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Harmonic and $(\phi, \acute{\phi})$-holomorphic maps on some generalization of contact metric manifolds

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Abstract. We study $(\phi, \acute{\phi})$-holomorphic and harmonic maps on almost contact metric manifolds satisfying $d\eta + d\eta o\phi \otimes \phi = 2\Phi$. It should be noted that these manifolds are not necessarily contact metric manifolds. Examples of such manifolds are nearly Sasakain and quasi contact metric manifolds.

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1. Introduction

J. Eells and J. H. Sampson studied harmonic maps on Riemannian manifolds ([8]). Also, they proved that a holomorphic map between Kähler manifolds is a harmonic map. In odd dimension, the almost contact metric manifolds represent the analogue of almost Hermitian manifolds (see [2], [3]). S. Ianus and A. M. Pastore, considered $(\phi, \acute{\phi})$-holomorphic maps between contact metric (or Sasakian) manifolds and proved that such maps are harmonic ([9]). Contact metric manifolds are (ϕ, ξ, η, g)- almost contact metric manifolds in which $d\eta = \Phi$.

In this paper, we study $(\phi, \acute{\phi})$-holomorphic maps between two classes of almost contact metric manifolds in which $d\eta + d\eta o\phi \otimes \phi = 2\Phi$. Nearly Sasakian and quasi contact metric manifolds are examples of these kind of manifolds (contact metric manifolds, satisfy this relation too, because in contact metric manifolds we have $d\eta o\phi \otimes \phi = d\eta$). Also, conformal transformation on these manifolds are considered in this paper. In [6], Tanno studied conformal transformation on contact metric manifolds.

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2. Preliminaries

A contact manifold is a C^∞ manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. In a contact manifold (M, η) we can choose a vector field ξ, called characteristic vector field, such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$.

We say that a Riemannian manifold (M, g) has an almost contact metric structure, if it admits a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η satisfying the following relations

$$\phi^2 = -I + \eta \otimes \xi, \quad \mathrm{g(X,\, \xi)} = \eta(X), \quad \mathrm{g}(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$

It is proved that given a contact manifold (M, η) with characteristic vector field ξ, there exists an almost contact metric structure(ϕ, ξ, η, g) such that $d\eta(X, Y) = \Phi(X, Y)$, in which $\Phi(X, Y) = g(X, \phi Y)$. In this case, we say that M is equipped with a (ϕ, ξ, η, g) contact metric structure. We mention that it is possible to have a contact manifold $M = (M, \eta)$ with characteristic vector field ξ and with (ϕ, ξ, η, g)-almost contact metric structure , same ξ and η , in which $d\eta(X, Y) \neq g(X, \phi Y)$, then $M = (M, \phi, \xi, \eta, g)$ is contact but not contact metric.

Y. Tashiro studied almost contact manifolds as hypersurfaces of almost complex manifolds ([16]). He showed that a hypersurface in an almost Hermitian manifold has an almost contact metric structure and when the almost Hermitian manifold is quasi Kähler, he called it's associated almost contact metric hypersurface, $contact\ O^*-manifold$. Then J. H. Kim, J. H. Park and K. Sekigawa called such an almost contact metric manifold a *quasi contact* metric manifold ([10]). They proved that an almost contact metric manifold $M = (M, \phi, \xi, \eta, g)$ is quasi contact metric if and only if it satisfies the following relation:

$$(\nabla_X \phi)Y + (\nabla_{\phi X} \phi)\phi Y = 2g(X, Y)\xi - \eta(Y)(X + \eta(X)\xi + hX), \tag{1}$$

for every vector fields X and Y on M, in which $h := \frac{1}{2} L_\xi \phi$.

The above relation holds in every contact metric manifold ([3], page116), thus quasi contact metric manifolds can be regarded as a generalization of contact metric manifolds.

An almost contact metric structure (ϕ, ξ, η, g) is said to be nearly Sasakian iff

$$(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X \tag{2}$$

for every vector fields X, Y on M.

Nearly Sasakian manifolds were introduced in 1976 by Blair and his collaborators[4]. They proved that every Sasakian manifold is nearly Sasakian, but the converse statement fails in general. In addition they proved that a normal nearly Sasakian structure is Sasakian. They also showed that a hypersurface of a nearly kähler manifold is nearly Sasakian if and only if it is quasi-umbilical with respect to the almost contact form. In [5] it is proved that nearly Sasakian manifolds are contact. In [1], [4], [7], [11], [12], [13] and [15], some other authors studied nearly Sasakian manifolds.

In this paper, we study $(\phi, \acute{\phi})$-holomorphic maps between two quasi contact metric (or nearly Sasakain) manifolds. They are two classes of almost contact metric manifolds in which $d\eta + d\eta o\phi \otimes \phi = 2\Phi$.

Conformal transformation, holomorphic map and harmonic map have many applications in physical problems, medical physics, engineering and applied science. Regarding to the importance of these maps, in this study, conformal transformation, holomorphic map and harmonic map on these manifolds are considered.

3. Some results on quasi contact metric and nearly Sasakian manifolds

There are some important properties on quasi contact metric and nearly Sasakian manifolds which we need for the next section.

In the following lemma we list some basic properties of quasi contact metric manifolds which can be easily proved by using (1) (some of them are proved in [3] and [10]).

Lemma 1. *In a quasi contact metric manifold $M = (M, \phi, \xi, \eta, g)$ the following properties are hold*

$(a)\nabla_\xi \phi = 0$

$(b)\nabla_\xi \xi = 0$

$(c)\eta o h = 0, \quad h\xi = 0$

$(d)(\nabla_X \eta)Y = g(X + hX, \phi Y)$

$(e)\phi h + h\phi = 0.$

We know that in a (ϕ, ξ, η, g) contact metric manifold, it satisfies $d\eta = \Phi$. It should be very useful if we have an expression for $d\eta$ with respect to Φ in quasi contact metric and nearly Sasakian manifolds. The following theorem gives the desired formula, and then we give a number of important properties on these manifolds by using this formula.

Theorem 1. *In a quasi contact metric manifold $M = (M, \phi, \xi, \eta, g)$ we have*

$$d\eta(X, Y) = \Phi(X, Y) + \frac{1}{2}[g(hX, \phi Y) - g(\phi X, hY)].$$

Proof. Using Lemma 1(d), we have

$$d\eta(X, Y) \quad = \quad \frac{1}{2}[(\nabla_X \eta)Y - (\nabla_Y \eta)X]$$

$$
\begin{aligned}
&= \frac{1}{2}[g(X+hX,\phi Y) - g(Y+hY,\phi X)] \\
&= g(X,\phi Y) + \frac{1}{2}[g(hX,\phi Y) - g(\phi X, hY)].
\end{aligned}
$$

It is remarkable to note that the above theorem also shows that if h is symmetric, then the quasi contact metric manifold is contact metric.

In nearly Sasakian manifold (M, ϕ, ξ, η, g), a tensor field h' of type (1, 1) is defined by ([5]):

$$h'X := \nabla_X \xi + \phi X.$$

It is skew-symmetric and anticommutes with ϕ. Moreover, $h'\xi = 0$ and $\eta o h' = 0$ ([5]), and the vanishing of h' causes the manifold to be Sasakian ([13]).

Theorem 2. *In a nearly Sasakian manifold $M = (M, \phi, \xi, \eta, g)$, we have*

$$d\eta(X,Y) = \Phi(X,Y) + g(h'X, Y).$$

Proof. We have

$$
\begin{aligned}
d\eta(X,Y) &= \frac{1}{2}[(\nabla_X \eta)Y - (\nabla_Y \eta)X] \\
&= \frac{1}{2}[g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X)] \\
&= \frac{1}{2}[g(-\phi X + h'X, Y) - g(-\phi Y + h'Y, X)] \\
&= \Phi(X,Y) + g(h'X, Y).
\end{aligned}
$$

Lemma 2. *In a quasi contact metric (or nearly Sasakian) manifold $M = (M, \phi, \xi, \eta, g)$ we have*

$$d\eta(\phi X, \phi Y) + d\eta(X,Y) = 2\Phi(X,Y).$$

Proof. By Theorem 1 we have:

$$
\begin{aligned}
d\eta(\phi X, \phi Y) + d\eta(X,Y) &= \Phi(\phi X, \phi Y) + \frac{1}{2}[\Phi(h\phi X, \phi Y) + \Phi(\phi X, h\phi Y)] \\
&\quad + \Phi(X,Y) + \frac{1}{2}[\Phi(hX, Y) + \Phi(X, hY)] \\
&= 2\Phi(X,Y) + \frac{1}{2}[-\Phi(hX,Y) - \Phi(X,hY) \\
&\quad + \Phi(hX,Y) + \Phi(X,hY)] \\
&= 2\Phi(X,Y).
\end{aligned}
$$

(By Theorem 2 and applying the same technique the result for nearly Sasakian manifolds follows.)

Theorems 1 and 2, show that $d\eta(X,\xi) = 0$, and then we deduce:

Corollary 1. *In a (ϕ, ξ, η, g)- quasi contact metric (or nearly Sasakian) manifolds, $\xi \in ker\, d\eta$.*

Theorem 3. *In a quasi contact metric (or nearly Sasakian) manifold $M = (M, \phi, \xi, \eta, g)$, η is a contact form.*

Proof. Let $M = (M, \phi, \xi, \eta, g)$ be a $(2n+1)$-dimensional quasi contact metric (or nearly Sasakain) manifold. We prove that $rankd\eta = 2n$.

Let $p \in M$ and $X \perp \xi_p$ be a nonzero vector in T_pM. If $X \in kerd\eta$ then $d\eta(X, .) = 0$. Thus by Lemma 3.4, we have

$$0 = d\eta(X, \phi X) = \Phi(X, \phi X) = -|X|^2,$$

which is a contradiction. Thus $dim(kerd\eta) \leq 1$. Also by Corollary 1 we have, $\xi \in kerd\eta$, and then $dim(kerd\eta) = 1$, that results $rankd\eta = 2n$. Thus $\eta \wedge (d\eta)^n \neq 0$ on T_pM and the proof is completed.

4. Harmonic and $(\phi, \acute{\phi})$-holomorphic maps on quasi contact metric, and nearly Sasakian manifolds

It is known that a contact manifold with contact form η carries an associated almost contact metric structure (ϕ, ξ, η, g) with $\Phi = d\eta$, called a contact metric structure, but in this section, we consider almost contact metric manifolds in which $d\eta + d\eta o\phi \otimes \phi = 2\Phi$ and treat conformal transfomations and holomorphic maps on these kind of manifolds.

Let (M, g, ∇) and $(\acute{M}, \acute{g}, \acute{\nabla})$ be two Riemannian manifolds in which ∇ and $\acute{\nabla}$ are the Levi-Civita connections on M and $\acute{M}$ respectively. For a differentiable map $f : M \to \acute{M}$ between Riemannian manifolds its tension field $\tau(f)$ is the trace of the second fundamental form B_f of f:

$$\tau(f) := \Sigma_i B_f(e_i, e_i) = \Sigma_i(\nabla^{f^{-1}T\acute{M}}_{e_i} df(e_i) - df(\nabla^{e_i}_{e_i})), \tag{3}$$

where $\{e_i\}$ is an orthonormal basis for the tangent space T_xM at $x \in M$, and $\nabla^{f^{-1}T\acute{M}}$ denotes the pull-back of the Levi-Civita connection $\acute{\nabla}$ on $\acute{M}$ to the pull-back bundle $f^{-1}T\acute{M} \longrightarrow M$, and $df : TM \longrightarrow f^{-1}T\acute{M}$ is the differential of f.

Definition 1. *([8]) A differentiable map $f : M \to \acute{M}$ between Riemannian manifolds M and $\acute{M}$ is called a harmonic map if $\tau(f) = 0$.*

Definition 2. *([8]) Let $M = (M, \phi, \xi, \eta, g)$ and $\acute{M} = (\acute{M}, \acute{\phi}, \acute{\xi}, \acute{\eta}, \acute{g})$ be two almost contact metric manifolds. We say that a differentiable map $f : M \to \acute{M}$ is $(\phi, \acute{\phi})$-holomorphic if $dfo\phi = \acute{\phi}odf$ and f is $(\phi, \acute{\phi})$-antiholomorphic if $dfo\phi = -\acute{\phi}odf$.*

Proposition 1. *Let $M = (M, \phi, \xi, \eta, g)$ be a quasi contact metric (or nearly Sasakain) manifold. If a transformation μ on M leaves ϕ invariant that means $\phi\mu = \mu\phi$, then there exists $\alpha \in \mathbb{R}$ such that the following relations are hold:*
(a) $\mu^*\eta = \alpha\eta$
(b) $\mu\xi = \alpha\xi$
(c) $\mu^*\Phi = \alpha\Phi$

Proof. First we prove (a) and (b) for some $\alpha \in C^\infty(M)$, then we prove that α must be constant.
(a) From $\eta o \phi = 0$ and $\phi\mu = \mu\phi$, we have $\eta o \mu\phi = 0$. Thus at any point x of M we have:

$$(\mu^*\eta)(\phi X)_x = 0, \quad X \in \chi(M).$$

So,

$$(\mu^*\eta)_x = \alpha(x)\eta_x,$$

for some $\alpha \in C^\infty(M)$.
(b) From $\phi\xi = 0$ and $\phi\mu = \mu\phi$, we have $\phi\mu\xi = 0$. So,

$$\mu\xi = \beta\xi,$$

for some $\beta \in C^\infty(M)$. Therefore $\eta(\mu\xi) = \eta(\beta\xi)$ and then $(\mu^*\eta)\xi = \beta\eta(\xi)$. By (a) it follows that $\alpha = \beta$.

Now, we show that α must be constant. By differentiation (a), we get:

$$d\mu^*\eta = d\alpha \wedge \eta + \alpha d\eta. \tag{4}$$

Since d and μ^* commute and by Corollary 1 and (4), we have:

$$\begin{aligned} 0 &= d\eta(\alpha\xi, \mu\xi) \\ &= d\eta(\mu\xi, \mu Y) \\ &= \mu^* d\eta(\xi, Y) \\ &= d\mu^*\eta(\xi, Y) \\ &= d\alpha \wedge \eta(\xi, Y) + \alpha d\eta(\xi, Y) \\ &= d\alpha(\xi)\eta(Y) - d\alpha(Y), \qquad Y \in \chi(M). \end{aligned}$$

So,

$$d\alpha(\xi)\eta = d\alpha. \tag{5}$$

Therefore, $d\alpha(\xi)\eta \wedge \eta = d\alpha \wedge \eta$. Then, we have $d\alpha \wedge \eta = 0$, and by differentation, we obtain $d\alpha \wedge d\eta = 0$. So (5) implies $d\alpha(\xi)\eta \wedge d\eta = 0$ and by Theorem 3 , it follows that $d\alpha(\xi) = 0$. Again, condition (5) gives $d\alpha = 0$, i.e, α is constant.

(c) Now by (4), it follows that $\mu^* d\eta = \alpha d\eta$. By lemma 2 ($2\Phi = d\eta + d\eta\ o\ \phi \otimes \phi$) we have:

$$\begin{aligned}(\mu^*\Phi) &= \frac{1}{2}(\mu^* d\eta + \mu^* d\eta\ o\ \phi \otimes \phi)\\ &= \frac{\alpha}{2}(d\eta + d\eta\ o\ \phi \otimes \phi)\\ &= \alpha\Phi.\end{aligned}$$

In ([6]), Tanno studied conformal transformation on contact metric manifold. In the following theorem, the same result is proved for quasi contact metric and nearly Sasakain manifolds.

Theorem 4. *Let $M = (M, \phi, \xi, \eta, g)$ be a quasi contact metric (or nearly Sasakain) manifold. If a transformation μ on M leaves ϕ invariant, then μ is a homothety (that means $\mu^* g = c^2 g$ for some nonzero scalar c) on $\Gamma(D)$ in which $D = \{X \in TM; \eta(X) = 0\}$).*

Proof. For an arbitrary $x \in M$ and $X, Y \in \Gamma(D)$. We have:

$$(\mu^*\Phi)_x(X, Y) = \Phi(\mu X, \mu Y).$$

So,

$$\alpha\Phi_x(X, Y) = g(\mu X, \mu\phi Y) = (\mu^* g)_x(X, \phi Y).$$

Thus we have

$$\alpha g_x(X, \phi Y) = (\mu^* g)_x(X, \phi Y).$$

Substituting Y by $-\phi X$ and we get $(\mu^* g)(X, X) = \alpha g(X, X)$, then $\alpha > 0$.

Theorem 5. *Let $M = (M, \phi, \xi, \eta, g)$ and $M' = (M', \phi', \xi', \eta', g')$ be two quasi contact metric manifolds. Any (ϕ, ϕ')-holomorphic (or (ϕ, ϕ')-antiholomorphic) submersion $f : M \to M'$ is a harmonic map.*

Proof. Let ∇ be the Riemannian connection on M. Substituting Y by ϕX in (1), we obtain:

$$(\nabla_X \phi)\phi X = (\nabla_{\phi X}\phi)X$$

for any $X \in \Gamma(D)$ in which $D = \{X \in TM; \eta(X) = 0\}$, or equivalently,

$$\nabla_X X + \nabla_{\phi X}\phi X = \phi[\phi X, X].$$

Let $\nabla^{f^{-1}TM'}$ denotes the pull-back of the Levi-Civita connection ∇' on M' to the pull-back bundle $f^{-1}TM' \longrightarrow M$. Since f is (ϕ, ϕ')-holomorphic, we also have on M':

$$\nabla_X^{f^{-1}TM'} df(X) + \nabla_{\phi X}^{f^{-1}TM'} df(\phi X) = \phi'[\phi' df(X), df(X)].$$

If B_f is the second fundamental form of f, that means B_f satisfies $B_f(X,Y) = \nabla_Y^{f^{-1}TM'} df(X) - df(\nabla_Y^X)$, it follows that for any $X \in \Gamma(D)$, we have:

$$\begin{aligned} B_f(X,X) + B_f(\phi X, \phi X) &= \nabla_X^{f^{-1}TM'} df(X) - df(\nabla_X^X) + \nabla_{\phi X}^{f^{-1}TM'} df(\phi X) - df(\nabla_{\phi X}^{\phi X}). \\ &= \phi' [\phi' \, df(X), df(X] - df\phi[\phi X, X] \\ &= 0. \end{aligned}$$

Let $(e_1, ..., e_n; \phi e_1, ..., \phi e_n, \xi)$ be a local orthonormal ϕ-basis where $\{e_i, \phi e_i\} \in \Gamma(D)$, then:

$$\tau(f) = \Sigma_{i=1}^n B_f(e_i, e_i) + \Sigma_{i=1}^n B_f(\phi e_i, \phi e_i) + B_f(\xi, \xi) = B_f(\xi, \xi).$$

Since $dfo\phi = \pm\phi' odf$, then $\phi'(df(\xi)) = \pm df(\phi\xi) = 0$, implies that there exists a function k on M such that $df_x(\xi_x) = k(x)\xi'_{f(x)}$ for any $x \in M$. On the other hand, by Lemma 3.1(b), we have respectively on M and M', $\nabla_\xi^\xi = 0$ and $\nabla_{\xi'}^{\xi'} = 0$, it follow that $B_f(\xi, \xi) = 0$ and the proof is complete.

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