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The Group of Units of Integral Group Rings of Extra-Special 2-Groups

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Abstract. One of the main problems on group rings is to determine its group of units. In this paper, we describe the group of units of integral group rings of two extra-special 2-groups: one of order 32, the central product of two copies of D_4, and another of order 128, the central product of three copies of D_4.

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1. Introduction

A important problem on group rings is to describe precisely the group of units $\mathcal{U}(RG)$ of a group ring RG, where R is a commutative ring with identity and G is a finite group. The high degree of complexity of this problem became evident in the seventies, when it was proven that, in general, the group of units contains a non abelian free subgroup. Many researchers, using techniques of group representation theory and algebraic number theory, presented an explicit description of the group of units, a description of the general structure of $\mathcal{U}(RG)$ or a set of generators of a finite index subgroup of $\mathcal{U}(RG)$. On these subjects we could quote A. K. Bhandari and I. S. Luthar [1], A. Bovdi and F. C. Polcino Milies [2], R. A. Ferraz [3], A. Giambruno and S. K. Seghal [4], E. G. Goodaire and E. Jespers [5], E. Jespers and G. Leal [8], [9], E. Jespers and G. Leal and F. C. Polcino Milies [10], E. Jespers and M. M. Parmenter and S. K. Sehgal [11], F. C. Polcino Milies [12], J. Ritter and S. K. Sehgal [13], [14], [15] and two new books by E. Jespers and A. del Rio [6], [7], and many others. In [6, 7], for many finite groups G, methods are given to describe all the rational representations of G. In particular, for nilpotent nite groups G the Wedderburn decomposition of the rational group algebra $\mathbb{Q}G$ is explicitly given via the construction of a complete set of matrix units of $\mathbb{Q}G$. From this one obtains an explicit set of nitely many generators for a subgroup of nite index of the unit group $\mathcal{U}(\mathbb{Z}G)$; these

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generators are the so called Bass units and bicyclic units. Actually, it known that for many nite groupsG the Bass units together with the bicyclic units generate a subgroup of nite index, the groups excluded are determined by the existence of exceptional simple components of $\mathbb{Q}G$ (such as non-commutative division algebras and $M_2(\mathbb{Q})$). In general it remains a problem to describe the full unit group $\mathcal{U}(\mathbb{Z}G)$. In [6] this has been done for several examples of nite groups.

F. C. Polcino Milies [10] was the first to describe the unit group $\mathcal{U}(\mathbb{Z}D_4)$, where D_4 denotes the dihedral group of order 8. Later, E. Jespers and G. Leal [6] described the same group using a different method that also was applied to other 2-groups. In this paper we describe* the group of units of the integral group rings of two extra-special 2-groups: G_1 of order 32, the central product of two copies of D_4, and G_2 of order 128, the central product of three copies of D_4.

2. Notation and Terminology

Before we begin, we will recall the definition of extra-special p-group, where p is a prime number:

Definition 1. *A p-group G is called extra-special if it is not abelian and its commutator subgroup G' coincides with its center $Z(G)$ and is of order p. In particular, every extra-special p-group is the central product of non-abelian subgroups of order p^3.*

Let $D_4 = \langle b, v \,|\, b^2 = v^4 = 1 \text{ and } bvbv = 1\rangle$ be the dihedral group of order 8. We also adopt the following notation for elements of D_4: $a = bv^2$, $s = v^2$, $t = bv$, $u = vb$, $w = v^3$.

Let D, D^1 and D^2 be groups isomorphic D_4, where the indices used are necessary to differentiate the elements. For example, we denote by b_i and v_i, $1 \le i \le 2$, the elements of D^i wich correspond respectively to b and v in D. The distinction of the elements is essential for the proofs we make in this work. With this notation, $s_i = v_i^2$, $1 \le i \le 2$, and the correspondence of the other elements is obvious. Thus we have the extra-special 2-group G_1 of order 32, the central product of two copies of D_4, and the 2-group extra-special G_2 of order 128, the central product of three copies of D_4:

$$G_1 = D \times D^1 \Big/ \{1, ss_1\}, \quad G_2 = D \times D^1 \times D^2 \Big/ \{1, ss_1, ss_2, s_1s_2\}.$$

The elements of G_1/G_1', where $G_1' = \{1, s\}$ is the commutator subgroup of G_1, and the elements of G_2/G_2', where $G_2' = \{1, s\}$ is the commutator subgroup of G_2, are denoted as follows:

$$G_1/G_1' = \left\{\overline{1},\, \overline{b},\, \overline{u},\, \overline{v},\, \overline{b_1},\, \overline{u_1},\, \overline{v_1},\, \overline{bb_1},\, \overline{bu_1},\, \overline{bv_1},\, \overline{ub_1},\, \overline{uu_1},\, \overline{uv_1},\, \overline{vb_1},\, \overline{vu_1},\, \overline{vv_1}\right\};$$

*The calculations given are complete and are independent of the general frame work given in [1, 2].

$$G_2/G_2' = \left\{ \begin{array}{l} \overline{1},\ \overline{b},\ \overline{u},\ \overline{v},\ \overline{b_1},\ \overline{u_1},\ \overline{v_1},\ \overline{b_2},\ \overline{u_2},\ \overline{v_2},\ \overline{bb_1},\ \overline{bu_1},\ \overline{bv_1},\ \overline{bb_2},\ \overline{bu_2},\ \overline{bv_2}, \\ \overline{ub_1},\ \overline{uu_1},\ \overline{uv_1},\ \overline{ub_2},\ \overline{uu_2},\ \overline{uv_2},\ \overline{vb_1},\ \overline{vu_1},\ \overline{vv_1},\ \overline{vb_2},\ \overline{vu_2},\ \overline{vv_2}, \\ \overline{b_1b_2},\ \overline{b_1u_2},\ \overline{b_1v_2},\ \overline{u_1b_2},\ \overline{u_1u_2},\ \overline{u_1v_2},\ \overline{v_1b_2},\ \overline{v_1u_2},\ \overline{v_1v_2},\ \overline{bb_1b_2}, \\ \overline{bb_1u_2},\ \overline{bb_1v_2},\ \overline{bu_1b_2},\ \overline{bu_1u_2},\ \overline{bu_1v_2},\ \overline{bv_1b_2},\ \overline{bv_1u_2}, \overline{bv_1v_2},\ \overline{ub_1b_2}, \\ \overline{ub_1u_2},\ \overline{ub_1v_2},\ \overline{uu_1b_2},\ \overline{uu_1u_2},\ \overline{uu_1v_2},\ \overline{uv_1b_2},\ \overline{uv_1u_2},\ \overline{uv_1v_2}, \\ \overline{vb_1b_2},\ \overline{vb_1u_2},\ \overline{vb_1v_2},\ \overline{vu_1b_2},\ \overline{vu_1u_2},\ \overline{vu_1v_2},\ \overline{vv_1b_2},\ \overline{vv_1u_2},\ \overline{vv_1v_2} \end{array} \right\}.$$

In this paper we describe $\mathcal{U}(\mathbb{Z}G_1)$ and $\mathcal{U}(\mathbb{Z}G_2)$, the group of units of $\mathbb{Z}G_1$ and $\mathbb{Z}G_2$, respectively. To this end, we still need to fixe more notations:

(i) If G is a finite extra-special 2-group and $G' = \{1, s\}$ is its commutator subgroup, then U_2 denotes the subgroup of $\mathcal{U}(\mathbb{Z}G)$ defined by

$$U_2 = \mathcal{U}(\mathbb{Z}G) \cap \Big(\mathbb{Q}G\Big(\frac{1-s}{2}\Big) + \Big(\frac{1+s}{2}\Big)\Big).$$

(ii) Let R be a domain and $GL_n(R)$ the group of invertible n by n matrices with coeficients in R. If S is a subset of $GL_n(R)$, then $S_{det=1}$ denotes a set of matrix units of S with determinant 1. Similarly, $S_{det=\pm1}$ denotes a set of matrix units of S with determinant ±1. By I_n we denote the identity matrix in $GL_n(R)$, and if $S_{det=1}$ is a multiplicative group, then

$$S_{\overline{det=1}} = S_{det=1}/(\{I_n, -I_n\} \cap S_{det=1}).$$

Similarly,

$$S_{\overline{det=\pm1}} = S_{det=\pm1}/(\{I_n, -I_n\} \cap S_{det=\pm1}).$$

3. Auxiliary Results

For the main results, we will need the following:

Proposition 1. [6, *Lemma* 4.2] *The group ring $\mathbb{Q}D_4$ admits the following decomposition:*

$$\mathbb{Q}D_4 = \mathbb{Q}D_4\Big(\frac{1+s}{2}\Big) \oplus \mathbb{Q}D_4\Big(\frac{1-s}{2}\Big),$$

where $\mathbb{Q}D_4\Big(\frac{1+s}{2}\Big) \cong \mathbb{Q}^4$ and $\mathbb{Q}D_4\Big(\frac{1-s}{2}\Big) \cong M_2(\mathbb{Q})$.

For the sequence of this work, we will fix the following representation of D_4 on $M_2(\mathbb{Q})$:

- $b \ \mapsto\ e_{11} - e_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$

- $v \ \mapsto\ -e_{12} + e_{21} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$

Proposition 2. [6, *Proposition* 3.2] *Writing $E = \dfrac{1-s}{2}$ and using the above fixed representation, an elementary $\mathbb{Q}$-basis matrix of $\mathbb{Q}D_4\Big(\dfrac{1-s}{2}\Big) \cong M_2(\mathbb{Q})$ is:*

$$e_{11} = \left(\frac{1+b}{2}\right)E, \qquad e_{12} = \left(\frac{vb-v}{2}\right)E,$$
$$e_{21} = \left(\frac{v+vb}{2}\right)E, \qquad e_{22} = \left(\frac{1-b}{2}\right)E.$$

Proposition 3. [6, *Proposition* 2.4] *Let G be an finite extra-special 2-group, $G' = \{1, s\}$ the commutator subgroup of G and ε the augmentation mapping on $\mathbb{Q}G$. Then:*

(i) $U_2 = \{u = 1 + \alpha(1-s) \,|\, u \in \mathcal{U}(\mathbb{Z}G),\ \alpha \in \mathbb{Z}G\}$;

(ii) If $V = \{u = 1 + \alpha\,(1-s) \,|\, u \in \mathcal{U}(\mathbb{Z}G),\ \alpha \in \mathbb{Z}G$ *and* $\varepsilon(\alpha)$ *is even*$\}$, *then* $V \cong U_2/G'$;

(iii) If G/G' *has exponent at most* 4, *then* $\mathcal{U}(\mathbb{Z}G) = \pm GV$.

Theorem 1. [6, *Theorem* 4.3] $\mathcal{U}(\mathbb{Z}D_4) = \pm D_4 V$ *and V is isomorphic to the group of 2-by-2 matrices*

$$\begin{bmatrix} 2\mathbb{Z}+1 & 4\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}_{\overline{det=1}}.$$

Proposition 4. *Let G be an finite extra-special 2-group and $G' = \{1, s\}$ its commutator subgroup. Se $\widehat{G'} = \dfrac{1+s}{2}$, then the component $\mathbb{Q}G(1-\widehat{G'}) = \mathbb{Q}G\Big(\dfrac{1-s}{2}\Big)$ in the decomposition of $\mathbb{Q}G$ is simple.*

Proof. Let $E = \dfrac{1-s}{2}$ and e be a non-trivial central idempotent of $\mathbb{Q}G(E)$. If $\mathcal{Z}(G)$ denotes the center of G and $\mathcal{C}_g$ denotes the class of conjugation of $g \in G$, then

$$e = \sum_{g \in \mathcal{Z}(G)} \alpha_g g + \sum_{g \notin \mathcal{Z}(G)} \alpha_g C_g,$$

where $C_g = \sum_{x \in \mathcal{C}_g} x$. Since $G' = \{1, s\}$, if g is not central, then $\mathcal{C}_g = \{g, gs\}$. Consequently,

$$e = \sum_{g \in \mathcal{Z}(G)} \alpha_g g + \sum_{g \notin \mathcal{Z}(G)} \alpha_g (g + gs) = \sum_{g \in \mathcal{Z}(G)} \alpha_g g + (1+s) \sum_{g \notin \mathcal{Z}(G)} \alpha_g g.$$

Since e is an idempotent of $\mathbb{Q}G(E)$, it follows $eE = e$. Hence, as $(1+s)E = 0$, $e = \Big(\sum_{g \in \mathcal{Z}(G)} \alpha_g g\Big)E$ and $e \in \mathbb{Q}(\mathcal{Z}(G))$. Since $\mathcal{Z}(G) = \{1, s\}$, the only possibilities for e are

$$0,\ 1,\ \frac{1+s}{2} \quad \text{and} \quad \frac{1-s}{2}.$$

Therefore, $e = E$ and the component $\mathbb{Q}G(1-\widehat{G'}) = \mathbb{Q}G(E)$ is simple, as we wanted to prove.

4. Main Results

Now that we have introduced the terminology, fix the notation and display the auxiliary results, we are able to present the main results. We begin describing $\mathcal{U}(\mathbb{Z}G_1)$, the group of units of $\mathbb{Z}G_1$.

4.1. The Group of Units of Group Ring $\mathbb{Z}G_1$

Proposition 5. *The group ring* $\mathbb{Q}G_1$ *admits the following decomposition:*

$$\mathbb{Q}G_1 = \mathbb{Q}G_1\Big(\frac{1+s}{2}\Big) \oplus \mathbb{Q}G_1\Big(\frac{1-s}{2}\Big),$$

where $\mathbb{Q}G_1\Big(\frac{1+s}{2}\Big) \cong \mathbb{Q}^{16}$ *and* $\mathbb{Q}G_1\Big(\frac{1-s}{2}\Big) \cong M_4(\mathbb{Q})$.

Proof. It follows from *Proposition 1* and *Proposition 4*.

For the next results, $\otimes$ denotes the *Kronecker product* and we use the representation of D_4 on $M_2(\mathbb{Q})$ previously fixed and we obtain a representation of G_1 on $M_4(\mathbb{Q})$:

- $b \;\mapsto\; e_{11}+e_{22}-e_{33}-e_{44} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$
- $v \;\mapsto\; -e_{13}-e_{24}+e_{31}+e_{42} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$
- $b_1 \;\mapsto\; e_{11}-e_{22}+e_{33}-e_{44} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$
- $v_1 \;\mapsto\; -e_{12}+e_{21}-e_{34}+e_{43} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Thus, the following proposition gives us an elementary $\mathbb{Q}$-basis matrix of $\mathbb{Q}G_1\Big(\frac{1-s}{2}\Big) \cong M_4(\mathbb{Q})$.

Proposition 6. *Writing* $E = \dfrac{1-s}{2}$ *and using the above representation, an elementary* $\mathbb{Q}$*-basis matrix of* $\mathbb{Q}G_1\Big(\frac{1-s}{2}\Big) \cong M_4(\mathbb{Q})$ *is:*

$$\begin{aligned}
e_{11} &= \Big(\frac{1+b}{2}\Big)\Big(\frac{1+b_1}{2}\Big)E, & e_{12} &= \Big(\frac{1+b}{2}\Big)\Big(\frac{v_1b_1-v_1}{2}\Big)E,\\
e_{13} &= \Big(\frac{vb-v}{2}\Big)\Big(\frac{1+b_1}{2}\Big)E, & e_{14} &= \Big(\frac{vb-v}{2}\Big)\Big(\frac{v_1b_1-v_1}{2}\Big)E,\\
e_{21} &= \Big(\frac{1+b}{2}\Big)\Big(\frac{v_1+v_1b_1}{2}\Big)E, & e_{22} &= \Big(\frac{1+b}{2}\Big)\Big(\frac{1-b_1}{2}\Big)E,\\
e_{23} &= \Big(\frac{vb-v}{2}\Big)\Big(\frac{v_1+v_1b_1}{2}\Big)E, & e_{24} &= \Big(\frac{vb-v}{2}\Big)\Big(\frac{1-b_1}{2}\Big)E,\\
e_{31} &= \Big(\frac{v+vb}{2}\Big)\Big(\frac{1+b_1}{2}\Big)E, & e_{32} &= \Big(\frac{v+vb}{2}\Big)\Big(\frac{v_1b_1-v_1}{2}\Big)E,\\
e_{33} &= \Big(\frac{1-b}{2}\Big)\Big(\frac{1+b_1}{2}\Big)E, & e_{34} &= \Big(\frac{1-b}{2}\Big)\Big(\frac{v_1b_1-v_1}{2}\Big)E,\\
e_{41} &= \Big(\frac{v+vb}{2}\Big)\Big(\frac{v_1+v_1b_1}{2}\Big)E, & e_{42} &= \Big(\frac{v+vb}{2}\Big)\Big(\frac{1-b_1}{2}\Big)E,\\
e_{43} &= \Big(\frac{1-b}{2}\Big)\Big(\frac{v_1+v_1b_1}{2}\Big)E, & e_{44} &= \Big(\frac{1-b}{2}\Big)\Big(\frac{1-b_1}{2}\Big)E.
\end{aligned}$$

For the next result, we will need the following:

Definition 2. *Let* $\mathcal{A} = \Big[2[X_{ij}] + \mathcal{I}_4\Big]$ *be a 4-by-4 matrix with* $X_{ij} \in \mathbb{Z}$*, for all* i, j*,* $1 \leq i, j \leq 4$*. We define 4 distinct blocks of* $\mathcal{A}$*:*

$$\begin{aligned}\mathfrak{B}_1 &= \{X_{11}, X_{22}, X_{33}, X_{44}\},\\ \mathfrak{B}_2 &= \{X_{12}, X_{21}, X_{34}, X_{43}\},\\ \mathfrak{B}_3 &= \{X_{13}, X_{24}, X_{31}, X_{42}\},\\ \mathfrak{B}_4 &= \{X_{14}, X_{23}, X_{32}, X_{41}\}.\end{aligned}$$

In particular, $\mathfrak{B}_k = \big\{X_{ij_i} | \, 1 \leq i \leq 4 \text{ and } j_i \neq j_{i'}, \text{ if } i \neq i'\big\}$, $1 \leq k \leq 4$.

Finally we are in a position to give a description of $\mathcal{U}(\mathbb{Z}G_1)$ in a similar vein as the description given for $\mathcal{U}(\mathbb{Z}D_4)$ in *Theorem 1*:

Theorem 2. *Let* $G_1 = D \times D^1 \Big/ \{1, ss_1\}$ *be the extra-special 2-group of order 32, the central product of two copies of* $D_4 = \big\langle b, v | \, b^2 = v^4 = 1 \text{ and } bvbv = 1\big\rangle$*. Then* $\mathcal{U}(\mathbb{Z}G_1) = \pm G_1 V_1$ *and* V_1 *is isomorphic to the group*

$$\Big[2\,[X_{ij}] + \mathcal{I}_4\Big]_{\overline{det=1}}$$

of 4-by-4 matrices, where, for each k*,* $1 \leq k \leq 4$*, the integers* $X_{ij_i} \in \mathfrak{B}_k$ *have the same parity and 4 divides the sum* $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$*.*

Proof. By *Proposition 3*, $\mathcal{U}(\mathbb{Z}G_1) = \pm G_1 V_1$, where $V_1 \cong U_2/G_1'$ and

$$\begin{aligned}U_2 = \mathcal{U}(\mathbb{Z}G_1) \cap \Big(\mathbb{Q}G_1\Big(\frac{1-s}{2}\Big) + \Big(\frac{1+s}{2}\Big)\Big) =\\ = \{u = 1 + \alpha(1-s) | \, u \in \mathcal{U}(\mathbb{Z}G_1), \, \alpha \in \mathbb{Z}G_1\}.\end{aligned}$$

Thus, to describe $\mathcal{U}(\mathbb{Z}G_1)$ we need a complete description of $V_1 \cong U_2/G_1'$ of the U_2, that is, we need to describe completly the subgroup U_2 of the $\mathcal{U}(\mathbb{Z}G_1)$.

Let $u \in U_2$. Then, by *Proposition 3* and writing $E = \dfrac{1-s}{2}$,

$u = 1 + \alpha(1-s) = 1 + 2(\alpha_1 + \alpha_2\, b + \alpha_3\, v + \alpha_4\, b_1 + \alpha_5\, v_1 + \alpha_6\, u + \alpha_7\, u_1 + \alpha_8\, bb_1 + \alpha_9\, bu_1 + \alpha_{10}\, bv_1 + +\alpha_{11}\, ub_1 + \alpha_{12}\, uu_1 + \alpha_{13}\, uv_1 + \alpha_{14}\, vb_1 + \alpha_{15}\, vu_1 + \alpha_{16}\, vv_1)E$.

The *Proposition 5* gives $\mathbb{Q}G_1(E) \cong M_4(\mathbb{Q})$ and using the elementary basis matrix of $\mathbb{Q}G_1(E) \cong M_4(\mathbb{Q})$ and the given representation of G_1 in $M_4(\mathbb{Q})$, we obtain

$u = \Big(\dfrac{1+s}{2}\Big) + (e_{11} + e_{22} + e_{33} + e_{44}) + 2[\alpha_1(e_{11} + e_{22} + e_{33} + e_{44}) + \alpha_2(e_{11} + e_{22} - e_{33} - e_{44}) + \alpha_3(-e_{13} - e_{24} + e_{31} + e_{42}) + \alpha_4(e_{11} - e_{22} + e_{33} - e_{44}) + \alpha_5(-e_{12} + e_{21} - e_{34} + e_{43}) + \alpha_6(e_{13} + e_{24} + e_{31} + e_{42}) + \alpha_7(e_{12} + e_{21} + e_{34} + e_{43}) + \alpha_8(e_{11} - e_{22} - e_{33} + e_{44}) + \alpha_9(e_{12} + e_{21} - e_{34} - e_{43}) + \alpha_{10}(-e_{12} + e_{21} + e_{34} - e_{43}) + \alpha_{11}(e_{13} - e_{24} + e_{31} - e_{42}) + \alpha_{12}(e_{14} + e_{23} + e_{32} + e_{41}) + \alpha_{13}(-e_{14} + e_{23} - e_{32} + e_{41}) + \alpha_{14}(-e_{13} + e_{24} + e_{31} - e_{42}) + \alpha_{15}(-e_{14} - e_{23} + e_{32} + e_{41}) + \alpha_{16}(e_{14} - e_{23} - e_{32} + e_{41})]$.

Hence

$u = \left(\dfrac{1+s}{2}\right) + [2(\alpha_1+\alpha_2+\alpha_4+\alpha_8)+1]e_{11} + 2(-\alpha_5+\alpha_7+\alpha_9-\alpha_{10})e_{12} + 2(-\alpha_3+\alpha_6+\alpha_{11}-\alpha_{14})e_{13} + 2(\alpha_{12}-\alpha_{13}-\alpha_{15}+\alpha_{16})e_{14} + 2(\alpha_5+\alpha_7+\alpha_9+\alpha_{10})e_{21} + [2(\alpha_1+\alpha_2-\alpha_4-\alpha_8)+1]e_{22} + 2(\alpha_{12}+\alpha_{13}-\alpha_{15}-\alpha_{16})e_{23} + 2(-\alpha_3+\alpha_6-\alpha_{11}+\alpha_{14})e_{24} + 2(\alpha_3+\alpha_6+\alpha_{11}+\alpha_{14})e_{31} + 2(\alpha_{12}-\alpha_{13}+\alpha_{15}-\alpha_{16})e_{32} + [2(\alpha_1-\alpha_2+\alpha_4-\alpha_8)+1]e_{33} + 2(-\alpha_5+\alpha_7-\alpha_9+\alpha_{10})e_{34} + 2(\alpha_{12}+\alpha_{13}+\alpha_{15}+\alpha_{16})e_{41} + 2(\alpha_3+\alpha_6-\alpha_{11}-\alpha_{14})e_{42} + 2(\alpha_5+\alpha_7-\alpha_9-\alpha_{10})e_{43} + [2(\alpha_1-\alpha_2-\alpha_4+\alpha_8)+1]e_{44}$.

Therefore u can be written as an integral invertible matrix $\mathcal{U} = [u_{ij}], 1 \le i,j \le 4$, where

$$
\begin{array}{ll}
u_{11} = 2(\alpha_1+\alpha_2+\alpha_4+\alpha_8)+1, & u_{12} = 2(-\alpha_5+\alpha_7+\alpha_9-\alpha_{10}),\\
u_{13} = 2(-\alpha_3+\alpha_6+\alpha_{11}-\alpha_{14}), & u_{14} = 2(\alpha_{12}-\alpha_{13}-\alpha_{15}+\alpha_{16}),\\
u_{21} = 2(\alpha_5+\alpha_7+\alpha_9+\alpha_{10}), & u_{22} = 2(\alpha_1+\alpha_2-\alpha_4-\alpha_8)+1,\\
u_{23} = 2(\alpha_{12}+\alpha_{13}-\alpha_{15}-\alpha_{16}), & u_{24} = 2(-\alpha_3+\alpha_6-\alpha_{11}+\alpha_{14}),\\
u_{31} = 2(\alpha_3+\alpha_6+\alpha_{11}+\alpha_{14}), & u_{32} = 2(\alpha_{12}-\alpha_{13}+\alpha_{15}-\alpha_{16}),\\
u_{33} = 2(\alpha_1-\alpha_2+\alpha_4-\alpha_8)+1, & u_{34} = 2(-\alpha_5+\alpha_7-\alpha_9+\alpha_{10}),\\
u_{41} = 2(\alpha_{12}+\alpha_{13}+\alpha_{15}+\alpha_{16}), & u_{42} = 2(\alpha_3+\alpha_6-\alpha_{11}-\alpha_{14}),\\
u_{43} = 2(\alpha_5+\alpha_7-\alpha_9-\alpha_{10}), & u_{44} = 2(\alpha_1-\alpha_2-\alpha_4+\alpha_8)+1.
\end{array}
$$

Thus we produce the monomorphism

$$
\varphi : U_2 \;\to \begin{bmatrix} 2\mathbb{Z}+1 & 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 & 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z}+1 \end{bmatrix}_{det=\pm 1}
$$

defined by $\varphi(u) = \Big[2(X_{ij}) + \mathcal{I}_4\Big]$, where

$$
\begin{array}{ll}
X_{11} = \alpha_1+\alpha_2+\alpha_4+\alpha_8, & X_{12} = -\alpha_5+\alpha_7+\alpha_9-\alpha_{10},\\
X_{13} = -\alpha_3+\alpha_6+\alpha_{11}-\alpha_{14}, & X_{14} = \alpha_{12}-\alpha_{13}-\alpha_{15}+\alpha_{16},\\
X_{21} = \alpha_5+\alpha_7+\alpha_9+\alpha_{10}, & X_{22} = \alpha_1+\alpha_2-\alpha_4-\alpha_8,\\
X_{23} = \alpha_{12}+\alpha_{13}-\alpha_{15}-\alpha_{16}, & X_{24} = -\alpha_3+\alpha_6-\alpha_{11}+\alpha_{14},\\
X_{31} = \alpha_3+\alpha_6+\alpha_{11}+\alpha_{14}, & X_{32} = \alpha_{12}-\alpha_{13}+\alpha_{15}-\alpha_{16},\\
X_{33} = \alpha_1-\alpha_2+\alpha_4-\alpha_8, & X_{34} = -\alpha_5+\alpha_7-\alpha_9+\alpha_{10},\\
X_{41} = \alpha_{12}+\alpha_{13}+\alpha_{15}+\alpha_{16}, & X_{42} = \alpha_3+\alpha_6-\alpha_{11}-\alpha_{14},\\
X_{43} = \alpha_5+\alpha_7-\alpha_9-\alpha_{10}, & X_{44} = \alpha_1-\alpha_2-\alpha_4+\alpha_8.
\end{array}
$$

Let $\mathcal{A} = [2(X_{ij}) + \mathcal{I}_4]$, with $X_{ij} \in \mathbb{Z}$, for all i, j, $1 \le i, j \le 4$. Then

$$\det \mathcal{A} = 1 + 2\beta_1 + 4\beta_2 + 8\beta_3 + 16\beta_4,$$

where $\beta_r \in \mathbb{Z}$, $1 \le r \le 4$. In particular, $\beta_1 = X_{11} + X_{22} + X_{33} + X_{44}$. Furthermore, $\mathcal{A} \in \varphi(U_2)$ if and only if $\det \mathcal{A} = 1$ and, for each k, $1 \le k \le 4$, the integers $X_{ij_i} \in \mathfrak{B}_k$ have the same parity and 4 divides the sum $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$. Indeed, $\mathcal{A} \in \varphi(U_2)$ if and only if $\det \mathcal{A} = \pm 1$ and, for each k, $1 \le k \le 4$, with $X_{ij_i} \in \mathfrak{B}_k$, 4 divides the sum

$X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$, this is, $4|(X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4})$, and $2|(X_{1j_1} + X_{2j_2})$, $2|(X_{1j_1} + X_{3j_3})$, $2|(X_{1j_1} + X_{4j_4})$, $2|(X_{2j_2} + X_{3j_3})$, $2|(X_{2j_2} + X_{4j_4})$ and $2|(X_{3j_3} + X_{4j_4})$. In particular, as $4|(X_{11} + X_{22} + X_{33} + X_{44})$, it follows that $\det \mathcal{A} = 1$.

So

$$\varphi(U_2) = \Big[2(X_{ij}) + \mathcal{I}_4\Big]_{det=1}$$

is a group of 4-by-4 matrices, where for each k, $1 \leq k \leq 4$, the integers $X_{ij_i} \in \mathfrak{B}_k$ have the same parity and 4 divides the sum $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$. Since

$$\varphi(s) = \varphi(1 + s(1 - s)) = -\mathcal{I}_4,$$

it follows that the mapping φ induces an isomorphism from V_1 onto the group

$$\Big[2(X_{ij}) + \mathcal{I}_4\Big]_{\overline{det=1}}$$

of 4-by-4 matrices, where for each k, $1 \leq k \leq 4$, the integers $X_{ij_i} \in \mathfrak{B}_k$ have the same parity and 4 divides the sum $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$.

4.2. The Group of Units of Group Ring $\mathbb{Z}G_2$

Proposition 7. *The group ring $\mathbb{Q}G_2$ admits the following decomposition:*

$$\mathbb{Q}G_2 = \mathbb{Q}G_2\Big(\frac{1+s}{2}\Big) \oplus \mathbb{Q}G_2\Big(\frac{1-s}{2}\Big),$$

where $\mathbb{Q}G_2\Big(\frac{1+s}{2}\Big) \cong \mathbb{Q}^{64}$ *and* $\mathbb{Q}G_2\Big(\frac{1-s}{2}\Big) \cong M_8(\mathbb{Q})$.

Proof. It follows from *Proposition 4* and *Proposition 5*.

For the next results, we use the representation of D_4 on $M_2(\mathbb{Q})$ previously fixed and we obtain a representation of G_2 on $M_8(\mathbb{Q})$:

- $b \mapsto e_{11} + e_{22} + e_{33} + e_{44} - e_{55} - e_{66} - e_{77} - e_{88} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,
- $v \mapsto -e_{15} - e_{26} - e_{37} - e_{48} + e_{51} + e_{62} + e_{73} + e_{84} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,
- $b_1 \mapsto e_{11} + e_{22} - e_{33} - e_{44} + e_{55} + e_{66} - e_{77} - e_{88} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $v_1 \mapsto -e_{13} - e_{24} + e_{31} + e_{42} - e_{57} - e_{68} + e_{75} + e_{86} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $b_2 \mapsto e_{11} - e_{22} + e_{33} - e_{44} + e_{55} - e_{66} + e_{77} - e_{88} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $v_2 \mapsto -e_{12} + e_{21} - e_{34} + e_{43} - e_{56} + e_{65} - e_{78} + e_{87} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Just as extend the *Proposition 2* can also extend the *Proposition 6* and get an elementary $\mathbb{Q}$-basis matrix of $\mathbb{Q}G_2\Big(\frac{1-s}{2}\Big) \cong M_8(\mathbb{Q})$, that will be used in next result. Even to the next result, we will need the following definition:

Definition 3. *Let* $\mathcal{A} = \Big[2(X_{ij}) + \mathcal{I}_8\Big]$ *be a 8-by-8 matrix with* $X_{ij} \in \mathbb{Z}$*, for all* i, j*,* $1 \leq i, j \leq 8$*. We define 8 distinct blocks of* $\mathcal{A}$*:*

$$\begin{aligned}
\mathfrak{B}_1 &= \{X_{11}, X_{22}, X_{33}, X_{44}, X_{55}, X_{66}, X_{77}, X_{88}\},\\
\mathfrak{B}_2 &= \{X_{12}, X_{21}, X_{34}, X_{43}, X_{56}, X_{65}, X_{78}, X_{87}\},\\
\mathfrak{B}_3 &= \{X_{13}, X_{24}, X_{31}, X_{42}, X_{57}, X_{68}, X_{75}, X_{86}\},\\
\mathfrak{B}_4 &= \{X_{14}, X_{23}, X_{32}, X_{41}, X_{58}, X_{67}, X_{76}, X_{85}\},\\
\mathfrak{B}_5 &= \{X_{15}, X_{26}, X_{37}, X_{48}, X_{51}, X_{62}, X_{73}, X_{84}\},\\
\mathfrak{B}_6 &= \{X_{16}, X_{25}, X_{38}, X_{47}, X_{52}, X_{61}, X_{74}, X_{83}\},\\
\mathfrak{B}_7 &= \{X_{17}, X_{28}, X_{35}, X_{46}, X_{53}, X_{64}, X_{71}, X_{82}\},\\
\mathfrak{B}_8 &= \{X_{18}, X_{27}, X_{36}, X_{45}, X_{54}, X_{63}, X_{72}, X_{81}\}.
\end{aligned}$$

In particular, $\mathfrak{B}_k = \big\{X_{ij_i} \,|\, 1 \leq i \leq 8 \text{ and } j_i \neq j_{i'}, \text{ if } i \neq i'\big\}$, $1 \leq k \leq 8$.

Finally, with the same idea that we use to extend the *Theorem 1*, extend the *Theorem 2* and describe completly $\mathcal{U}(\mathbb{Z}G_2)$:

Theorem 3. *Let* $G_2 = D \times D^1 \times D^2 \Big/ \{1, ss_1, ss_2, s_1s_2\}$ *be the extra-special 2-group of order 128, the central product of three copies of*

$$D_4 = \big\langle b, v \,|\, b^2 = v^4 = 1 \text{ and } bvbv = 1\big\rangle.$$

Then $\mathcal{U}(\mathbb{Z}G_2) = \pm G_2V_2$ *and* V_2 *is isomorphic to the group*

$$\Big[2(X_{ij}) + \mathcal{I}_8\Big]_{\overline{det=1}}$$

of 8-by-8 matrices, where, for each k*,* $1 \leq k \leq 8$*, the integers* $X_{ij_i} \in \mathfrak{B}_k$ *satisfy the following conditions:*

(i) *All integers* X_{ij_i}*,* $1 \leq i \leq 8$*, have the same parity;*

(ii) *4 divides the sums* $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$ *and* $X_{5j_5} + X_{6j_6} + X_{7j_7} + X_{8j_8}$*;*

(iii) *If* X_{1j_1} *and* $X_{(1+k)j_{(1+k)}}$*,* $1 \leq k \leq 3$*, are congruentes module 4, then* X_{5j_5} *e* $X_{(5+k)j_{(5+k)}}$ *also are;*

(iv) *8 divides the sum* $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4} + X_{5j_5} + X_{6j_6} + X_{7j_7} + X_{8j_8}$.

Proof. By *Proposition 3*, $\mathcal{U}(\mathbb{Z}G_2) = \pm G_2V_2$, where $V_2 \cong U_2/G_2'$ and

$$\begin{aligned}
U_2 = \mathcal{U}(\mathbb{Z}G_2) \cap \Big(\mathbb{Q}G_2\Big(\frac{1-s}{2}\Big) + \Big(\frac{1+s}{2}\Big)\Big) = \\
= \{u = 1 + \alpha(1-s) \,|\, u \in \mathcal{U}(\mathbb{Z}G_2),\ \alpha \in \mathbb{Z}G_2\}.
\end{aligned}$$

Thus, to describe $\mathcal{U}(\mathbb{Z}G_2)$ we need a complet description of $V_2 \cong U_2/G_2'$ of the U_2, that is, we need to describe completly the subgroup U_2 of the $\mathcal{U}(\mathbb{Z}G_2)$.

Let $u \in U_2$. Then, by *Proposition 3* and writing $E = \dfrac{1-s}{2}$,

$u = 1 + \alpha(1-s) = 1 + 2(\alpha_1 + \alpha_2\, b + \alpha_3\, v + \alpha_4\, b_1 + \alpha_5\, v_1 + \alpha_6\, b_2 + \alpha_7\, v_2 + \alpha_8\, u + \alpha_9\, u_1 + +\alpha_{10}\, u_2 + \alpha_{11}\, bb_1 + \alpha_{12}\, bu_1 + \alpha_{13}\, bv_1 + \alpha_{14}\, ub_1 + \alpha_{15}\, uu_1 + \alpha_{16}\, uv_1 + \alpha_{17}\, vb_1 + \alpha_{18}\, vu_1 + \alpha_{19}\, vv_1 + +\alpha_{20}\, bb_2 + \alpha_{21}\, bu_2 + \alpha_{22}\, bv_2 + \alpha_{23}\, ub_2 + \alpha_{24}\, uu_2 + \alpha_{25}\, uv_2 + \alpha_{26}\, vb_2 + \alpha_{27}\, vu_2 + \alpha_{28}\, vv_2 + +\alpha_{29}\, b_1b_2 + \alpha_{30}\, b_1u_2 + \alpha_{31}\, b_1v_2 + \alpha_{32}\, u_1b_2 + \alpha_{33}\, u_1u_2 + \alpha_{34}\, u_1v_2 + \alpha_{35}\, v_1b_2 + \alpha_{36}\, v_1u_2 + +\alpha_{37}\, v_1v_2 + \alpha_{38}\, bb_1b_2 + \alpha_{39}\, bb_1u_2 + \alpha_{40}\, bb_1v_2 + \alpha_{41}\, bu_1b_2 + \alpha_{42}\, bu_1u_2 + \alpha_{43}\, bu_1v_2 + +\alpha_{44}\, bv_1b_2 + \alpha_{45}\, bv_1u_2 + \alpha_{46}\, bv_1v_2 + \alpha_{47}\, ub_1b_2 + \alpha_{48}\, ub_1u_2 + \alpha_{49}\, ub_1v_2 + \alpha_{50}\, uu_1b_2 + +\alpha_{51}\, uu_1u_2 + \alpha_{52}\, uu_1v_2 + \alpha_{53}\, uv_1b_2 + \alpha_{54}\, uv_1u_2 + \alpha_{55}\, uv_1v_2 + \alpha_{56}\, vb_1b_2 + \alpha_{57}\, vb_1u_2 + +\alpha_{58}\, vb_1v_2 + \alpha_{59}\, vu_1b_2 + \alpha_{60}\, vu_1u_2 + \alpha_{61}\, vu_1v_2 + \alpha_{62}\, vv_1b_2 + \alpha_{63}\, vv_1u_2 + \alpha_{64}\, vv_1v_2)E.$

The *Proposition 7* gives $\mathbb{Q}G_2(E) \cong M_8(\mathbb{Q})$ and using the elementary matrix basis of $\mathbb{Q}G_1(E) \cong M_8(\mathbb{Q})$ and the give representation of G_2 in $M_8(\mathbb{Q})$, u can be written as an integral invertible matrix $[u_{ij}]\,, 1 \le i,j \le 8$. Thus we produce the monomorphism defined by $\varphi(u) = \Big[2(X_{ij}) + \mathcal{I}_8\Big]$.

Let $\mathcal{A} = [2(X_{ij}) + \mathcal{I}_8]$, with $X_{ij} \in \mathbb{Z}$, for all i, j, $1 \le i, j \le 8$. Then

$$\det \mathcal{A} = 1 + 2\beta_1 + 4\beta_2 + 8\beta_3 + 16\beta_4 + 32\beta_5 + 64\beta_6 + 128\beta_7 + 256\beta_8,$$

where $\beta_r \in \mathbb{Z}$, $1 \le r \le 8$. In particular,

$$\beta_1 = X_{11} + X_{22} + X_{33} + X_{44} + X_{55} + X_{66} + X_{77} + X_{88}.$$

Furthermore, $\mathcal{A} \in \varphi(U_2)$ if and only if $\det \mathcal{A} = 1$ and, for each k, $1 \le k \le 8$, the integers $X_{ij_i} \in \mathfrak{B}_k$ satisfy the following conditions:

(i) All integers X_{ij_i}, $1 \le i \le 8$, have the same parity;

(ii) 4 divides the sums $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4}$ and $X_{5j_5} + X_{6j_6} + X_{7j_7} + X_{8j_8}$;

(iii) If X_{1j_1} and $X_{(1+k)j_{(1+k)}}$, $1 \le k \le 3$, are congruentes module 4, then X_{5j_5} e $X_{(5+k)j_{(5+k)}}$ also are;

(iv) 8 divides the sum $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4} + X_{5j_5} + X_{6j_6} + X_{7j_7} + X_{8j_8}$.

Indeed, $\mathcal{A} \in \varphi(U_2)$ if and only if $\det \mathcal{A} = 1$ and, for each k, $1 \le k \le 8$, the integers $X_{ij_i} \in \mathfrak{B}_k$ satisfy the following conditions:

- 2 divides the sum $X_{ij_i} + X_{i'j_{i'}}$, for all $1 \le i, i' \le 8$;
- 4 divides the following sums:

  · $X_{1j_1} + X_{2j_2} + X_{ij_i} + X_{(i+1)j_{(i+1)}}$, with $i \in \{3, 5, 7\}$;

  · $X_{1j_1} + X_{3j_3} + X_{ij_i} + X_{(i+2)j_{(i+2)}}$, with $i \in \{5, 6\}$;

  · $X_{1j_1} + X_{4j_4} + X_{5j_5} + X_{8j_8}$;

  · $X_{1j_1} + X_{4j_4} + X_{6j_6} + X_{7j_7}$;

  · $X_{2j_2} + X_{3j_3} + X_{5j_5} + X_{8j_8}$;

· $X_{2j_2} + X_{3j_3} + X_{6j_6} + X_{7j_7}$;

· $X_{2j_2} + X_{4j_4} + X_{ij_i} + X_{(i+2)j_{(i+2)}}$, with $i \in \{5, 6\}$;

· $X_{3j_3} + X_{4j_4} + X_{ij_i} + X_{(i+1)j_{(i+1)}}$, with $i \in \{5, 7\}$;

· $X_{5j_5} + X_{6j_6} + X_{7j_7} + X_{8j_8}$.

- 8 divides the sum $X_{1j_1} + X_{2j_2} + X_{3j_3} + X_{4j_4} + X_{5j_5} + X_{6j_6} + X_{7j_7} + X_{8j_8}$.

In particular, as 8 divides $X_{11}+X_{22}+X_{33}+X_{44}+X_{55}+X_{66}+X_{77}+X_{88}$, we have that $\det \mathcal{A} = 1$. Furthermore, $\varphi(s) = \varphi(1 + s(1 - s)) = -\mathcal{I}_8$. So, it follows that the mapping φ induces the isomorphism wanted.

The reader may notice that the idea used in this paper can be extended to describe the group of units of any integral group ring of a finite extra-special 2-group of order higher than 128, that is a central product of copies of D_4.

References

[1] A K Bhandari and I S Luthar. Torsion Units of Integral Group Rings of Metacyclic Groups. *J. Number Theory*, 17:170–183, 1983.

[2] A Bovdi and F C Polcino Milies. Normal Subgroups of the Group of Units in Group Rings of Torsion Groups. *Publ. Math. Debrecen*, 59:235–242, 2001.

[3] R A Ferraz. Groups Generated by a Bass Cyclic Unit and a Bicyclic Unit in the Units of $\mathbb{Z}G$. *J. Group Theory*, 7:421–430, 2004.

[4] A Giambruno and S K Sehgal. Generators of Large Subgroups of Units of Integral Group Rings of Nilpotent Groups. *J. of Algebra*, 174:150–156, 1995.

[5] E G Goodaire and E Jespers. Determining Units in Some Integral Group Rings. *Canad. Math. Bull.*, 33(2):242–1246, 1990.

[6] E Jespers and A Del Rio. *Group Ring Groups, Vol. 1: Orders and Generic Constructions of Units.* De Gruyter, Berlin, 2015.

[7] E Jespers and A Del Rio. *Group Ring Groups, Vol. 2: Structure Theorems of Unit Groups.* De Gruyter, Berlin, 2015.

[8] E Jespers and G Leal. Describing Units of Integral Group Rings of Some 2-Groups. *Comm. Algebra*, 19(6):1809–1827, 1991.

[9] E Jespers and G Leal. Generators of Large Subgroups of the Units of Integral Group Rings. *Manuscripta Math.*, 78:303–315, 1993.

[10] E Jespers, G Leal and F C Polcino Milies. Units of Integral Group Rings of Some Metacyclic Groups. *Canad. Math. Bull.*, 37(2):228–237, 1994.

[11] E Jespers, M M Parmenter and S K Sehgal. Central Units of Integral Group Rings of Nilpotent Groups. *Proc. Amer. Math Soc.*, 124:1007–1012, 1996.

[12] F C Polcino Milies. The units of integral group ring $\mathbb{Z}D_4$. *Bol. Soc. Brasileira De Mat.*, 4:85–92, 1972.

[13] J Ritter and S K Sehgal. Generators of subgroups of $\mathcal{U}(\mathbb{Z}G)$. *Contemp. Math.*, 93:331–347, 1989.

[14] J Ritter and S K Sehgal. Construction of units in integral group rings of finite nilpotent groups. *Trans. Amer. Math. Soc.*, 324:602–621, 1991.

[15] J Ritter and S K Sehgal. Construction of units in integral group rings of monomial and symmetric groups. *J. Algebra*, 142:511–526, 1991.