}{c(\tau)} F_k(\tau; u, a, b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)} ds} d\tau \right.$$

$$\left. + \int_0^t \frac{1}{c(\tau)} F_k(\tau; u, a, b) e^{-\int_\tau^t \frac{\lambda_k^2}{c(s)} ds} d\tau \right] ((-1)^k h_1(t) - h_2(t)). \tag{37}$$

Analogously [10], the following lemma was proved.

Lemma 2. *Let* $\{u(x,t), a(t), b(t)\}$ *be an arbitrary solution of (1)-(3), (12)-(14), then the functions*

$$u_k(t) = m_k \int_0^1 u(x,t) \cos \lambda_k x dx \quad (k = 0, 1, 2, \ldots)$$

satisfy system (30) and (31) on the interval $[0, T]$.

Remark 1. *From Lemma 2 it follows that to prove the uniqueness of the solution of problem (1)-(3),(12)-(14), it is suffices to prove the uniqueness of the solution of system (32),(36) and (37).*

Now, consider the following space.

Denote by $B^3_{2,T}$ [10] the set of all functions of the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x \quad (\lambda=k\pi),$$

considered in domain Q_T, where the functions $u_k(t)$ $(k=0,1,2,...)$ are continuous on the interval $[0,T]$ and satisfies the following condition

$$\|u_0(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}<+\infty.$$

In the space $B^3_{2,T}$ the operations addition and multiplication by a scalar, we define in the usual way, and the norm defined by the following formula

$$\|u(x,t)\|_{B^3_{2,T}}=\|u_0(t)\|_{C[0,T]}+\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}.$$

We denote by E^3_T, the Banach space $B^3_{2,T}\times C[0,T]\times C[0,T]$ of vector functions $z(x,t)=\{u(x,t),a(t),b(t)\}$ with norm

$$\|z(x,t)\|_{B^3_{2,T}}=\|u(x,t)\|_{B^3_{2,T}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is known that $B^3_{2,T}$ and E^3_T are Banach spaces.

Now consider the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\}$$

in the space E^3_T, where

$$\Phi_1(u,a,b)=\tilde{u}(x,t)\equiv\sum_{k=0}^{\infty}\tilde{u}_k(t)\cos\lambda_k x,\quad \Phi_2(u,a,b)=\tilde{a}(t),\quad \Phi_3(u,a,b)=\tilde{b}(t),$$

and the functions $\tilde{u}_0(t)$, $\tilde{u}_k(t)$ $(k=1,2,...)$, $\tilde{a}(t)$ and $\tilde{b}(t)$ are equal to the right-hand sides of (30),(31),(36) and (37) respectively.

Using simple transformations from (30),(31),(36) and (37) we obtain

$$\|\tilde{u}_0(t)\|_{C[0,T]}\le(1+\delta)^{-1}\left[|\varphi_1|+\delta\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left(\sqrt{T}\left(\int_0^T|f_0(\tau)|^2d\tau\right)^{\frac{1}{2}}\right.\right.$$

$$\left.\left.+T\|a(t)\|_{C[0,T]}\|u_0(t)\|_{C[0,T]}+\sqrt{T}\|b(t)\|_{C[0,T]}\left(\int_0^T|g_0(\tau)|^2d\tau\right)^{\frac{1}{2}}\right)\right]$$

$$+\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\left[\sqrt{T}\left(\int_0^T |f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}}+T\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]}\right.$$

$$\left.+\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T |g_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}}\right], \qquad (38)$$

$$\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|\tilde{u}_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}} \le \sqrt{5}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\,|\varphi_k|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{5}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\sqrt{5}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{2T}(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|g_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}, \qquad (39)$$

$$\|\tilde{a}(t)\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\times\{\left\|(c(t)h_1'(t)-f(0,t))g(1,t)-(c(t)h_2'(t)-f(1,t))g(0,t))\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||g(0,t)|+|g(1,t)|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\,|\varphi_k|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|f_k(\tau)|\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\,\|u_k(t)\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\left.\left.+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\,\|b(t)\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\,|g_k(\tau)d\tau|\right)^2\right)^{\frac{1}{2}}\right]\right\}, \qquad (40)$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]} \le \left\|[h(t)]^{-1}\right\|_{C[0,T]}$$

$$\times\{\left\|h_1(t)(c(t)h_2'(t)-f(1,t))-h_2(t)(c(t)h_1'(t)-f(0,t))\right\|_{C[0,T]}$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\||h_1(t)|+|h_2(t)|\|_{C[0,T]}\left[\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_k\right|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\sum_{i=1}^{2}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|f_k(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}$$

$$+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\left(\lambda_k^3\left\|u_k(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$\left.+(1+\delta)\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\sqrt{T}\left\|b(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}\left(\lambda_k^3\left|g_k(\tau)d\tau\right|\right)^2\right)^{\frac{1}{2}}\right]\Bigg\}. \tag{41}$$

Suppose that the data for problem (1)-(3), and (12)-(14) satisfy the following conditions

A_1) $\varphi(x)\in C^2[0,1]$, $\varphi'''(x)\in L_2(0,1)$, $\varphi'(0)=\varphi'(1)=0$;

A_2) $f(x,t)$, $f_x(x,t)$, $f_{xx}(x,t)\in C^2[0,1]$, $f_{xxx}(x,t)\in L_2(Q_T)$,
$f_x(0,t)=f_x(1,t)=0$ $(0\le t\le T)$;

A_3) $g(x,t)$, $g_x(x,t)$, $g_{xx}(x,t)\in C^2[0,1]$, $g_{xxx}(x,t)\in L_2(Q_T)$,
$g_x(0,t)=g_x(1,t)=0$ $(0\le t\le T)$;

A_4) $\delta\ge 0$, $h_i(t)\in C^1[0,T]$ $(i=1,2)$, $h(t)=h_1(t)g(1,t)-h_2(t)g(0,t)\neq 0$ $(0\le t\le T)$.

Then from (33)-(35) we find that

$$\|\tilde{u}(x,t)\|_{B_{2,T}^3}\le A_1(T)+B_1(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+D_1(T)\|b(t)\|_{C[0,T]}, \tag{42}$$

$$\|\tilde{a}(t)\|_{C[0,T]}\le A_2(T)+B_2(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+D_2(T)\|b(t)\|_{C[0,T]}, \tag{43}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\le A_3(T)+B_3(T)\|a(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}+D_3(T)\|b(t)\|_{C[0,T]}, \tag{44}$$

where

$$A_1(T)=(1+\delta)^{-1}\left(2\|\varphi(x)\|_{L_2(0,1)}+2\delta\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f(x,t)\|_{L_2(Q_T)}\right)$$

$$+\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f(x,t)\|_{L_2(Q_T)}+\sqrt{5}\left\|\varphi'''(x)\right\|_{L_2(0,1)}$$

$$+\sqrt{5T}\left\|\frac{1}{c(t)}\right\|_{C[0,T]}\|f_{xxx}(x,t)\|_{L_2(Q_T)},$$

$$B_1(T) = \big(\delta(1+\delta)^{-1} + 1\big)\, T \left\| \frac{1}{c(t)} \right\|_{C[0,T]},$$

$$D_1(T) = (\delta(1+\delta)^{-1} + \sqrt{5})\sqrt{T} \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \|g_{xxx}(x,t)\|_{L_2(Q_T)},$$

$$A_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]}$$

$$\times \{ \left\| (c(t)h_1'(t) - f(0,t))g(1,t) - (c(t)h_2'(t) - f(1,t))g(0,t)) \right\|_{C[0,T]}$$

$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \| |g(x_1,t)| + |g(x_2,t)| \|_{C[0,T]}$$

$$\times \left[\left\| \varphi'''(x) \right\|_{L_2(0,1)} + (1+\delta) \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \sqrt{T} \, \|f_{xxx}(x,t)\|_{L_2(Q_T)} \right] \right\},$$

$$B_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2}) \right)^{\frac{1}{2}} \| |g(x_1,t)| + |g(x_2,t)| \|_{C[0,T]} (1+\delta) T \left\| \frac{1}{c(t)} \right\|_{C[0,T]},$$

$$D_2(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2}) \right)^{\frac{1}{2}} \| |g(1,t)| \, |g(0,t)| \|_{C[0,T]}$$

$$\times (1+\delta) \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \sqrt{T} \, \|f_{xxx}(x,t)\|_{L_2(Q_T)},$$

$$A_3(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \{ \left\| h_1(t)(c(t)h_2'(t) - f(1,t)) - h_2(t)(c(t)h_1'(t) - f(0,t)) \right\|_{C[0,T]}$$

$$+ \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]}$$

$$\times \left[2 \left\| \varphi'''(x) \right\|_{L_2(0,1)} + 2(1+\delta) \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \sqrt{T} \, \|f_{xxx}(x,t)\|_{L_2(Q_T)} \right] \right\},$$

$$B_3(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2}) \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]} (1+\delta) T \left\| \frac{1}{c(t)} \right\|_{C[0,T]},$$

$$D_3(T) = \left\| [h(t)]^{-1} \right\|_{C[0,T]} \left(\sum_{k=1}^{\infty} (\lambda_k^{-2}) \right)^{\frac{1}{2}} \| |h_1(t)| + |h_2(t)| \|_{C[0,T]}$$

$$\times (1+\delta) \left\| \frac{1}{c(t)} \right\|_{C[0,T]} \sqrt{T} \, \|f_{xxx}(x,t)\|_{L_2(Q_T)}.$$

From the inequalities (42)-(44) we conclude that

$$\|\tilde{u}(x,t)\|_{B^3_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} + \left\|\tilde{b}(t)\right\|_{C[0,T]}$$

$$\leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + D(T)\,\|b(t)\|_{C[0,T]}\,, \tag{45}$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T),$$
$$D(T) = D_1(T) + D_2(T) + D_3(T).$$

Theorem 2. *Suppose that the conditions $A_1) - A_4)$ and the inequality*

$$[B(T)(A(T)+2) + D(T)](A(T)+2) < 1 \tag{46}$$

hold, then problem (1)-(3), (12)-(14) has a unique solution in the ball $K = K_R(\|z\|_{E^3_T} \leq R \leq A(T)+2)$ of the space E^3_T.

Proof. In the space E^3_T, we consider the equation

$$z = \Phi z, \tag{47}$$

where $z = \{u, a, b\}$, and components $\Phi_i(u,a,b)\ \ (i = 1,2,3)$, of operator $\Phi(u,a,b)$, defined by the right side of equations (32), (36) and (37) respectively.

Consider the operator $\Phi(u,a,b)$, in the ball $K = K_R$ of the space E^3_T. Similarly, with the aid of (45) we get that for any $z_1, z_2, z_3 \in K_R$ the following inequalities hold

$$\|\Phi z\|_{E^3_T} \leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^3_{2,T}} + D(T)\,\|b(t)\|_{C[0,T]}$$

$$\leq A(T) + B(T)(A(T)+2)^2 + D(T)(A(T)+2) < A(T)+2, \tag{48}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^3_T} \leq B(T)R\left(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B^3_{2,T}}\right)$$
$$+ D(T)\,\|b_1(t) - b_2(t)\|_{C[0,T]}\,. \tag{49}$$

Then by (46), from (48) and (49) it is clear that the operator Φ on the set $K = K_R$ satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\} = \{u, a, b\}$, in the ball $K = K_R$, which is a solution of equation (47); i.e. in the ball $K = K_R$ is the unique solution of the systems (32), (36) and (37). Then the function $u(x,t)$, as an element of space $B^3_{2,T}$, is continuous and has continuous derivatives $u_x(x,t)$ and $u_{xx}(x,t)$ in Q_T.

From the equation (28) we obtain

$$\left(\sum_{k=1}^{\infty} (\lambda_k \left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}} < +\infty.$$

Hence it follows that the function $u(x,t)$ is continuous in Q_T. Further, it is possible to verify that equation (1), and conditions (2), (3), (12), (13), (14), are satisfied in the usual sense. Consequently, $\{u(x,t), a(t), b(t)\}$ is a solution of (1)-(3), (11)-(12), and due to the Lemma 2, it is unique in the ball $K = K_R$. The theorem is thus proved.

From Theorem 2 and Theorem 1, it follows directly that the following assertion is valid.

Theorem 3. *Suppose that all assumptions of Theorem 2, and the compatibility conditions (10) and (11) hold. If*

$$\int_0^1 f(x,t)dx = 0, \quad \int_0^1 g(x,t)dx = 0 \quad (0 \le t \le T),$$

then problem (1)-(6) has a unique classical solution in the ball $K = K_R$.

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