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The study of multidimensional mixed problem for one class of third order semilinear psevdohyperbolic equations

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Abstract. This work is dedicated to the study of existence in small of classical solution of multidimensional mixed problem for one class of third order semilinear psevdohyperbolic equations. Conception of classical solution for mixed problem under consideration is introduced. After applying Fourier method, the solution of original problem is reduced to the solution of some countable system of nonlinear inteqro-differential equations in unknown Fourier coefficients of the sought solution. Besides, existence theorem in small of classical solution of the mixed problem is proved by contracted mappings principle.

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1. Introduction

This work is dedicated to the study existence of classical solution for the following multidimensional mixed problem:

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial t}(L(u(t,x))) =$$
$$= F(t,x,u(t,x),u_t(t,x),\nabla u(t,x),\nabla u_t(t,x),\nabla^2 u(t,x)) \;\; (t\in[0,T],\;\, x\in\Omega), \tag{1}$$

$$u(0,x)=\varphi(x) \;\; (x\in\Omega), \;\; u_t(0,x)=\psi(x) \;\; (x\in\Omega), \tag{2}$$

$$u(t,x)|_\Gamma = 0, \tag{3}$$

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where $0 < T < +\infty$; $x = (x_1, ...x_n)$, Ω is a bounded n -dimensional domain with an enough smooth boundary S; $\Gamma = [0, T] \times S$;

$$L(u(t,x)) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) - a(x) \cdot u(t,x), \tag{4}$$

functions $a_{ij}(x)$ $(i, j = \overline{1, n})$ and $a(x)$ are measurable and bounded in Ω and satisfy in Ω the following conditions:

$$a_{ij}(x) = a_{ji}(x), a(x) \geq 0, \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \alpha \cdot \sum_{i=1}^{n} \xi_i^2 \qquad (\alpha = const > 0),$$

where ξ_i $(i = 1, ..., n)$ are arbitrary real numbers; F, φ, ψ are the given functions, and $u(t, x)$ is a sought function; we define the classical solution of problem (1)-(3) as a function $u(t, x)$ if it is continuous on domain $\overline{Q}_T = [0, T] \times \overline{\Omega}$ together with the derivatives $u_t(t, x)$, $u_{x_i}(t, x)$ $(i = 1, ..., n)$, $u_{tx_i}(t, x)$ $(i = 1, ..., n)$, $u_{x_ix_j}(t, x)$ $(i, j = 1, ..., n)$, $u_{tt}(t, x)$, $u_{tx_ix_j}(t, x)$ $(i, j = 1, ...n)$, for which satisfies the equation (1) on the closed domain $\overline{Q}_T$, initial conditions (2) on $\overline{\Omega}$ and boundary condition (3) in the usual sense.

It must note that, many problems in elasticity theory, in particular the problems of longitudinal vibration of the viscoelastic non-homogeneous bar, some wave problems for elastic-viscidal liquid and etc. lead to the problems type (1)-(3).

In the works [4-6] considered a special case of the equation (1), when $L = \Delta$, $F(t,\ x,\ 0,\ ...,\ 0) = 0$ and proven a theorems of existence and uniqueness of the classical solution for initial functions $\varphi(x)$, $\psi(x)$ with sufficiently small (in a certain metric) norms.

We mention the work [1] in which is introduced the definition of almost everywhere solution of the problem (1)-(3) for arbitrary dimension n (i.e. for arbitrary number of the variables) and proven the local existence and global uniqueness theorems for the almost everywhere solution.

We mention also the results of [2] which is complete progression of the results of [1]. In particular in the work [2] the apriori estimates for the almost everywhere solution of the considered mixed problem are esthablished in three steps, which are getting stronger from step to step.

In the work [10] is proven the existence and uniqueness of the strong global solution of the one special case of the problem (1)-(3), when $L = \Delta$, $n \leq 3$ and $F = \Delta u + f(u)$.

Finally we mention the work [8] in which proven theorems about existence and uniqueness of the generalized, almost everywhere and classical solution for one special one-dimensional case of the problem (1)-(3), when $n = 1$, $\Omega = (0, 1)$, $Lu = \alpha \cdot u_{xx}$.

2. Auxiliaries

In this section, we introduce a number of concepts, notations and facts to be used later.

1. We denote by $\dot{D}(\Omega)$ the class of all continuously differentiable functions on Ω which vanished near the boundary of Ω. The closure of $\dot{D}(\Omega)$ with respect to the norm of $W_2^1(\Omega)$ we denote by $\overset{\circ}{D}(\Omega)$. Hence $\overset{\circ}{D}(\Omega) \subset W_2^1(\Omega)$.

For investigation of the problem (1)-(3) we recall one property of the operator L, generating by the differential expression (4) and boundary condition (3): there are denumerable number of negative eigenvalues

$$0 > -\lambda_1^2 \geq -\lambda_2^2 \geq ... \geq -\lambda_s^2 \geq ..., \quad (0 < \lambda_s \to +\infty \quad as \ \ s \to \infty)$$

with the corresponding generalized eigenfunctions $\upsilon_s(x)$ which are complete and orthonormal in $L_2(\Omega)$. We call function $\upsilon_s(x) \in \overset{\circ}{D}(\Omega)$ a generalized eigenfunction of the operator L, if it is not identically zero and

$$\int\limits_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial \upsilon_s(x)}{\partial x_i} \cdot \frac{\partial \Phi(x)}{\partial x_j} + a(x)\upsilon_s(x)\Phi(x) \right\} dx = \lambda_s^2 \cdot \int\limits_{\Omega} \upsilon_s(x)\Phi(x)dx \tag{5}$$

for any function $\Phi(x) \in \overset{\circ}{D}(\Omega)$.

2. We denote by $B^{\alpha_0,...,\alpha_l}_{\beta_0,...\beta_l,T}$ a totality of all the functions of the from

$$u(t,x) = \sum_{s=1}^{\infty} u_s(t)\upsilon_s(x)$$

considered in $Q_T = (0,\,T) \times \Omega$, where $u_s(t) \in C^{(l)}\left([0,\,T]\right)$ for all s and

$$N_T(u) \equiv \sum_{i=0}^{l} \left\{ \sum_{s=1}^{\infty} \left(\lambda_s^{\alpha_i} \cdot \max_{0 \leq t \leq T} \left| u_s^{(i)}(t) \right| \right)^{\beta_i} \right\}^{\frac{1}{\beta_i}} < +\infty,$$

with $\alpha_i \geq 0$, $1 \leq \beta_i \leq 2$ $(i = 0,\,1,\,...,\,n)$. We define the norm in this set as $\|u\| = N_T(u)$. It is evident that all these spaces are Banach spaces.

3. Let $\forall t \in [0,\,T]$ $A_i(t,x)$ $(i = 0,\,1,\,...,\,n)$, $B(t,\,x) \in L_2(\Omega)$. Then the following inequality hold ([7, p. 135]):

$$\sum_{s=1}^{\infty} \left\{ \int\limits_{\Omega} \left[\sum_{i,j=1}^{n} a_{ij}(x) A_i(t,x) \frac{\partial}{\partial x_j} \left(\frac{\upsilon_s(x)}{\lambda_s} \right) + a(x)B(t,x)\frac{\upsilon_s(x)}{\lambda_s} \right] dx \right\}^2$$

$$\leq \int\limits_{\Omega} \left\{ \sum_{i,j=1}^{n} a_{ij}(x) A_i(t,x) A_j(t,x) + a(x)B^2(t,x) \right\} dx. \tag{6}$$

4. We will also often use the following lemma from [9, p.88, lemma 1] by O.A. Ladyzhenskaya:

Lemma 1. *Let the following conditions be satisfied for a natural number* $m \geq 2$*:*

a) $a_{ij}(x)$ $(i,j = 1,2,...,n) \in C^{m-1}(\bar{\Omega}))$, $a(x) \in C^{m-2}(\bar{\Omega})$, $\partial\Omega \equiv S \in C^m$, $\upsilon(x) \in C^m(\bar{\Omega})$*;*

b) $\upsilon(x)|_s = L(\upsilon(x))|_s = ... = L^{\left[\frac{m-1}{2}\right]}(\upsilon(x))\Big|_s = 0.$

Then the following inequalities hold:

$H_m(\upsilon) \equiv \sum\limits_{k=0}^{m} J_k(\upsilon) \leq C \left\{ \sum\limits_{s=0}^{r} J_0(L^s\upsilon) + \sum\limits_{s=0}^{r-1} J_1(L^s\upsilon) \right\}$ *for* $m = 2r$*,*

$H_m(\upsilon) \leq C \left\{ \sum\limits_{s=0}^{r} J_1(L^s\upsilon) + \sum\limits_{s=0}^{r} J_0(L^s\upsilon) \right\}$ *for* $m = 2r+1$*, where*

$$J_k(u,\upsilon) = \int\limits_\Omega \sum_{\substack{1 \leq \alpha_1,...,\alpha_k \leq n \\ 1 \leq \beta_1,...,\beta_k \leq n}} a_{\alpha_1\beta_1}...a_{\alpha_k\beta_k} \frac{\partial^k u(x)}{\partial x_{\alpha_1}...\partial x_{\alpha_k}} \cdot \frac{\partial^k \upsilon(x)}{\partial x_{\beta_1}...\partial x_{\beta_k}} dx$$

$$J_k(u) = J_k(u,u)(k \geq 1); J_0(u) = J_0(u,u), J_0(u,\upsilon) = \int\limits_\Omega u(x)\upsilon(x)dx,$$

$C > 0$ *is some constant not depending on* $\upsilon(x)$*.*

It is evident that for the function of the form $u(t,x) = \sum\limits_{s=1}^{\infty} u_s(t)\upsilon_s(x)$

$$J_0(u(t,x)) = \int\limits_\Omega u^2(t,x)dx = \sum_{s=1}^{\infty} u_s^2(t), \tag{7}$$

hold and, due to the equality (5) we have

$$J_1(u(t,x)) = \int\limits_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(t,x)}{\partial x_i} \cdot \frac{\partial u(t,x)}{\partial x_j} \right\} dx$$

$$\leq \int\limits_\Omega \left\{ \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(t,x)}{\partial x_i} \cdot \frac{\partial u(t,x)}{\partial x_j} + a(x)u^2(t,x) \right\} dx = \sum_{s=1}^{\infty} \lambda_s^2 u_s^2(t). \tag{8}$$

Besides, for any function $u(x) \in W_2^k(\Omega)$, the following inequality hold [9, p.84, inequality (18)]:

$$A_k \cdot \int\limits_\Omega \sum_{1\leq\alpha_1,...,\alpha_k\leq n} \left(\frac{\partial^k u(x)}{\partial x_{\alpha_1}...\partial x_{\alpha_k}} \right)^2 dx \leq J_k(u)$$

$$\leq B_k \cdot \int\limits_\Omega \sum_{1\leq\alpha_1,...,\alpha_k\leq n} \left(\frac{\partial^k u(x)}{\partial x_{\alpha_1}...\partial x_{\alpha_k}} \right)^2 dx \quad (k \geq 1) \tag{9}$$

where $A_k > 0$, $B_k > 0$ are some constants not depending on $u(x) \in W_2^k(\Omega)$.

5. As the system $\{\upsilon_s(x)\}_{s=1}^{\infty}$ is complete orthonormal in $L_2(\Omega)$, then it is evident that every classical solution of problem (1)-(3) has the following form:

$$u(t,x) = \sum_{s=1}^{\infty} u_s(t)\upsilon_s(x),$$

where

$$u_s(t) = \int_{\Omega} u(t,x)\upsilon_s(x)dx \quad (s = 1,\, 2,\, ...).$$

Then, after applying the Fourier method, finding the unknown Fourier coefficients $u_s(t)$ $(s = 1,\, 2,\, ...)$ for the classical solution $u(t,\, x)$ of the problem (1)-(3) is reduced to the solution of the following countable system of nonlinear integro-differential equations:

$$u_s(t) = \varphi_s + \frac{1}{\lambda_s^2}\left(1 - e^{-\lambda_s^2 t}\right)\psi_s$$

$$+\frac{1}{\lambda_s^2}\int_0^t \int_{\Omega} \Im(u(\tau,x))\left[1 - e^{\lambda_s^2(t-\tau)}\right]\upsilon_s(x)dxd\tau \quad (s = 1,\, 2,\, ...;\ \ t \in [0,\, T]), \tag{10}$$

where

$$\varphi_s = \int_{\Omega} \varphi(x)\upsilon_s(x)dx, \qquad \psi_s = \int_{\Omega} \psi(x)\upsilon_s(x)dx \quad (s = 1,\, 2,\, ...),$$

$$\Im(u(\tau,x)) \equiv F(\tau, x, u(\tau,x),\ u_\tau(\tau,x),\ u_x(\tau,x),\ u_{\tau x}(\tau,x),\ u_{xx}(\tau,x)). \tag{11}$$

Proceeding from the definition of classical solution of problem (1)-(3), it is easy to prove the following

Lemma 2. *If $u(t,x) = \sum\limits_{s=1}^{\infty} u_s(t)\upsilon_s(x)$ is any classical solution of problem (1)-(3) and the generalized derivatives $\frac{\partial}{\partial x_k}a_{ij}(x)$ $(i,j,k = 1,2,...,n)$ are bounded on Ω, then functions $u_s(t)$ $(s = 1,2,...)$ satisfy system (10).*

Proof. Let $u(t,x) = \sum\limits_{s=1}^{\infty} u_s(t)\upsilon_s(x)$ be any classical solution of problem (1)-(3). Then it is evident that

$$\int_0^T \int_{\Omega} \left\{u_{tt}(t,x) - \frac{\partial}{\partial t}(L(u(t,x))) - \Im(u(t,x))\right\}\Phi(t,x)dxdt = 0 \tag{12}$$

for each $\Phi(t,x) \in L_2(Q_T)$, and $\Im$ is defined by (11).

If, in particular, we take

$$\Phi(t,x) = \begin{cases} (t-\tau)^2 \upsilon_s(x) & \text{for } 0 \le t \le \tau, \quad x \in \Omega, \\ 0 & for\ \tau < t \le T, \quad x \in \Omega, \end{cases}$$

where $s = 1, 2, ...$ and $\tau \in [0, T]$, then with the help of integration by parts with respect to t twice in the first term and once in the second term of (12) and taking the initial conditions (2) into consideration we easily get

$$2\int_0^\tau u_s(t)dt - 2\lambda_s^2 \int_0^\tau (t-\tau)u_s(t)dt - \int_0^\tau (t-\tau)^2 \Im_s(u,t)dt$$

$$-2\tau\varphi_s - \tau^2\psi_s - \lambda_s^2\tau^2\varphi_s = 0, \tag{13}$$

where

$$\Im_s(u,t) \equiv \int_\Omega \Im(u(t,x))\upsilon_s(x)dx.$$

Differentiating (13) three times with respect to τ we have the next problem

$$\begin{cases} u_s''(\tau) + \lambda_s^2 u_s'(\tau) = \Im_s(u,\tau) & (s = 1, 2, ...;\ t \in [0,T]), \\ u_s(0) = \varphi_s, \qquad u_s'(0) = \psi_s, \end{cases}$$

which is obviously equivalent to system (10). Lemma is proved.

6. We agree to assume that all the quantities throughout this work are real, all the functions are real –valued, and all the integrals are understood in the sense of Lebesgue.

3. Main Result

In this section, using contracted mappings principle, the following existence in small (i.e. for sufficiently small values of T) theorem for the classical solution of problem (1)-(3) is proved for n:

Theorem 1. *Let*

1. $a_{ij}(x) \in C^{\left[\frac{n}{2}\right]+3}(\bar{\Omega})$ *$(i, j = 1, 2, ..., n)$; $a(x) \in C^{\left[\frac{n}{2}\right]+2}(\bar{\Omega})$; $S \in C^{\left[\frac{n}{2}\right]+4}$; the eigenfunctions $\upsilon_s(x)$ of the operator L under boundary condition $\upsilon_s(x)\big|_s = 0$ be $\left[\frac{n}{2}\right]+4$ times continuously differentiable on $\bar{\Omega}$.*

2. $\varphi(x) \in W_2^{\left[\frac{n}{2}\right]+4}(\Omega)$, $\varphi(x), L\varphi(x), ..., L^{\left[\frac{n+2}{4}\right]+1}\varphi(x) \in \overset{\circ}{D}(\Omega)$; $\psi(x) \in W_2^{\left[\frac{n}{2}\right]+3}(\Omega)$, $\psi(x), L\psi(x), ..., L^{\left[\frac{n}{4}\right]+1}\psi(x) \in \overset{\circ}{D}(\Omega)$.

3. *a)* $\partial^k F(t,\xi_1,...,\xi_{\tilde N}) \Big/ \partial\xi_1^{\alpha_1}...\partial\xi_{\tilde N}^{\alpha_{\tilde N}} \in C(\bar Q_T\times(-\infty,\infty)^N$ *(*$k=0,1,...,\left[\frac{n}{2}\right]+2$*);*

   b) $\partial^k F(t,\xi_1,...,\xi_n,0,0,\xi_{n+3},...,\xi_{\tilde N}) \Big/ \partial\xi_1^{\alpha_1}...\partial\xi_{\tilde N}^{\alpha_{\tilde N}} \equiv 0$ *(*$k=0,1,...,2\left[\frac{n+2}{4}\right]$*)*

   $\forall t\in[0,\,T]\,,\ (\xi_1...,\xi_n)\in S,\ \ \xi_{n+3},...,\xi_N\in(-\infty,\infty),\ N=2+2n+n^2,\ \tilde N=n+N.$

4. $\forall R>0$ *in* $\bar Q_T\times(-\infty,\infty)^N$

$$\left|\partial^{2\left[\frac{n+2}{4}\right]}F(t,x,u_1,\dots u_N)\Big/\partial x^\alpha\partial u_1^{\gamma_1}...\partial u_N^{\gamma_N}-\right.$$

$$\left.-\partial^{2\left[\frac{n+2}{4}\right]}F(t,x,\tilde u_1,\,...,\,\tilde u_N)\Big/\partial x^\alpha\partial u_1^{\gamma_1}...\partial u_N^{\gamma_N}\right|\le C_R\sum_{i=1}^N|u_i-\tilde u_i|\,,$$

where $\alpha=(\alpha_1,...,\alpha_n)$*,* $|\alpha|+\sum\limits_{i=1}^N\gamma_i=2\left[\frac{n+2}{4}\right]$*, and* $C_R>0$ *is a constant.*

Then there exists in small (i.e. for sufficiently small values of T*) a unique in large (i.e. for any finite value of* T*) classical solution of problem (1)-(3).*

Proof. We consider the following operator Q in space $B_{2,2,T}^{\left[\frac{n}{2}\right]+4,\left[\frac{n}{2}\right]+3}$:

$$Q(u(t,x))=W(t,x)+\mathrm{P}(u(t,x)) \tag{14}$$

where

$$W(t,x)=\sum_{s=1}^\infty\left\{\varphi_s+\frac{1}{\lambda_s^2}\left[1-e^{-\lambda_s^2t}\right]\psi(s)\right\}\upsilon_s(x),$$

$$\mathrm{P}(u(t,x))=\sum_{s=1}^\infty\frac{1}{\lambda_s^2}\int_0^t\int_\Omega\Im(u(\tau,x))\left[1-e^{-\lambda_s^2(t-\tau)}\right]\upsilon_s(x)dxd\tau\,\cdot\upsilon_s(x),$$

and operator $\Im$ is defined by (11). It is easy to see that under condition 3 of this theorem $\forall u(t,x)\in B_{2,2,T}^{\left[\frac{n}{2}\right]+4,\left[\frac{n}{2}\right]+3}$:

$$\begin{aligned}\mathrm{P}(u(t,x))=\sum_{s=1}^\infty\frac{(-1)^{r+1}}{\lambda_s^{\left[\frac{n}{2}\right]+4}}\int_0^t\int_\Omega L^{r+1}(\Im(u(\tau,x)))\left[1-e^{-\lambda_s^2(t-\tau)}\right]\\ \times\upsilon_s(x)dxd\tau\,\cdot\upsilon_s(x)\quad\text{for}\ \ n=4r,\ 4r+1,\end{aligned}\tag{15}$$

$$\mathrm{P}(u(t,x))=\sum_{s=1}^\infty\frac{(-1)^{r+1}}{\lambda_s^{\left[\frac{n}{2}\right]+4}}\int_0^t\int_\Omega\left\{\sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}L^{r+1}(\Im(u(\tau,x)))\right]$$

$$\times\frac{\partial}{\partial x_j}\left(\frac{\upsilon_s(x)}{\lambda_s}\right)+a(x)L^{r+1}(\Im(u(\tau,x)))\times\frac{\upsilon_s(x)}{\lambda_s}\right\}$$

$$\times\left[1-e^{-\lambda_s^2(t-\tau)}\right]dxd\tau\cdot\upsilon_s(x)\quad\text{for } n=4r+2,\ 4r+3. \tag{16}$$

To ease our writings, we will use the following notations:

$$\|u\|_{B_{2,\,2,\,t}^{\left[\frac{n}{2}\right]+4,\left[\frac{n}{2}\right]+3}}=\|u\|_{E_t}\quad(t\in[0,\ T]).$$

Next, using inequalities (9), lemma 1, and conditions 1,2 of this theorem, we obtain that $W(t,x)\in E_T$, because

$$\|W(t,x)\|_{E_T}\leq\left\{2\sum_{s=1}^{\infty}\left(\lambda_s^{\left[\frac{n}{2}\right]+4}\cdot\varphi_s\right)^2+2\sum_{s=1}^{\infty}\left(\lambda_s^{\left[\frac{n}{2}\right]+2}\cdot\psi_s\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{s=1}^{\infty}\left(\lambda_s^{\left[\frac{n}{2}\right]+3}\cdot\psi_s\right)^2\right\}^{\frac{1}{2}}$$

$$\leq\begin{cases}C\cdot\left(\left\|L^{r+2}\varphi\right\|_{L_2(\Omega)}+\left\|L^{r+1}\psi\right\|_{L_2(\Omega)}+\sqrt{D\left(L^{r+1}\psi\right)}\right)<+\infty & \text{for } n=4r,\ 4r+1,\\ C\cdot\left(\sqrt{D\left(L^{r+2}\varphi\right)}+\sqrt{D\left(L^{r+1}\psi\right)}+\left\|L^{r+2}\psi\right\|_{L_2(\Omega)}\right)<+\infty & \text{for } n=4r+2,\ 4r+3,\end{cases}$$

where $C>0$ is some constant,

$$D(z(t,x))\equiv\int_\Omega\left\{\sum_{i,j=1}^n a_{ij}(x)\frac{\partial z(t,x)}{\partial x_i}\cdot\frac{\partial z(t,x)}{\partial x_j}+a(x)\cdot z^2(t,x)\right\}dx.$$

Now we consider operator Q defined by (14) in a closed ball K_r of space E_T with a center in zero and a radius $r>\|W(t,x)\|_{E_T}$. Then, using Bessel's inequality (for $n=4r,\ 4r+1$) and inequality (6) (for $n=4r+2,\ 4r+3$), where functions $A_i(t,x)$ $(i=1,\ 2,\ ...,\ n)$ and $B(t,\ x)$ must be replaced by $\frac{\partial}{\partial x_i}L^{r+1}\left(\Im(u(t,x))\right)(i=1,\ 2,\ ...,\ n)$ and $L^{r+1}(\Im(u(t,x)))$, respectively, we obtain from (14), (15) and (16) that $\forall u\in E_T$:

$$\|Q(u)\|_{E_T}\leq\|W(t,x)\|_{E_T}+\left\{(2T+1)\cdot\int_0^T\int_\Omega\left[L^{r+1}\left(\Im(u(t,x))\right)\right]^2dxdt\right\}^{\frac{1}{2}}$$

$$=\|W(t,x)\|_{E_T}+\sqrt{2T+1}\cdot\left\|L^{r+1}(\Im(u(t,x)))\right\|_{L_2(Q_T)}$$

$$\leq\|W(t,x)\|_{E_T}+\tilde{C}\cdot\sqrt{2T+1}\cdot\|\Im(u(t,x))\|_{W_{t,x,2}^{0,2(r+1)}(Q_T)}\quad\text{for } n=4r,4r+1, \tag{17}$$

$$\|Q(u)\|_{E_T}\leq\|W(t,x)\|_{E_T}+\left\{(2T+1)\cdot\int_0^T\int_\Omega\left\{\sum_{i,j=1}^n a_{ij}(x)\frac{\partial}{\partial x_i}L^{r+1}\left(\Im(u(t,x))\right)\right.\right.$$

$$\left.\left.\times\frac{\partial}{\partial x_j}L^{r+1}\left(\Im(u(t,x))\right)+a(x)\left[L^{r+1}(u(t,x))\right]^2\right\}dxdt\right\}^{\frac{1}{2}}=\|W(t,x)\|_{E_T}$$

$$+\sqrt{2T+1}\cdot\left\{\left\|D\left(L^{r+1}(\Im(u(t,x)))\right)\right\|_{L(0,T)}\right\}^{\frac{1}{2}}$$

$$\leq\|W(t,x)\|_{E_T}+\tilde{\tilde{C}}\cdot\sqrt{2T+1}\cdot\|\Im(u(t,x))\|_{W^{0,2(r+1)+1}_{t,x,2}(Q_T)}\quad\text{for } n=4r+2,4r+3,\tag{18}$$

where $\tilde{C}>0$ and $\tilde{\tilde{C}}>0$ are some constants.

Due to estimates (17) and (18), we have for every n and $\forall u\in E_T$:

$$\|Q(u)\|_{E_T}\leq\|W(t,x)\|_{E_T}+C\cdot\sqrt{2T+1}\cdot\|\Im(u(t,x))\|_{W^{0,\left[\frac{n}{2}\right]+2}_{t,x,2}(Q_T)}.\tag{19}$$

Using Sobolev's imbedding theorems, inequalities (9), lemma 1, estimates (7), (8), and the structure of space E_T, we have for every $u\in E_T$ and $t\in[0,T]$:

$$\left\|D_t^kD^\alpha u(t,x)\right\|_{L_2(\Omega)}\leq C\cdot\|u\|_{E_t}\leq C\cdot\|u\|_{E_T}\,(k=0,1;\quad 0\leq k+|\alpha|\leq\left[\frac{n}{2}\right]+4),\tag{20}$$

$$\left\|D_t^kD^\alpha u(t,x)\right\|_{C(\bar{Q}_T)}\leq C\cdot\|u\|_{E_T}\,(k=0,1;\quad 0\leq k+|\alpha|\leq 3),\tag{21}$$

$$\left\|D_t^kD^\alpha u(t,x)\right\|_{L_q(\Omega)}\leq C_q\cdot\|u\|_{E_t}\leq C_q\cdot\|u\|_{E_T}\left(k=0,1;\quad 0\leq k+|\alpha|\leq\left[\frac{n}{2}\right]+4\right),\tag{22}$$

where $C>0$, $C_q>0$ are some constants not depending on u and t, and

$$1\leq q\leq\frac{2n}{n-2\left(\left[\frac{n}{2}\right]+4-k-|\alpha|\right)}=\frac{2n}{n-2\left(\left[\frac{n}{2}\right]+4\right)+2\left(k+|\alpha|\right)},\quad q<+\infty.\tag{23}$$

Due to estimates (21) and condition 3_a of this theorem, $\forall u\in K_r$:

$$\left\|\partial^sF(t,x,u(t,x),\,u_t(t,x),\,u_x(t,x),\,u_{tx}(t,x),\,u_{xx}(t,x)/\partial x^\alpha\partial u_1^{\gamma_1}...\partial u_N^{\gamma_n}\right\|_{C(\bar{Q}_T)}$$

$$\leq\tilde{A}_r\qquad\left(0\leq|\alpha|+\sum_{i=1}^N\gamma_i=s\leq\left[\frac{n}{2}\right]+2\right),\tag{24}$$

where $A_r>0$ is some constant depending on the radius of closed sphere K_r in space E_T with a center in zero and a radius r. Next, due to estimates (24) and (21), to have an estimate for

$$\frac{\partial^{\left[\frac{n}{2}\right]+2}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}\left\{\Im(u(t,x))\right\}$$

with $u \in K_r$, it suffices to estimate in $L_2(Q_T)$ the products of the following form:

$$\prod_{i=1}^{l} D_t^{k_{il}} D^{\alpha_{il}} u(t,x) \Big(l = 1,2,...,\left[\frac{n}{2}\right]+2; \quad 4 \le k_{il}+|\alpha_{il}|;$$

$$\sum_{i=1}^{l}(k_{il}+|\alpha_{il}|) \le l\cdot 3 + \left(\left[\frac{n}{2}\right]+2-l\right) = \left[\frac{n}{2}\right]+2l+2; \ k_{il} \le 1 \Big). \tag{25}$$

Using estimate (22) and taking (23) into account, for products of the form (25) with any $u \in K_r$ and $t \in [0, T]$ we have:

$$\left\|\prod_{i=1}^{l} D_t^{k_{il}} D^{\alpha_{il}} u(t,x)\right\|_{L_2(\Omega)} \le \prod_{i=1}^{l}\left\|D_t^{k_{il}} D^{\alpha_{il}} u(t,x)\right\|_{L_{2p_i}(\Omega)}$$

$$\le C\cdot\|u\|_{E_t}^{l} \le C\cdot\|u\|_{E_T}^{l} \le C\cdot r^l, \tag{26}$$

where $C > 0$ is some constant and

$$1 \le p_i \le \frac{n}{n-2\left(\left[\frac{n}{2}\right]+4\right)+2\left(k_{il}+|\alpha_{il}|\right)}, \quad p_i < +\infty, \quad \sum_{i=1}^{l}\frac{1}{p_i} = 1, \tag{27}$$

the possibility of last equality following from the relation below:

$$\sum_{i=1}^{l}\frac{n-2\left(\left[\frac{n}{2}\right]+4\right)+2\left(k_{il}+|\alpha_{il}|\right)}{n} = \frac{1}{n}\left\{\left(n-2\left(\left[\frac{n}{2}\right]+4\right)\right)\cdot l + 2\sum_{i=1}^{l}(k_{il}+|\alpha_{il}|)\right\}$$

$$\le \frac{1}{n}\left\{\left(n-2\left(\left[\frac{n}{2}\right]+4\right)\right)\cdot l + 2\left(\left[\frac{n}{2}\right]+2l+2\right)\right\}$$

$$= \frac{1}{n}\left\{n+(l-1)\left(n-2\left[\frac{n}{2}\right]-4\right)\right\} \le 1, \tag{28}$$

when $l \ge 2$, the latter part of (28) becomes a strict inequality because $n-2\left[\frac{n}{2}\right]-4 < 0$.

Now, using estimates (24), (21) and (26), from (19) we obtain that $\forall u \in K_r$:

$$\|Q(u)\|_{E_T} \le \|W\|_{E_T} + \sqrt{T}\cdot\sqrt{2T+1}\cdot C_r, \tag{29}$$

where $C_r > 0$ is some number depending on r.

Next, similar to (19), $\forall u,\ \tilde{u} \in K_r$ we have:

$$\|Q(u)-Q(\tilde{u})\|_{E_T} \le C\cdot\sqrt{2T+1}\cdot\|\Im(u(t,x))-\Im(\tilde{u}(t,x))\|_{W_{t,x,2}^{0,\left[\frac{n}{2}\right]+2}(Q_T)}. \tag{30}$$

Due to estimates (21) and condition 4 of this theorem, $\forall u,\ \tilde{u} \in K_r$:

$$\left\| \partial^s F(t,x,u(t,x)u_t(t,x),\ u_x(t,x),\ u_{tx}(t,x),\ u_{xx}(t,x) / \partial x^\alpha \partial u_1^{\gamma_1} ... \partial u_N^{\gamma_N} \right.$$

$$\left. -\partial^s F(t,x,\tilde{u}(t,x)\tilde{u}_t(t,x),\ \tilde{u}_x(t,x),\ \tilde{u}_{tx}(t,x),\ \tilde{u}_{xx}(t,x) / \partial x^\alpha \partial u_1^{\gamma_1} ... \partial u_N^{\gamma_N} \right\|_{C(\bar{Q}_T)}$$

$$\leq \tilde{A}_r \cdot \|u - \tilde{u}\|_{E_T} \quad \left(0 \leq |\alpha| + \sum_{i=1}^{N} \gamma_i = s \leq \left[\frac{n}{2}\right] + 2 \right), \tag{31}$$

where $\tilde{A}_r > 0$ is some constant depending on r.

Next, using estimate (22) and taking (23) into account, similar to (26) we have for every $t \in [0,\ T]$:

$$\left\| \prod_{i=1}^{l} D_t^{k_{il}} D^{\alpha_{il}} u_i(t,x) \right\|_{L_2(\Omega)} \leq \prod_{i=1}^{l} \left\| D_t^{k_{il}} D^{\alpha_{il}} u_i(t,x) \right\|_{L_{2p_i}(\Omega)}$$

$$\leq C \cdot \prod_{i=1}^{l} \|u_i\|_{E_t} \leq C \cdot \prod_{i=1}^{l} \|u_i\|_{E_T} \leq C \cdot r^{l-1} \cdot \|u - \tilde{u}\|_{E_T}, \tag{32}$$

where the conditions (27) are satisfied, one of functions u_i $(i = 1, 2, ..., l)$ is equal to $u - \tilde{u}$ while the other u_i's are equal to u and $\tilde{u}$, with $u,\ \tilde{u} \in K_r$.

Now, using estimates (31), (21) and (32) (in (21) u must be replaced by $u - \tilde{u}$), from (30) we obtain that $\forall u,\ \tilde{u} \in K_r$:

$$\|Q(u) - Q(\tilde{u})\|_{E_T} \leq \sqrt{T} \cdot \sqrt{2T+1} \cdot \tilde{C}_r \cdot \|u - \tilde{u}\|_{E_T}, \tag{33}$$

where $\tilde{C}_r > 0$ is some number depending on r.

It can be seen from the inequalities (29) and (33) that for sufficiently small values of T the operator Q is a contraction in sphere K_r, and, consequently, has a unique fixed point $u(t,x)$ in K_r. Then it is evident that

$$u(t,x) = Q(u(t,x)) = W(t,x) + \mathrm{P}(u(t,x)) = W(t,x)$$

$$+ \sum_{s=1}^{\infty} \frac{1}{\lambda_s^2} \int_0^t \int_\Omega \Im(u(t,x)) \cdot \left[1 - e^{-\lambda_s^2 (t-\tau)}\right] \upsilon_s(x) dx d\tau \cdot \upsilon_s(x).$$

Thus, Fourier coefficients $u_s(t)$ of the found function

$$u(t,x) \in K_r \subset E_T = B_{2,2,T}^{\left[\frac{n}{2}\right]+4, \left[\frac{n}{2}\right]+3}$$

satisfy system (10).

And now let's show that the function $u(t,x)$ is a classical solution of problem (1)-(3). As

$$\sum_{s=1}^{\infty} u_s(t)\upsilon_s(x) = u(t,x) \in B_{2,2,T}^{\left[\frac{n}{2}\right]+4,\left[\frac{n}{2}\right]+3},$$

then it is evident that the functions $u_{pq}(t,x) = \sum\limits_{s=p}^{q} u_s(t)\upsilon_s(x)$ $(1 \le p \le q)$ satisfy conditions of lemma 1. Then, from (9) (for $u = u_{pq}(t,x) = \sum\limits_{s=p}^{q} u_s(t)\upsilon_s(x)$ and $k \le \left[\frac{n}{2}\right] + 4$), due to lemma 1 (for $m = \left[\frac{n}{2}\right] + 4$, $\upsilon = u_{pq}(t,x)$), we obtain $\forall t \in [0,\, T]$:

$$\|u_{pq}(t,x)\|^2_{W_2^{\left[\frac{n}{2}\right]+4}(\Omega)} \le C \cdot \sum_{s=p}^{q} \left[\lambda_s^{\left[\frac{n}{2}\right]+4} u_s(t)\right]^2 \le C \cdot \sum_{s=p}^{q} \left(\lambda_s^{\left[\frac{n}{2}\right]+4} \max_{0\le t\le T} |u_s(t)|\right)^2, \quad (34)$$

where $C > 0$ is some constant.

From (34), due to convergence of number series

$$\sum_{s=1}^{\infty} \left(\lambda_s^{\left[\frac{n}{2}\right]+4} \max_{0\le t\le T} |u_s(t)|\right)^2$$

it follows that $\|u_{pq}(t,x)\|_{W_2^{\left[\frac{n}{2}\right]+4}(\Omega)} \to 0$ uniformly with regard to $t \in [0,\, T]$ as $p,q \to \infty$.

Similar to the foregoing discussions, it is easy to show that $\left\|\frac{\partial u_{pq}(t,x)}{\partial t}\right\|_{W_2^{\left[\frac{n}{2}\right]+3}(\Omega)} \to 0$ uniformly with regard to $t \in [0,\, T]$ as $p,q \to \infty$, because numerical series

$$\sum_{s=1}^{\infty} \left(\lambda_s^{\left[\frac{n}{2}\right]+3} \max_{0\le t\le T} \left|u_s'(t)\right|\right)^2$$

is convergent.

So we got that the series $\sum\limits_{s=1}^{\infty} u_s(t)\upsilon_s(x)$, $\sum\limits_{s=1}^{\infty} u_s'(t)\upsilon_s(x)$ and the ones obtained from them by differentiating them with regard to $x_1, ..., x_n$ up to $\left[\frac{n}{2}\right] + 4$ and $\left[\frac{n}{2}\right] + 3$ times, respectively, converge in $L_2(\Omega)$ uniformly with regard to $t \in [0,\, T]$. Then it is evident that

$$u(t,x) \in C\left([0,\, T]\,;\; W_2^{\left[\frac{n}{2}\right]+4}(\Omega)\right), \quad u_t(t,x) \in C\left([0,\, T]\,;\; W_2^{\left[\frac{n}{2}\right]+3}(\Omega)\right). \quad (35)$$

From (35), due to S.L. Sobolev's imbedding theorems, we obtain that each of the functions

$$u(t,x),\; u_t(t,x),\; u_{x_i}(t,x)\; (i = 1,2,...n),\;\; u_{t\,x_i}(t,x)\; (i = 1,2,...n),\; u_{x_i x_j}(t,x)\; (i,j = 1,2,...n),$$

$$u_{tx_ix_j}(t,x) \quad (i,j=1,2,...n),\, u_{x_ix_jx_k}(t,x) \quad (i,j,k=1,2,...n)$$

is continuous in the closed domain $\bar{Q}_T$.

It is easy to show that for each fixed $t \in [0,\, T]$ and for all $x \in \bar{\Omega}$:

$$\Im(u(t,x)) = \sum_{s=1}^{\infty}\left(\int_{\Omega} \Im(u(t,x))\upsilon_s(x)dx\right)\upsilon_s(x),$$

$$\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial t}(L(u(t,x))) = \Im(u(t,x)). \tag{36}$$

From (36), it follows that $u_{tt}(t,x) \in C(\bar{Q}_T)$ and the function $u(t,x)$ satisfies the equation (1) on the closed domain $\bar{Q}_T$, initial conditions (2) on $\bar{\Omega}$ and boundary condition (3) in the usual sense.

Moreover,

$$\|u(t,x)-\varphi(x)\|_{W_2^{\left[\frac{n}{2}\right]+4}(\Omega)} \to 0,\, \|u_t(t,x)-\psi(x)\|_{W_2^{\left[\frac{n}{2}\right]+3}(\Omega)} \to 0 \text{ as } t \to +0.$$

Thus, function $u(t,x)$ is a classical solution of problem (1)-(3)

And now we prove the uniqueness (in large) of the classical solution of problem (1)-(3). Let $u(t,x) = \sum\limits_{s=1}^{\infty} u_s(t)\upsilon_s(x)$ and $\tilde{u}(t,x) = \sum\limits_{s=1}^{\infty} \tilde{u}_s(t)\upsilon_s(x)$ be two arbitrary classical solutions of problem (1)-(3). Then, due to lemma 2, from system (10) we obtain that

$$\|u-\tilde{u}\|^2_{B^{2,1}_{2,2,T}} \le (2T+1)\int_0^T \|\Im(u(t,x))-\Im(\tilde{u}(t,x))\|^2_{L_2(\Omega)}\, dt, \tag{37}$$

where operator $\Im$ is defined by (11). Then it is evident that $u(t,x)-\tilde{u}(t,x) \in B^{2,1}_{2,2,T}$, because $\Im(u(t,x)),\ \Im(\tilde{u}(t,x)) \in C(\bar{Q}_T)$.

Next, similar to (37), from system (10) we have for every $t \in [0,\, T]$:

$$\|u-\tilde{u}\|^2_{B^{2,1}_{2,2,T}} \le (2T+1)\int_0^t \|\Im(u(\tau,x))-\Im(\tilde{u}(\tau,x))\|^2_{L_2(\Omega)}\, d\tau. \tag{38}$$

From (38), due to condition 4 of this theorem and using the structure of spase $B^{2,1}_{2,2,T}$, for every $t \in [0,\, T]$ we have:

$$\|u-\tilde{u}\|^2_{B^{2,1}_{2,2,t}} \le C\int_0^t \|u-\tilde{u}\|^2_{B^{2,1}_{2,2,\tau}}\, d\tau,$$

where $C > 0$ is some constant.

From here, on applying Bellman's inequality ([3], pp. 188-189), we obtain that $\forall t \in [0,\, T]$ $\|u-\tilde{u}\|^2_{B^{2,1}_{2,2,t}} = 0$. Hence, $u = \tilde{u}$. Theorem is proved.

References

[1] S.J.Aliyev ; On local existence and global uniqueness of the almost everywhere solution of a multidimensional mixed problem for one class of third order nonlinear differential equations, Report of Baku State University, Phys. Mat. Ser. **3** (2003), 1-7 (in Russian).

[2] S.J.Aliyev; On global existence of almost everywhere solution of a multidimensional mixed problem for one class third order nonlinear differential equations, Report of Baku State University, Phys. Mat. Ser. **4** (2003), 1-10 (in Russian).

[3] E.Beckenbach, R. Bellman; Inequalities, Mir, 1965, 276p. (in Russian).

[4] Ebihara Yukiyoshi; On some nonlinear evolution equations with the strong dissipation, J. Different. Equat. **30** (2) (1978), 149-164.

[5] Ebihara Yukiyoshi; On some nonlinear evolution equations with the strong dissipation, II J. Different. Equat. **34** (3)(1979), 329-352.

[6] Ebihara Yukiyoshi; On some nonlinear evolution equations with the strong dissipation, III J. Different. Equat. **45** (3) (1982), 332-355.

[7] K.I. Khudaverdiyev; Multidimensional mixed problem for nonlinear hyperbolic equations, Az. gostekn. university publ. Baku, 2011, 611p. (in Russian).

[8] K.I. Khudaverdiyev, A.A. Veliyev; Study of the onedimensional mixed problem for a class of third order psevdohyperbolic equations with nonlinear operator right side, Chashioglu, Baku, 2010, 167p. (in Russian).

[9] O.A. Ladyzhenskaya; Mixed problem for hyperbolic equations, Gostekhizdat, 1953, 279p. (in Russian).

[10] G.F. Webb; Existence and asymptotic behavior for a strongly damped nonlinear wave equation, Can. J. Math., **3** (1980), 631-643.