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Selberg-Type Generalized Quadratic Forms Gamma and Beta Integrals

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Abstract. Although Selberg-type single positive definite symmetric matrices gamma and beta integrals have been evaluated by several authors, see e.g., Askey and Richards [1], Gupta and Kabe [2, 4], Mathai [8], and elsewhere in the vast multivariate statistical analysis literature. However, several other types of Selberg-type integrals appear to have been neglected in the literature. Thus e.g., Selberg-type integrals associated with inverse Wishart densities, inverse multivariate beta densities, their noncentral counterparts, etc, have not been explored as yet. The present paper records Selberg-type generalized quadratic forms gamma and beta integrals. Our methodology is based on hypercomplex (HC) multivariate normal distribution theory, Kabe [6].

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1. Introduction

The HC multivariate normal distribution is defined as follows. Let $x_1,\ x_2\ ,...,x_{4t}$, $t=\frac{1}{4},\frac{1}{2},1,2$ be 4t $p\times n$ real random matrices and for $t=2$, i.e., the octonions case, set

$$Y = x_1 + ix_2, +jx_3 + kx_4 + lx_5 + mx_6 + nx_7 + rx_8, \tag{1}$$

where the base octonions i,j,k,l,m,n,r satisfy the multiplication rule

$$i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = r^2 = -1 = ijk = ilm = irn = jmr = kjr = knm. \tag{2}$$

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For $t = 1, t = 2, t = 4, t = 4i$ is the bioctonion case, Hypercomplex variables do not form a field, they are known to form Clifford Algebras. The octonions conjugate of Y is defined by

$$\bar{Y} = x_1 - ix_2 - jx_3 - kx_4 - lx_5 - mx_6 - nx_7 - rx_8. \tag{3}$$

Note that $Y\bar{Y}'$ is a positive definite HC Hermitian matrix (HCHM).

Next set

$$\sum = \sum_1 + i\sum_2 + j\sum_3 + k\sum_4 + l\sum_5 + m\sum_6 + n\sum_7 + r\sum_8, \tag{4}$$

where $\sum_1$ is a $p \times p$ positive definite symmetric matrix, and $\sum_2, ..., \sum_8$ are real $p \times p$ skew symmetric matrices. Note that $\sum^{-1} = \sum$ and $\sum$ is HCHM. Now setting $dy = dx_1...dx_8$, Kabe [6] shows that the pn variate HC multivariate normal density of Y can be written as

$$f(Y) = \pi^{-2pnt}|\sum|^{-2nt}exp\{-tr\sum{}^{-1}Y\bar{Y}\}, \tag{5}$$

and hence the HC Wishart density of the $p \times p$ HCHM $G = Y\bar{Y}'$ is

$$f(G) = \{\Gamma_p(2nt)\}^{-1}|\sum|^{-2nt}|G|^{-2t(n-p+1)-1}exp\{-tr\sum{}^{-1}G\}, \tag{6}$$

where

$$\Gamma_p(a) = \pi^{tp(p-1)}\Pi_{i=1}^{p}\Gamma(a - 2t(p-i)). \tag{7}$$

Further for given two $p \times p$ HCHM matrices A and B, having HC Wishart densities with n and q degrees of freedom, the density of the $p \times p$ HCHM R defined by

$$R = G^{-\frac{1}{2}}AG^{-\frac{1}{2}}, A + B = G, \tag{8}$$

is given by the expression

$$f(R) = \{B_p(2nt, 2qt)\}^{-1}|I - R|^{2t(n-p+1)-1}|R|^{2t(q-p+1)-1}, \tag{9}$$

where (see [5]),

$$B_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}. \tag{10}$$

If now $\wedge$ is the $p \times p$ diagonal matrix of the roots of R, then Kabe [6, p.68, equation (21)] shows that the Jacobian

$$J(R : \wedge) = \Pi_{i<j}^{p}(x_i - x_j)^{4t}, \wedge = diag(\lambda_1, ..., \lambda_p), \tag{11}$$

and the HC multivariate beta density of $\wedge$ is

$$f(\wedge) = \{B_p(2nt, 2qt)^{-1}|I - \wedge|^{2t(n-p+1)-1}|\wedge|^{2t(q-p+1)-1}\Pi_{i<j}^{p}(\lambda_i - \lambda_j)^{4t}, \tag{12}$$

with a similar result for the density of the roots matrix of (6).

Now our paper proceeds as follows. The next section derives the real generalized quadratic forms Wishart density (GQFWD), and section 3 develops the real generalized quadratic forms multivariate beta density (GQFMBD). Section 4 records the gamma integrals, and section 5 records the beta integrals of the context.

2. GQFWD

Let X be a $p \times n$ matrix of rank $p \leq n$, and $\Delta = diag(\delta_1, ..., \delta_n)$ $n \times n$ diagonal matrix, then GQFWD of $p \times p$ T is defined by the integral

$$f(T) = k \int_{XX'=T} exp\{-trX\Delta X'\}dX, \tag{13}$$

where k, as a generic letter, denotes the normalizing constants of density functions in this paper.

The moment generating function of T is

$$\phi(\theta) = k \int exp\{-tr(X\Delta X' - \theta XX')\}dX = \Pi_{i=1}^{n} \ |\delta_i I - \theta|^{-\frac{1}{2}}, \tag{14}$$

where θ, in the usual sense, is $p \times p$ positive definite symmetric matrix.

Indeed, Mathai [8, p.353], derives the density (13); however, his density function is not suitable in our context. In our context inverting (14) we find that

$$f(T) = |\Delta|^{-\frac{1}{2}p}|\delta_1|^{-\frac{1}{2}pn}exp\{-\delta_1 trT\}|T|^{\frac{1}{2}(n-p-1)}\{\Gamma_p(\frac{1}{2}n)\}^{-1}$$

$$\cdot {}_1F_1(\frac{1}{2}(n-1); \frac{1}{2}n; ((n-1)\delta_1 - \delta_2 - ... - \delta_n)T) \tag{15}$$

where $\delta_1 = max(\delta_1, ..., \delta_n)$. It is possible to use Mathai's [8] GQFWD and GQFMBD results to derive Selberg-type gamma and beta integrals of our context; however, our expressions for the GQFWD and GQFMBD appear to be simpler than the ones given by Mathai [8], and Mathai, Provost and Hayakawa [7, Chapter 5]

We proceed to prove (15) . Given the joint density of n gamma variates to be

$$f(y_1, ..., y_n) = kexp\{-(\alpha_1 y_1 + \ + \alpha_n y_n)\}y_1^{g_1-1}...y_n^{g_n-1}, \tag{16}$$

the density of $t = y_1 + ... + y_n$ is desired, where $\alpha's$ are distinct real positive constants. The moment generating function $\Psi(\theta)$ of t is,

$$\begin{aligned}\Psi(\theta) &= (\alpha_1 - \theta)^{-g_1}...(\alpha_n - \theta)^{-g_n} \\ &= (\alpha_1 - \theta)^{-g_1}((\alpha_1 - \theta) - (\alpha_1 - \alpha_2))^{-g_2}...((\alpha_1 - \theta) - (\alpha_1 - \alpha_n))^{-g_n} \\ &= (\alpha_1 - \theta)^{-(g_1+...+g_n+r_2+...r_n)} \textstyle\sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \binom{g_2+r_2-1}{r_2}...\binom{g_n+r_n-1}{r_n}(\alpha_1 - \alpha_2)^{r_2}... \\ &\qquad (\alpha_1 - \alpha_n)^{r_n},\end{aligned} \tag{17}$$

where $\alpha_1 = max\{\alpha_1, ..., \alpha_n\}$. And now inverting (17) we have that

$$f(T) = (\Gamma(g_1 + \ + g_n))^{-1}\Pi_{i=1}^{n}\alpha_j^{g_j}(\alpha_1)^{g_1+...+g_n}exp\{-\alpha_1 t\}t^{(g_1+...+g_n-1)}$$

$$\varphi(g_2,...,g_n;g_1+...+g_n;(\alpha_1-\alpha_2)t,...,(\alpha_1-\alpha_n)t). \tag{18}$$

Here, $\varphi = \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} \frac{\Gamma(g_2+r_2)...\Gamma(g_n+r_n)(\alpha_1-\alpha_2)^{r_2}(\alpha_1-\alpha_n)^{r_n}}{\Gamma(g_1+...+g_n+r_2+...+r_n)r_2!...r_n!}$

$$={}_1 F_1(g_2+...+g_n;g_1+...+g_n;((n-1)\alpha_1-\alpha_2-...-\alpha_n)t). \tag{19}$$

To prove (19) we proceed as follows. The sum of the two noncentral $p\times p$ Wishart matrices A and B, with n and q df, and noncentrality parameters Δ and Ω, is again noncentral Wishart with (n+q)df, and $(\Delta+\Omega)$ as noncentrality parameter. With 2g=(p+1), we write this result as

$$k\int exp\{-tr(A+B)\}|A|^{n-g}|B|^{q-g}\ {}_0F_1(n;\Delta A){}_0F_1(q;\Omega B)dA\ dB$$
$$= exp\{-trD\}|D|^{n-g}\ {}_0F_1(n+q;(\Delta+\Omega)D), \tag{20}$$

$${}_0F_1(n;\Delta A){}_0F_1(q;\Omega B) ={}_0 F_1(n+q;(\Delta+\Omega)(A+B)). \tag{21}$$

Now Mathai [8, p. 339, Theorem 5.5] defines

$$\varphi(b_1,b_2;c;X_1,X_2)$$

$$= \int |U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}\ {}_1F_1(b_1;d_1;X_1U_1)\ {}_1F_1(b_2;d_2;X_2U_2)dU_1dU_2$$

$$= \int |U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}exp\{-tr(Z_1+Z_2)\}$$
$$|Z_1|^{b_1-g}|Z_2|^{b_2-g}\ {}_0F_1(d_1;X_1U_1Z_1){}_0F_1(d_2;X_2U_2Z_2)dZ_1dZ_2dU_1dU_2$$

$$= \int |U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}exp\{-tr(Z_1+Z_2)\}$$
$$|Z_1|^{b_1-g}|Z_2|^{b_2-g}\ {}_0F_1(d_1+d_2;(X_1+X_2)(U_1+U_2)(Z_1+Z_2))dZ_1dZ_2dU_1dU_2$$

$$= \int |U_1|^{d_1-g}|U_2|^{d_2-g}|I-U_1-U_2|^{c-d_1-d_2-g}$$
$${}_1F_1(b_1+b_2;d_1+d_2;(X_1+X_2)(U_1+U_2))dU_1dU_2$$

$$= k_2F_2(d_1+d_2;b_1+b_2;d_1+d_2;c;X_1+X_2)$$

$$= k_1\ {}_1F_1(b_1+b_2;c;X_1+X_2), \tag{22}$$

and hence (15) and (19) follow. All matrices in (22) are $p\times p$ positive definite matrices.

3. GQFMBD

The GQFMBD of the $p\times p$ matrix M defined by the integral

$$f(M) = \int_R g(T_1)g(T_2)dT_1dT_2, \tag{23}$$

where$R = (T_1 + T_2)^{\frac{1}{2}} M (T_1 + T_2)^{\frac{1}{2}} = T_1$. The joint density of $p \times p$ T_1 and T_2 is

$$f(T_1, T_2) = |\Delta|^p (\delta_1)^{pn} exp\{-\delta_1 tr(T_1 + T_2)\} |T_1|^{\frac{1}{2}(n-p-1)} |T_2|^{\frac{1}{2}(q-p-1)}$$

$$_1F_1(\frac{1}{2}(n-1); \frac{1}{2}n; \delta T_1) {}_1F_1(\frac{1}{2}(q-1); \frac{1}{2}q; \delta T_2). \tag{24}$$

Note that the matrix M has a doubly noncentral multivariate beta density derived by Gupta and Kabe [3] as follows.

$$f(T_1, T_2) = k\ exp\{-\delta_1 tr(T_1 + T_2)\} |T_1|^{\frac{1}{2}(n-p-1)} |T_2|^{\frac{1}{2}(q-p-1)}$$
$$\int exp\{-tr(Z_1 + Z_2)\} |Z_1|^{\frac{1}{2}(n-1)-g} |Z_2|^{\frac{1}{2}(q-1)-g}$$
$$_0F_1(\frac{1}{2}n; \delta T_1, Z_1)\ {}_0F_1(\frac{1}{2}q; \delta T_2) dZ_1 dZ_2, \tag{25}$$

and hence the doubly noncentral multivariate beta density of M is

$$f(M) = k\ exp\{-tr(T_1 + T_2)\} |Z_1|^{\frac{1}{2}(n-1)-g} |Z_2|^{\frac{1}{2}(q-1)-g} |M|^{\frac{1}{2}(n-p-1)}$$
$$|I - M|^{\frac{1}{2}(q-p-1)} {}_1F_1(\tfrac{1}{2}(n+q); \tfrac{1}{2}n; \delta M(Z_1 + Z_2) dZ_1 dZ_2$$
$$= |\Delta|^p (\delta_1)^{\frac{1}{2}(n+q-2)} |M|^{\frac{1}{2}(n-p-1)} |I - M|^{\frac{1}{2}(q-p-1)} \{B_p(\tfrac{1}{2}n; \tfrac{1}{2}q)\}^{-1}$$
$$_2F_1(\frac{1}{2}(n+q-2); \frac{1}{2}(n+q); \frac{1}{2}n; \delta\delta_1^{-p} M). \tag{26}$$

We now proceed to write the HC counterparts of (15) and (25), with $\delta = ((n-1)\delta_1 - \delta_2 - \\ - \delta_n)$.

4. HCGFWD

The HCGFWD of (15) is

$$f(T) = |\Delta|^{2pt} exp\{-\delta_1 trT\} |T|^{2t(n-p+1)-1} \{\Gamma_p(2nt)\}^{-1} (\delta_1)^{2ptn} {}_1F_1(2t(n-1); 2tn; \delta T), \tag{27}$$

and hence the density of the roots matrix $p \times p \wedge$ is

$$f(\wedge) = |\Delta||^{2pt} exp\{-\delta_1 trT\} |T|^{2t(n-p+1)-1} \{\Gamma_p(2nt)\}^{-1} (\delta_1)^{2ptn}$$
$$_1F_1(2t(n-1); 2tn; \delta T) \Pi_{i<j}^p (\lambda_i - \lambda_j)^{4t} \tag{28}$$

and setting $(2t = h)$ we find that

$$\int exp\{-\delta_1 tr\wedge\} | \wedge |^{g-1}\ {}_1F_1((n-1)h; nh; \delta\wedge) \Pi_{i<j}^p (\lambda_i - \lambda_j)^{2h} d\wedge$$
$$= |\Delta|^{-ph} (\delta_1)^{pnh} \frac{\Gamma_p(g + hp - h)}{\Gamma(p+1)} \tag{29}$$

Note that in (28) the roots are ordered, but in Selberg-type they are unordered hence the factor $\Gamma(p+1)$ in (29).

5. HCGQFMBD

The HCGQFMBD of $p \times p$ M is

$$f(M) = |\Delta|^{4pt}(\delta_1)^{2pt(n-q+2)}\{B_p(2nt; 2qt)\}^{-1}|M|^{2t(n-p+1)-1}|I - M|^{2t(q-p+1)-1}$$
$$ {}_2F_1(2t(n+q-2); 2t(n+q); 2tn; \delta_1^{-4pt}\delta M),$$

$$f(\wedge) = \{B_p(2nt; 2qt)\}^{-1}|\Delta|^{4pt}(\delta_1)^{2pt(n-q+2)}| \wedge |^{2t(n-p+1)-1}|I - \wedge|^{2t(q-p+1)-1}$$
$$ {}_2F_1(2t(n+q-2); 2t(n+q); 2tn; \delta_1^{-4pt}\delta \wedge)\Pi_{i<j}^{p}(\lambda_i - \lambda_j)^{4t}$$

and hence setting $2t = h$, we have that

$$\int | \wedge |^{g-1}|I - \wedge|^{t-1}{}_2F_1(h(n+q-2); h(n+q); hn; \delta(\delta_1)^{-2hp}\wedge)\Pi_{i<j}^{p}(\lambda_i - \lambda_j)^{4t}d\wedge$$
$$= \frac{B_p(g + hp - h; t + hp - h)|\Delta|^{-2hp}(\delta_1)^{-hp(n-q+2)}}{\Gamma(p+1)} \tag{30}$$

Thus (29) and (30) are the Selberg-type integrals of our context.

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