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Some Coincidence and Fixed Point Results in Partially Ordered Complete Generalized D^*-Metric Spaces

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Abstract. In the present paper, several coincidence fixed point theorems established for mappings satisfying contractive conditions related to a non-Decreasing ϕ-maps in partially ordered complete generalized D^*-metric spaces where the cone is not necessarily normal which is the main result of our article.

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1. Introduction

The fixed point theorems in metric spaces are playing a fundamental role to construct methods in mathematical sciences. So the metric fixed point theorems (F.P.Ths.) has been researched extensively in the past two decades. The concept of cone metric spaces is a generalization of metric spaces (M.SPS). The Banach fixed point theorems [5] provides a technique for solving variety problems in mathematical science and engineering. In the literature there are several generalizations of the Banach's contraction principle, for some of these generalizations of the Banach's fixed point theorems and various contractive definitions that have been employed; we refer the readers to [1,6-8,11-14,17,21], and other references listed in the reference section of this article.
Recently,(F.P.Th.) has developed rapidly in partially ordered metric spaces such as [17, 18], Ran and Reurings [23] and [22] studied several new facts for contractions in partially ordered metric spaces. The authors in [15] generalized the conception of (M.SPS), substitute the R by an ordered Banach spaces (B.S) and defined cone-Metric spaces.
B.C.Dhage, [9] in 1992, defined D-Metric spaces as a generalization of (M.SPS) and he proved the existence of unique fixed point of a self-map satisfying a contractive condition in complete and bounded D-Metric spaces. In 2007, S. Shaban, etal [24] have been established the meaning of D^*-Metric spaces which as a probable modification of the definition of (D-Metric) established via the author in [9], and proved several basic properties in

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D^*-Metric spaces. Afterwards, many authors [25,26,16] proved several (F.P.Ths.) in these spaces. Fixed point problems have as well been considered partially ordered D^*-M.SPS, in [4] Alaa.M. AL. Jumaili and Xiao-Song Yang, They used the meaning of D^*-Metric spaces presented a new notion of the ∇^*-distance on a complete D^*-Metric spaces and established several (F.P.Ths.) in partially ordered D^*-metric spaces. Recently, the authors in [2] extension the concept of D^*-Metric spaces by changing R by a Real-(B.S) in D^*-Metric spaces, they established several (F.P.Th.) under certain contractive conditions.
The motivation of this article is to study several coincidence (F.P.Ths.) for functions satisfying contractive conditions concerning to a non-decreasing ϕ-maps [3,10] partially ordered complete generalized D^*-M.SPS, where the cone is not necessarily normal.

2. Preliminaries

Assume E is real-(B.S) and P is proper sub set of E. P is called an order cone (O.C) if:
a) P is closed, $P \neq \phi$ and $P \neq \{0\}$,
b) $ax + by \in P\ \forall\ x, y \in P$ and $a, b \in R^+$,
c) $x \in P$ and $-x \in P$ implies $x = 0$.
For an (O.C) $P \subset E$, we define a partial ordering $\preccurlyeq$ on E with respect to P via $x \preccurlyeq y$ iff $y - x \in P$. we shall using $x \prec y$ to indicate that $x \preccurlyeq y$ but $x \neq y$, while $x \ll y$ for $y - x \in intP$, where $intP$ refer to the interior of P.
The (O.C) P is called normal if $\exists$ a number $K > 0$ (s.t) $\forall x, y \in E, 0 \leq x \leq y \Rightarrow$,

$$\|x\| \leq K\|y\| \quad \text{..................................(2.1)}$$

Or equivalently,

$$inf\{\|x + y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0 \quad \text{..................................(2.2)}$$

We name the positive element K which satisfying (2.1) normal constant of P.
From (2.2) we can deduce that P is non-normal iff $\exists$ sequences $\{x_s\}, \{y_s\} \in P$ (s.t),

$$0 \leq \{x_s\} \leq \{x_s\} + \{y_s\}, lim_{s\to\infty}(\{x_s\} + \{y_s\}) = 0, \text{ but } lim_{s\to\infty}\{x_s\} \neq 0.$$

In this paper, E stands for a real-(B.S), P is a cone in E with $intP \neq \{0\}$ (such cones are called solid) and $\preccurlyeq$ is a partial-ordering (P-O) with respect to P, where the cone is not necessarily normal unless otherwise stated.

Now, recall several basic definitions and results of generalized D^*-Metric spaces, and for more details on D^*-Metric spaces and generalized D^*-Metric spaces, we refer the authors for review [24] and [2] respectively.

Definition 1. [2] *Let X be a non empty set. A generalized D^*-M.SP on a set* X *is a function, D^*: $X \times X \times X \to E$, that satisfies the following conditions $\forall x, y, z, a \in X$:*

A. $D^*(x, y, z) \geq 0$,

B. $D^*(x,y,z) = 0 \Leftrightarrow x = y = z$,
C. $D^*(x,y,z) = D^*(p\{x,y,z\})$,*(symmetry) where* p *is a permutation function,*
D. $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$.

In that case D^* *is called a generalized* D^**-Metric (*D^**-Cone metric) and* (X, D^*) *is called a generalized* D^**-M.SP (*D^**-Cone metric space).*

Remark 1. *It is obvious that the concept of a generalized* D^**-M.SP (*D^**-Cone metric space) is more general than that of* D^**-M.SP or Cone metric space. If* $E = R$ *and* $P = [0, +\infty)$ *after that a generalized* D^**-M.SP becomes* D^**-M.SP.*

Example 1. *Suppose* $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}, X = R$, *defined a function,* D^**:* $X^3 \to E$ *via :*
$D^*(x,y,z) = (|x-y|+|y-z|+|x-z|, \alpha(|x-y|+|y-z|+|x-z|)) \ni \alpha \geq 0$ *is a constant* $(see[2])$. *In that case* (X, D^*) *is a generalized* D^**-M.SP (*D^**-Cone metric space). over the normal cone P.*

Example 2. *let* $E = C^1_R[0,1]$ *with* $\|u\| = \|u\|_\infty + \|u'\|_\infty$ *and* $P = \{u \in E : u(t) \geq 0$ *on* $[0, 1]\}$ *(see, e.g., [27]). Let* $X = [0, +\infty)$ *and,*
$d(x,y) = |x-y|, g(x,y,z) = d(x,y)+d(y,z)+d(z,x)$ $\forall x,y,z \in X$, *defined a function* D^**:* $X^3 \to P$ *via* D^* *(x, y, z) = g (x, y, z)u where* $u \in P$ *is fixed. In that case* (X, D^*) *is a generalized* D^**-M.SP over the non-normal cone P.*

Lemma 1. [2] *Assume* (X, D^*) *is a generalized* D^**-M.SP, in that case* $\forall x, y \in X$, *obtain* $D^*(x, x, y)$ *equal to* $D^*(x, y, y)$.

Remark 2. *For the case of non-normal cones, the following remarks holed and useful in the sequel for elements* $u, v, w \in P$*:*
$(R_1)-$ *If* $u \leq v$ *and* $v \ll w$, *in that case* $u \ll w$.
$(R_2)-$ *If* $u \ll v$ *and* $v \leq w$, *in that case* $u \ll w$.
$(R_3)-$ *If* $0 \leq u \ll d$ $\forall$ $d \in intP$, *in that case* $u = 0$.

Definition 2. [2] *suppose* (X, D^*) *is a generalized* D^**-M.SP in that case:*
a) A sequence$\{x_s\}$ *in* X *is called Cauchy sequence if* $\forall$ *d belong to E with* $0 \ll d$, *there exist H (s.t)* $\forall$ $r, s, l \geq H$, $D^*(x_r, x_s, x_l) \ll d$.
b) If each Cauchy sequence is convergent in X*, in that case* X *is called complete generalized* D^**-Metric.*
c) A sequence$\{x_s\} \to x \in X$, *if* $\forall$ $d \in E$ *with* $0 \ll d$ *there exist H (s.t)* $\forall$ $r, s \geq H$, $D^*(x_r, x_s, x) \ll d$, *and x is the limit point of* $\{x_s\}$ *with indicate via* $x_s \to x$, *as* $(s \to \infty)$.

Proposition 1. [2] *Assume* (X, D^*) *is a generalized* D^**-M.SP in X, If* $x_s \to x$, *in that case* $\{x_s\}$ *is a Cauchy sequence.*

Proposition 2. [2] *Assume* (X, D^*) *is a generalized* D^**-M.SP, and P is normal cone with normal constant K. Suppose* $\{x_s\}$ *in X, in that case* $\{x_s\}$ *converges to* $x \Leftrightarrow D^*(x_r, x_s, x) \to 0$, *as* $(r, s \to \infty)$.

Proposition 3. [2] *Assume (X, D^*) is a generalized D^*-M.SP, and P is normal cone. Suppose $\{x_s\}$ in X and $x \in X$. in that case the following equivalent:*
a) $\{x_s\}$ is D^*-convergent to x;
b) $D^*(x_s, x_s, x)$ convergent to 0, when $(s \to \infty)$;
c) $D^*(x_s, x, x)$ convergent to 0, when $(s \to \infty)$.

Definition 3. *Suppose* $\mathrm{X} \neq \phi$, *in that case $(X, D^*, \preccurlyeq)$ is said to be an ordered generalized D^*-M.SP if the following hold:*
a) (X, D^*) is a generalized D^*-M.SP,
b) $(X, \preccurlyeq)$ is a partially ordered set $(P.O.S)$.

Recall that if $(X, \preccurlyeq)$ is a $(P.O.S)$, in that case $x, y \in X$ are said to be comparable [19] when $x \preccurlyeq y$ or $y \preccurlyeq x$ satisfies.
Also Nashine and Samet, in [20] introduced the following concept:
Let $\mathrm{X} \neq \phi$ *and let $\pounds : X \to X$, $\forall x \in X$, we denoted via $\pounds^{-1}(x)$ the sub set of X given through $\pounds^{-1}(x) := \{u \in X : \pounds u = x\}$.*

Definition 4. [19] *Assume $(X, \preccurlyeq)$ is a $(P.O.S)$ and let $T, G, \pounds : X \to X$ be given mappings (s.t) $TX \subseteq \pounds X$ and $GX \subseteq \pounds X$. Describe that G and T are weakly increasing with respect to $\pounds$ if for each $x \in X$, we obtain:*
$Tx \preccurlyeq Gy, \forall\, y \in \pounds^{-1}(Tx)$ and $Gx \preccurlyeq Ty, \forall\, y \in \pounds^{-1}(Gx)$. If $T = G$, we say that T is weakly increasing with related to $\pounds$.

Remark 3. *If $\pounds : X \to X$ is the identity mapping ($\pounds x = x$ $\forall x \in X$), in that case G and T are weakly increasing with respect to $\pounds \Leftrightarrow G$ and T are weakly increasing mappings [19], i.e., $Tx \preccurlyeq G(Tx)$ and $Gx \preccurlyeq T(Gx)$ hold $\forall x \in X$.*

Definition 5. *Let $(X, \preccurlyeq)$ be an ordered generalized D^*-M.SP, X is called regular [19] if the next condition holds:*
if $\{z_s\}$ is a non-Decreasing sequence in $(X, \preccurlyeq)$ (s.t) $z_s \to z \in X$, when $(s \to \infty)$, in that case $z_s \preccurlyeq z$, for each s belong to N.

3. Coincidence Fixed Point Theorems in Partially Ordered Complete Generalized D^*-M.SPS

In this section, we establish several coincidence (F.P.Ths.) in partially ordered complete generalized D^*-M.SPS. We start with the following definition (ϕ-maps).

Definition 6. [3, 10]. *Let P be $(O.C)$. A non-Decreasing function $\phi : P \to P$ is called an ϕ-maps if:*
a) $\phi(0) = 0$ and $0 < \phi(w) < w$ for $w \in P \backslash \{0\}$,
b) $w \in intP$ implies $w - \phi(w) \in intP$,
c) If $w \in P \backslash \{0\}$ and $d \in intP$, So $\exists\ s_0 \in N$ (s.t) $\phi^s(w) \ll d$ for each $s \geq s_0$.

Example 3 (3). (a) - *If P is an arbitrary cone in $(B.S)$ E and $\delta \in (0, 1)$, in that case $\phi : P \to P$, defined by $\phi(w) = \delta w$ for $w \in P$, is a ϕ-maps.*

(b) - let $\Psi : [0, +\infty) \to [0, +\infty)$ be any real valued ϕ-map and let P be a cone in (B.S) E and $\delta \in (0, 1)$, be fixed. In that case the function $\phi_\delta : P \to P$ defined by: $\phi_\delta(w) = \Psi(\delta)w$, is a ϕ-maps. Examples of this type are of particular interest in the case when the cone P is non-normal. (See Example 2), one can take,
$E = C^1_R[0, 1]$, $P = \{x \in E : x(t) \geq 0$ *on* $[0, 1]\}$ *as well* $\Psi(\delta) = \dfrac{\delta}{1 + \delta}; \delta \in (0, 1)$.

The following theorem is our first main results.

Theorem 1. *suppose $(X, \preccurlyeq)$ is $(P.O.S)$ with assume (X, D^*) is a generalized D^*-M.SP and P is $(O.C)$ with normal cone K.*
let $£, T : X \to X$ be two maps (s.t),
$D^*(Tx, Ty, Tz) \leq \phi(D^*(£x, £y, £z))$ .... *(3.1)*
And assume the following:
a) T is weakly increasing with respect to $£$,
b) $£X$ is a complete sub-space of X,
c) X is regular.
For each $x, y, z \in X$ with $£z \preccurlyeq £y \preccurlyeq £x$ where ϕ is a ϕ-map. In that case $£$ and T have a coincidence point.

Proof. Assume that a point $x_0 \in X$ is arbitrary. By definition (4) we have $TX \subseteq £X$, thus we can construct a sequence $\{x_s\}$ in X via: $£x_{s+1} = Tx_s, \forall s \in N_0$.
Since T is weakly increasing with respect to $£$ and $x_1 \in £^{-1}(Tx_0)$ and $x_2 \in £^{-1}(Tx_1)$, in that case we get: $£x_1 \preccurlyeq £x_2 \preccurlyeq £x_3 \preccurlyeq$ .... $\preccurlyeq £x_s \preccurlyeq £x_{s+1} \preccurlyeq$ ....

Now establish that $\{£x_s\}$ is a Cauchy sequence in $(£(X), D^*)$. We will discuss two cases:
(a)- There exists $s \in N$ (s.t) $£x_s = £x_{s+1}$. Using the considered contractive condition, get:
$Tx_s = Tx_{s+1}$, that is, $£x_{s+1} = £x_{s+2}$. Therefore we have $£x_r = £x_s$, $\forall\ r \geq s \Rightarrow \{£x_s\}$ is a Cauchy sequence in $(£(X), D^*)$.
(b) - The successive conditions of a sequence $\{£x_s\}$ are different. From the above inequality (3.1), we obtain:
$D^*(£x_s, £x_s, £x_{s+1})$
$\leq D^*(Tx_{s-1}, Tx_{s-1}, Tx_s)$
$\leq \phi(D^*(£x_{s-1}, £x_{s-1}, £x_s))$
$\leq \phi^2(D^*(£x_{s-2}, £x_{s-2}, £x_{s-1}))$.......... $\leq \phi^s(D^*(£x_0, £x_0, £x_1))$.
Fix d, $0 \ll d$. By means of the characteristic (c) of definition (6), $\exists\ s_0 \in N$ (s.t),
$\phi^s(D^*(£x_0, £x_0, £x_1)) \ll d\ \forall\ s \geq s_0$.
According the Remark $(2 - R_1)$, we get:
$D^*(£x_s, £x_s, £x_{s+1}) \ll d\ \forall\ s \geq s_0$. In the same method choose a natural number $s_1 \in N$ (s.t): $D^*(£x_r, £x_r, £x_{r+1}) < d - \phi(d)\ \forall\ r \geq s_1$ ....(3.2)

Now via induction on r we will prove the following claim:
$D^*(\pounds x_s, \pounds x_s, \pounds x_r) \ll d \; \forall \; r > s \geq s_1$ ....(3.3)

By using the inequality (3.2) and the truth that $d - \phi(d) < d$, we have the inequality (3.3) holds for $r = s + 1$. Assume that the inequality (3.3) holds for $r = h$.
For $r = h + 1$ and by using Remark (2) we obtain:
$D^*(\pounds x_s, \pounds x_s, \pounds x_{h+1})$
$\leq D^*(\pounds x_s, \pounds x_s, \pounds x_{s+1}) + D^*(\pounds x_{s+1}, \pounds x_{s+1}, \pounds x_{h+1})$
$\ll d - \phi(d) + \phi(D^*(\pounds x_s, \pounds x_s, \pounds x_h))$
$\ll d - \phi(d) + \phi(d) = d.$
By induction on r, we conclude that the inequality (3.3) holds $\forall \; r > s \geq s_1$. Now the part (D) of definition (1) implies that,
$D^*(x_r, x_s, x_l) \leq D^*(x_r, x_r, x_s) + D^*(x_s, x_l, x_l) \ll 2d$ holds for $r, s, l \geq s_1$.
Therefore, $\{\pounds x_s\}$ is $(D^* - Cauchysequence)$ in $(\pounds(X), D^*)$ which is complete by assumption. Thus, $\exists \; u = \pounds v$ and $z \in X$ (s.t),
$lim_{s\to\infty} \pounds x_s = u = \pounds z$..(3.4)
Since X is regular and $\{\pounds x_s\}$ is a non-Decreasing sequence, we get from (3.4) that $\pounds x_s \preccurlyeq \pounds z \; \forall \; s \in N$. Assume that $\pounds x_s \neq \pounds z$. Fix d, $0 \ll d$, choose, $s \in N$ (s.t),
$D^*(\pounds x_s, \pounds z, \pounds z) \ll \dfrac{d}{2}$ and $D^*(\pounds x_{s+1}, \pounds z, \pounds z) \ll \dfrac{d}{2}$.
Therefore, we can apply the considered contractive condition to obtain:
$D^*(Tz, Tz, \pounds z)$
$\leq D^*(Tz, Tz, Tx_s) + D^*(Tx_s, \pounds z, \pounds z)$
$\leq \phi(D^*(\pounds x_s, \pounds z, \pounds z)) + D^*(\pounds x_{s+1}, \pounds z, \pounds z)$ (Using (3.1))
$< D^*(\pounds x_s, \pounds z, \pounds z) + D^*(\pounds x_{s+1}, \pounds z, \pounds z)$
$\ll \dfrac{d}{2} + \dfrac{d}{d} = d.$
Since $d \in intP$, via Remark $(2 - R_1)$ it follows that $D^*(Tz, Tz, \pounds z) = 0$ such that $Tz = \pounds z$. Thus z is a coincidence point for $\pounds$ and T.

Next, remember the following case see [19] to explain the validity of Theorem (1).

Example 4. *Assume that (X, D^*) is a generalized D^*-M.SP. Consider Example (2), with the reverse order: $x \preccurlyeq y \Leftrightarrow x \geq y$. Define a maps $T : X \times X \to X$ and $\pounds : X \times X \to X$ as follows: $Tx = 2x$ and $\pounds x = 3x$ and a ϕ-map define by $\phi(w) = \dfrac{w}{2}, w \in P$. Hence all the conditions of Theorem (1) are satisfied; particularly one can be able to reduce condition (3.1) to: $2(|x - y| + |y - z| + |z - x|)u \geq \frac{3}{2}(|x - y| + |y - z| + |z - x|)u$, and holds $\forall \; x, y, z \in [0, +\infty)$. As well, T is weakly increasing with respect to $\pounds$, since $\pounds y = Tx \Rightarrow 3y = 2x \Rightarrow y = \frac{2x}{3}$, which mean implies $Tx = 2x \geq 2y = Ty$, it mean $Tx \preccurlyeq Ty$. Clear that, 0 a coincidence point of $\pounds$ and T.*

Corollary 1. *Assume $(X, \preccurlyeq)$ is $(P.O.S)$ with suppose that (X, D^*) is a generalized D^*-M.SP and P is $(O.C)$. Let $\pounds, T : X \to X$ be two non-Decreasing maps. Assume that $\forall \; x, y, z \in X$ with $z \preccurlyeq y \preccurlyeq x \; \exists$ some $h \in [0, 1)$ (s.t): $D^*(Tx, Ty, Tz) \leq hD^*(\pounds x, \pounds y, \pounds z)$ holds. Assume the following:*

a) T is weakly increasing with respect to $\pounds$,
b) $\pounds X$ is a complete sub-space of X,
c) X is regular.
In that case $\pounds$ and T have a coincidence point.

Proof. The proof is directly consequence from Theorem (1) when taking $\phi(w) = hw$.

Remark 4. *if the mapping $\pounds : X \to X$ is identity, we get the following $(F.P)$ result.*

Corollary 2. *Assume $(X, \preccurlyeq)$ is $(P.O.S)$ and suppose that (X, D^*) is a complete generalized D^*-M.SP and P is $(O.C)$. Let $T : X \to X$ be a mapping (s.t) $D^*(Tx, Ty, Tz) \leq \phi(D^*(x, y, z))$ holds $\forall\ x, y, z \in X$ with $z \preccurlyeq y \preccurlyeq x$ where ϕ is a ϕ-map. Assume the following:*
a) $Tx \preccurlyeq T(Tx)\ \forall\ x \in X$,
b) X is regular.
In that case T has a $(F.P)$.

Next, our result is the following generalization of Theorem (1).

Theorem 2. *Assume $(X, \preccurlyeq)$ is $(P.O.S)$ and suppose that (X, D^*) is a complete generalized D^*-M.SP and P is $(O.C)$, and Let $\pounds, T : X \to X$ be a non-Decreasing maps. Assume that $\forall\ x, y, z \in X$ with $\pounds z \preccurlyeq \pounds y \preccurlyeq \pounds x$ there exists,*
$\Phi(x, y, z) \in \{D^*(\pounds x, \pounds y, \pounds z), D^*(\pounds x, \pounds x, Tx), D^*(\pounds y, \pounds y, Ty), D^*(Tx, \pounds y, \pounds z)\}$ (s.t):
$D^*(Tx, Ty, Tz) \leq \phi(\Phi(x, y, z))$ where ϕ is a ϕ-map. Assume the following:
a) T is weakly increasing with respect to $\pounds$,
b) X is regular.
In that case $\pounds$ and T have a coincidence point.

Proof. Assume that a point $x_0 \in X$ is arbitrary. By definition (4) we have, $TX \subseteq \pounds X$, thus construct a sequence $\{x_s\}$ in X defined via:
$\pounds x_{s+1} = Tx_s$, for each $s \in N$. since $x_1 \in \pounds^{-1}(Tx_0)$ and $x_2 \in \pounds^{-1}(Tx_1)$, and via T is weakly increasing with respect to $\pounds$, we get that:
$\pounds x_1 = Tx_0 \preccurlyeq Tx_1 = \pounds x_2 \preccurlyeq Tx_2 = \pounds x_3$. Continuing this process, we obtain that:
$\pounds x_1 \preccurlyeq \pounds x_2 \preccurlyeq \pounds x_3 \preccurlyeq .... \preccurlyeq \pounds x_s \preccurlyeq \pounds x_{s+1} \preccurlyeq ....$
Assume that $\exists\ s_0 \in \{1, 2, 3, ...\}$ (s.t) $\Phi(x_{s_0}, x_{s_0}, x_{s_0-1}) = 0$ thus it is clear that $\pounds x_{s_0-1} = \pounds x_{s_0} = Tx_{s_0-1}$ therefore we are completed.
Next we can assume $\Phi(x_s, x_s, x_{s-1}) > 0\ \forall\ s \geq 1$.
Assume $\pounds x_{s-1} \neq \pounds x_s\ \forall\ s \in N$ therefore for $s \in N$ we obtain:
$D^*(\pounds x_s, \pounds x_s, \pounds x_{s+1}) = D^*(Tx_{s-1}, Tx_{s-1}, Tx_s) \leq \phi(\Phi(x_{s-1}, x_{s-1}, x_s))$ where,
$\Phi(x_{s-1}, x_{s-1}, x_s) \in \{D^*(\pounds x_{s-1}, \pounds x_{s-1}, \pounds x_s), D^*(\pounds x_{s-1}, \pounds x_{s-1}, Tx_{s-1}),$
$D^*(\pounds x_s, \pounds x_s, Tx_s), D^*(Tx_{s-1}, Tx_{s-1}, \pounds x_s)\} = \{D^*(\pounds x_{s-1}, \pounds x_{s-1}, \pounds x_s),$
$D^*(\pounds x_{s-1}, \pounds x_{s-1}, \pounds x_s), D^*(\pounds x_s, \pounds x_s, \pounds x_{s+1}), D^*(\pounds x_s, \pounds x_s, \pounds x_s)\}$
$= \{D^*(\pounds x_{s-1}, \pounds x_{s-1}, \pounds x_s), D^*(\pounds x_s, \pounds x_s, \pounds x_{s+1}), 0\}$.

I. If $\Phi(x_{s-1},x_{s-1},x_s)=D^*(£x_s,£x_s,£x_{s+1})$ therefore, $D^*(£x_s,£x_s,£x_{s+1})\leq\phi(D^*(£x_s,£x_s,£x_{s+1}))$, and via the characteristic of ϕ-map we get:
$D^*(£x_s,£x_s,£x_{s+1})<D^*(£x_s,£x_s,£x_{s+1})$ which is impossible.

II. If $\Phi(x_{s-1},x_{s-1},x_s)=0$, hence $D^*(£x_s,£x_s,£x_{s+1})\leq\phi(0)<0$ which is a contradiction. So, $\Phi(x_{s-1},x_{s-1},x_s)=D^*(£x_{s-1},£x_{s-1},£x_s)$, thus
$D^*(£x_s,£x_s,£x_{s+1})\leq\Phi(D^*(£x_{s-1},£x_{s-1},£x_s))$ after that for $s\in N$, we obtain:
$D^*(£x_s,£x_s,£x_{s+1})$
$=D^*(Tx_{s-1},Tx_{s-1},Tx_s)$
$\leq\phi(D^*(£x_{s-1},£x_{s-1},£x_s))$
$\leq\phi^2(D^*(£x_{s-2},£x_{s-2},£x_{s-1}))$.......... $\leq\phi^s(D^*(£x_0,£x_0,£x_1))$

We can show that $\{£x_s\}$ is a Cauchy sequence by similar method to that in the evidence of Theorem (1). Since X is D^*-complete, so $\{£x_s\}$ is convergent to a point u in X. Now we explain that $£u=Tu$.
Since $\{£x_s\}$ non-Decreasing sequence and $£x_s\to u$, therefore by regularity of X we have $£x_s\preccurlyeq u\ \forall\ s$. if $£x_s=u$ for some u, hence, by construction we obtain, $£x_{s+1}=u$ and u is $(F.P)$. So we presume that $£x_s\neq u$, thus for $s\in N$ we obtain:
$D^*(£u,Tu,Tu)$
$\leq D^*(£u,£x_s,£x_s)+D^*(£x_s,Tu,Tu)$
$=D^*(£u,£x_s,£x_s)+D^*(Tx_{s-1},Tu,Tu)$
$\leq D^*(£u,£x_s,£x_s)+\phi(\Phi(x_{s-1},u,u))$ where,
$\Phi(x_{s-1},u,u)\in\{D^*(£x_{s-1},£u,£u),D^*(£x_{s-1},£x_{s-1},Tx_{s-1}),$
$D^*(£x_{s-1},£x_{s-1},Tx_{s-1}),D^*(Tx_{s-1},£x_{s-1},£u)\}=$
$\{D^*(£x_{s-1},£u,£u),D^*(£x_{s-1},£x_{s-1},£x_s),D^*(£x_s,£x_{s-1},£u)\}$

Fix d, $0\ll d$, and choose $N_1\in N$ (s.t),
$D^*(£u,£x_s,£x_s)\ll\dfrac{d}{2}$ and $D^*(£x_{s-1},£u,£u)\ll\dfrac{d}{2},\forall\ s\geq N_1$.
We can discuss three cases as following:

A. If $\Phi(x_{s-1},u,u)=D^*(£x_{s-1},£u,£u)$, therefore we have:
$D^*(£u,Tu,Tu)$
$\leq D^*(£u,£x_s,£x_s)+\phi(D^*(£x_{s-1},£u,£u))$
$<D^*(£u,£x_s,£x_s)+D^*(£x_{s-1},£u,£u)$
$\ll\dfrac{d}{2}+\dfrac{d}{2}=d.$

B. If $\Phi(x_{s-1},u,u)=D^*(£x_{s-1},£x_{s-1},£x_s)$, after that we obtain:
$D^*(£u,Tu,Tu)$
$\leq D^*(£u,£x_s,£x_s)+\phi(D^*(£x_{s-1},£x_{s-1},£x_s))$
$<D^*(£u,£x_s,£x_s)+D^*(£x_{s-1},£x_{s-1},£u)$
$\ll\dfrac{d}{2}+\dfrac{d}{2}=d.$

C. If $\Phi(x_{s-1}, u, u) = D^*(\pounds x_s, \pounds x_{s-1}, \pounds u)$, thus we get:
$D^*(\pounds u, Tu, Tu)$
$\leq D^*(\pounds u, \pounds x_s, \pounds x_s) + \phi(D^*(\pounds x_s, \pounds x_{s-1}, \pounds u))$
$< D^*(\pounds u, \pounds x_s, \pounds x_s) + D^*(\pounds x_s, \pounds x_{s-1}, \pounds u)$
$\leq D^*(\pounds u, \pounds x_s, \pounds x_s) + D^*(\pounds x_s, \pounds x_s, \pounds x_{s-1}) + D^*(\pounds x_{s-1}, \pounds u, \pounds u) \ll d.$
Whenever $s \in N$, Therefore in all above cases we have:
$D^*(\pounds u, Tu, Tu) \ll d$ for arbitrary $d \in intP$. According the Remark $(2 - R_3)$, it follows that $D^*(\pounds u, Tu, Tu) = 0$, which implies that $\pounds u = Tu$.
Thus we conclude that u is a coincidence point for $\pounds$ and T.

Corollary 3. *Assume* $(X, \preccurlyeq)$ *is* $(P.O.S)$ *and suppose that* (X, D^*) *is a complete generalized* D^**-M.SP and P is* $(O.C)$*. Let* $\pounds, T : X \to X$ *be non-Decreasing mappings. Assume that for some* $h \in [0, 1)$*, and* $\forall\ x, y, z \in X$ *with* $Fz \preccurlyeq \pounds y \preccurlyeq \pounds x$, $\exists$
$\Phi(x, y, z) \in \{D^*(\pounds x, \pounds y, \pounds z), D^*(\pounds x, \pounds x, Tx), D^*(\pounds y, \pounds y, Ty), D^*(Tx, \pounds y, \pounds z)\}$
(s.t): $D^*(Tx, Ty, Tz) \leq h\Phi(x, y, z)$*. We assume the following:*
a) T is weakly increasing with respect to $\pounds$*,*
b) X is regular.
In that case $\pounds$ *and T have a coincidence point.*

Proof. The proof is direct result from Theorem (2).

Remark 5. *if the mapping* $\pounds : X \to X$ *is identity, so from Theorem (2) we get easily the following* $(F.P)$ *result.*

Corollary 4. *Assume* $(X, \preccurlyeq)$ *is (P.O.S) and suppose that* (X, D^*) *is a complete generalized* D^**-M.SP and P is (O.C). Let* $\pounds, T : X \to X$ *be non-Decreasing mappings, (s.t):*
$D^*(Tx, Ty, Tz) \leq \phi(\Phi(x, y, z))$ *where*
$\Phi(x, y, z) \in \{D^*(x, y, z), D^*(x, x, Tx), D^*(y, y, Ty), D^*(Tx, y, z)\}$ *and* $\forall\ x, y, z \in X$ *with* $z \preccurlyeq y \preccurlyeq x$*, where* ϕ *is a* ϕ*-map. Assume the following:*
a) $Tx \preccurlyeq T(Tx)\ \forall\ x \in X$*,*
b) X is regular.
In that case T has $(F.P)$*.*

Next, we present a sufficient condition for the unique-ness of the point of coincidence in the following result.

Theorem 3. *Assume that* $(X, \preccurlyeq)$ *is a totally ordered set and* (X, D^*) *is a complete generalized* D^**-M.SP, P is* $(O.C)$*. Let* $\pounds, T : X \to X$ *be non-Decreasing mappings. Assume that* $\forall\ x, y, z \in X$ *with* $\pounds z \preccurlyeq \pounds y \preccurlyeq \pounds x$ $\exists$*,*
$\Phi(x, y, z) \in \{D^*(\pounds x, \pounds y, \pounds z), D^*(\pounds x, \pounds x, Tx), D^*(\pounds y, \pounds y, Ty), D^*(Tx, \pounds y, \pounds z)\}$
(s.t): $D^*(Tx, Ty, Tz) \leq \phi(\Phi(x, y, z))$ *where* ϕ *is a* ϕ*-map. Assume the following:*
a) T is weakly increasing with respect to $\pounds$*,*
b) X is regular.
In that case $\pounds$ *and T have a unique coincidence point.*

Proof. Assume that $\pounds$ and T have two points of coincidence, u and w, (s.t) $\pounds u = Tu$ and $\pounds w = Tw, \pounds u \neq \pounds w$. As $(X, \preccurlyeq)$ is a totally ordered set and $u, w \in X$, assume that $u \prec w$. by means of the contractive condition we have that:
$D^*(\pounds u, \pounds w, \pounds w) = D^*(Tu, \pounds w, \pounds w) \leq \phi(\Phi(u, w, w))$ holds for some,
$\Phi(u, w, w) \in \{D^*(\pounds u, \pounds w, \pounds w), D^*(\pounds u, \pounds u, Tu), D^*(\pounds u, \pounds u, Tu), D^*(Tu, \pounds u, \pounds w)\}$
$= \{0, D^*(\pounds u, \pounds w, \pounds w)\}$
By means of characteristic of ϕ-mapping we get a contradiction. Therefore $\pounds u = \pounds w$. Thus $\pounds$ and T have a unique point of coincidence $\pounds u = Tu$.

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