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On weak graded rings

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Abstract. In this paper, we investigate some properties of weak graded rings that are rings graded by a set $\mathfrak{G}$ of coset representatives for the left action of a subgroup H on a group X. Moreover, a graded rings by using the product $H \times \mathfrak{G}$ are also discussed. A detailed example is given.

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1. Introduction

Let R be an associative ring and G be a semigroup. Recall that R is called G-graded if there is an additive subgroup R_g of R, for each $g \in G$, such that $R = \bigoplus_{g\in G} R_g$ and the inclusion property $R_g R_h \subseteq R_{gh}$ is satisfied for all $g, h \in G$.

Semigroup graded rings and modules as well as a lot of their properties were investigated by many mathematicians, see for example [1], [7], [10], [11], [12], [13] and [15].

The construction of group graded rings and their modules were deeply studied by Dade in [6] that enriched and extended the concept of the classical stable Clifford theory.

The work of Dade was an initial source for many researchers who were interested in the field of group graded rings and their modules, see for exampl [5], [8] and [9].

In [4], Beggs form a set $\mathfrak{G}$ of left coset representatives for the left action of a subgroup H on a group X and defined a binary operation on $\mathfrak{G}$ which has a left identity and the right division property. This binary operation is not associative, but the associativity can be obtained by a "cocycle" $f : \mathfrak{G} \times \mathfrak{G} \longrightarrow H$.

In [2], a new concept named the weak graded rings and modules were introduced. In more details, a graded ring R were constructed using a set $\mathfrak{G}$ of left coset representatives and some results were proved in the new setting. In addition, some properties of these graded rings and their modules were derived.

In this paper, some properties of weak graded rings, that are rings graded by a set $\mathfrak{G}$ of coset representatives for the left action of a subgroup H on a group X, are investigated.

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Moreover a graded rings by using the product $H \times \mathfrak{G}$ are also discussed. A detailed example is given.

Throughout this paper, for the sake of simplicity, it is assumed that all rings are associative, commutative and with unities, although for many results associative rings are only required.

2. Preliminaries

In this section some definitions and required results from [4] are presented.

Definition 1. *For a group X and a subgroup H, we call $\mathfrak{G} \subset X$ a set of left coset representatives if for every $x \in X$ there is a unique $s \in \mathfrak{G}$ such that $x \in Hs$. The decomposition $x = us$ for $u \in H$ and $s \in \mathfrak{G}$ is called the unique factorization of x.*

In what follows, it will be assumed that $\mathfrak{G} \subset X$ is a fixed set of left coset representatives for the subgroup $H \subset X$. Also, the identity in X will be denoted by e.

Definition 2. *For elements $s, t \in \mathfrak{G}$ we define $f(s,t) \in H$ and $s * t \in \mathfrak{G}$ by the unique factorization $st = f(s,t)(s*t)$ in X, where f is the cocycle map. Also, the functions $\triangleright : \mathfrak{G} \times H \to H$ and $\triangleleft : \mathfrak{G} \times H \to \mathfrak{G}$ are also defined by the unique factorization $su = (s \triangleright u)(s \triangleleft u)$ for $s, s \triangleleft u \in \mathfrak{G}$ and $u, s \triangleright u \in H$.*

It was proved that the binary operation $*$ on $\mathfrak{G}$ has a unique left identity $e_{\mathfrak{G}} \in \mathfrak{G}$ and also has the right division property which means that, there is a unique solution $p \in \mathfrak{G}$ satisfies the equation $p * s = t$ for all $s, t \in \mathfrak{G}$. It is noted that if $e \in \mathfrak{G}$ then $e_{\mathfrak{G}} = e$ is also a right identity.

Proposition 1. *For $s, t, p \in \mathfrak{G}$ and $u, v \in H$, the following identities between $(\mathfrak{G}, *)$ and f hold:*

$$s \triangleright (t \triangleright u) = f(s,t)\big((s*t) \triangleright u\big) f\big(s \triangleleft (t \triangleright u), t \triangleleft u\big)^{-1}$$

$$(s*t) \triangleleft u = \big(s \triangleleft (t \triangleright u)\big) * (t \triangleleft u)$$

$$s \triangleright uv = (s \triangleright u)\big((s \triangleleft u) \triangleright v\big)$$

$$s \triangleleft uv = (s \triangleleft u) \triangleleft v$$

$$f(p,s) f(p*s,t) = \big(p \triangleright f(s,t)\big) f\big(p \triangleleft f(s,t), s*t\big)$$

$$\big(p \triangleleft f(s,t)\big) * (s*t) = (p*s)*t.$$

Proposition 2. *For $t \in \mathfrak{G}$ and $v \in H$, the following identities between $(\mathfrak{G}, *)$ and f hold:*

$$e_{\mathfrak{G}} \triangleleft v = e_{\mathfrak{G}}, \qquad e_{\mathfrak{G}} \triangleright v = e_{\mathfrak{G}} v e_{\mathfrak{G}}^{-1}, \qquad t \triangleright e = e, \qquad t \triangleleft e = t,$$

$$f(e_{\mathfrak{G}}, t) = e_{\mathfrak{G}}, \qquad t \triangleright e_{\mathfrak{G}}^{-1} = f\big(t \triangleleft e_{\mathfrak{G}}^{-1}, e_{\mathfrak{G}}\big)^{-1}, \qquad \big(t \triangleleft e_{\mathfrak{G}}^{-1}\big) * e_{\mathfrak{G}} = t.$$

3. $\mathfrak{G}$-weak Graded Rings

Definition 3. *[2] Let X be a group, H be a subgroup of X and $(\mathfrak{G}, *)$ be a fixed set of left coset representatives for the subgroup H with the binary operation $*$ which is defined as in 2. A ring R is called a $\mathfrak{G}$-weak graded ring if*

$$R = \bigoplus_{s \in \mathfrak{G}} R_s \tag{1}$$

and

$$R_s R_t \subseteq R_{s*t} \qquad \text{for all } s,t \in \mathfrak{G}, \tag{2}$$

where R_s is an additive subgroup for each $s \in \mathfrak{G}$. If (2) is replaced by

$$R_s R_t = R_{s*t} \qquad \text{for all } s,t \in \mathfrak{G}, \tag{3}$$

then R is called a fully (or strongly) $\mathfrak{G}$-weak graded ring.

It can be noted that any ring R can be put into a $\mathfrak{G}$-weak graded ring by placing $R = R_{e_{\mathfrak{G}}}$ and $R_s = 0$ for all $e_{\mathfrak{G}} \neq s \in \mathfrak{G}$ which is called the trivial $\mathfrak{G}$-weak graded ring.

Example 1. *Consider the Morita ring*

$$T = \left\{ \begin{pmatrix} r & m \\ n & s \end{pmatrix} : r \in R, m \in M, n \in N and\ s \in S \right\},$$

with a Morita contex $(R, S, {}_R M_S, {}_S N_R, \phi, \varphi)$ where the bimodule homomorphisms

$$\phi : M \otimes_S N \longrightarrow R$$

$$\varphi : N \otimes_R M \longrightarrow S$$

satisfy $(mn)m' = m(nm')$ as $\phi(m,n) = mn$ and $\varphi(n,m) = nm$, i.e. $\phi(m \otimes n)m' = m\varphi(n \otimes m')$ and $\varphi(n \otimes m)n' = n\phi(m \otimes n')$ for all $m, m' \in M$ and $n, n' \in N$. It is well known that T with the usual matrix addition and multiplication forms a ring. Now to put the Morita ring $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ into a weak graded ring we consider the group $X = \mathbb{Z}_2 \times \mathbb{Z}_3$ under addition with the additive subgroup $H = \{(0,0),(1,0)\}$ of X. Take the set of left coset representatives to be $\mathfrak{G} = \{(1,0),(0,1),(1,2)\}$. Then the $*$ and f operations as well as the actions $\triangleleft, \triangleright$ are given by the following tables:

Table 1: $*$ and f operations.

$*$	$(1,0)$	$(0,1)$	$(1,2)$
$(1,0)$	$(1,0)$	$(0,1)$	$(1,2)$
$(0,1)$	$(0,1)$	$(1,2)$	$(1,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(0,1)$

f	$(1,0)$	$(0,1)$	$(1,2)$
$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
$(0,1)$	$(1,0)$	$(1,0)$	$(0,0)$
$(1,2)$	$(1,0)$	$(0,0)$	$(0,0)$

Table 2: $\triangleleft$ and $\triangleright$ actions.

$s \triangleright u$	$(0,0)$	$(1,0)$
$(1,0)$	$(0,0)$	$(1,0)$
$(0,1)$	$(0,0)$	$(1,0)$
$(1,2)$	$(0,0)$	$(1,0)$

$s \triangleleft u$	$(0,0)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,0)$
$(0,1)$	$(0,1)$	$(0,1)$
$(1,2)$	$(1,2)$	$(1,2)$

Hence the ring T can be written as $T = T_{(1,0)} \oplus T_{(0,1)} \oplus T_{(1,2)}$ where,

$$T_{(1,0)} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} : r \in R, ands \in S \right\}$$

$$T_{(0,1)} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : m \in M \right\}$$

$$T_{(1,2)} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} : n \in N \right\}$$

Next, to ensure that the inclusion property is satisfied, the following calculations are needed:

(1) $T_{(1,0)}T_{(1,0)} \subseteq T_{(1,0)*(1,0)}$ *as for all* $\begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} \in T_{(1,0)}$ *we have :*

$$\begin{pmatrix} r_1 & 0 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} r_1 r_2 & 0 \\ 0 & s_1 s_2 \end{pmatrix} \in T_{(1,0)} = T_{(1,0)*(1,0)}.$$

(2) $T_{(1,0)}T_{(0,1)} \subseteq T_{(1,0)*(0,1)}$ *as for all* $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{(1,0)}$, $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T_{(0,1)}$ *we have :*

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & rm \\ 0 & 0 \end{pmatrix} \in T_{(0,1)} = T_{(1,0)*(0,1)}.$$

It can be noted that $rm \in M$ as M is a left R-module.

(3) $T_{(1,0)}T_{(1,2)} \subseteq T_{(1,0)*(1,2)}$ *as for all* $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{(1,0)}$,$\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \in T_{(1,2)}$ *we have :*

$$T_{(1,0)}T_{(1,2)} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ sn & 0 \end{pmatrix} \in T_{(1,2)} = T_{(1,0)*(1,2)}.$$

It can be noted that $sn \in N$ as N is a left S-module.

(4) $T_{(0,1)}T_{(1,0)} \subseteq T_{(0,1)*(1,0)}$ *for all* $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T_{(0,1)}$, $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{(1,0)}$ *we have :*

$$T_{(0,1)}T_{(1,0)} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & ms \\ 0 & 0 \end{pmatrix} \in T_{(0,1)} = T_{(0,1)*(1,0)}.$$

It can be noted that $ms \in M$ as M is a right S-module.

(5) $T_{(0,1)}T_{(0,1)} \subseteq T_{(0,1)*(0,1)}$ *as for all* $\begin{pmatrix} 0 & m_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ 0 & 0 \end{pmatrix} \in T_{(0,1)}$ *we have :*

$$T_{(0,1)}T_{(0,1)} = \begin{pmatrix} 0 & m_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in T_{(1,2)} = T_{(0,1)*(0,1)}.$$

(6) $T_{(0,1)}T_{(1,2)} \subseteq T_{(0,1)*(1,2)}$ *as for all* $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T_{(0,1)}$, $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \in T_{(1,2)}$ *we have :*

$$T_{(0,1)}T_{(1,2)} = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} = \begin{pmatrix} mn & 0 \\ 0 & 0 \end{pmatrix} \in T_{(1,0)} = T_{(0,1)*(1,2)}.$$

(7) $T_{(1,2)}T_{(1,0)} \subseteq T_{(1,2)*(1,0)}$ *as for all* $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \in T_{(1,2)}, \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \in T_{(1,0)}$ *we have :*

$$T_{(1,2)}T_{(1,0)} = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ nr & 0 \end{pmatrix} \in T_{(1,2)} = T_{(1,2)*(1,0)}.$$

It can be noted that $nr \in N$ *as* N *is a left* R*-module*

(8) $T_{(1,2)}T_{(0,1)} \subseteq T_{(1,2)*(0,1)}$ *as for all* $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \in T_{(1,2)}$, $\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in T_{(0,1)}$ *we have :*

$$T_{(1,2)}T_{(0,1)} = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & nm \end{pmatrix} \in T_{(1,0)} = T_{(1,2)*(0,1)}.$$

(9) $T_{(1,2)}T_{(1,2)} \subseteq T_{(1,2)*(1,2)}$ *as for all* $\begin{pmatrix} 0 & 0 \\ n_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ n_2 & 0 \end{pmatrix} \in T_{(1,2)}$ *we have:*

$$T_{(1,2)}T_{(1,2)} = \begin{pmatrix} 0 & 0 \\ n_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ n_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in T_{(0,1)} = T_{(1,2)*(1,2)}.$$

Thus, T *is a* $\mathfrak{G}$*-weak graded ring. However, it is not a fully (strongly)* $\mathfrak{G}$*-weak graded ring. For instance,* $T_{(1,2)}T_{(1,2)} \neq T_{(1,2)*(1,2)}$ *as* $T_{(0,1)} = T_{(1,2)*(1,2)} \nsubseteq T_{(1,2)}T_{(1,2)}$.

4. Some properties of $\mathfrak{G}$-weak Graded Rings

In this section, in the light of [4], some properties of $\mathfrak{G}$-weak graded rings are proved.

Proposition 3. *Let* X *be a group,* H *be a subgroup of* X, $\mathfrak{G} \subset X$ *be a set of left coset representatives and* R *be a* $\mathfrak{G}$*-weak graded ring. Then for any* $s, t, p \in \mathfrak{G}$ *and* $u, v \in H$*, the following properties are satisfied:*

(i) $R_{s\triangleright(t\triangleright u)} = R_{f(s,t)\big((s*t)\triangleright u\big)f\big(s\triangleleft(t\triangleright u),t\triangleleft u\big)^{-1}}$.

(ii) $R_{(s*t)\triangleleft u} = R_{\big(s\triangleleft(t\triangleright u)\big)*(t\triangleleft u)}$.

(iii) $R_{s\triangleright uv} = R_{(s\triangleright u)\big((s\triangleleft u)\triangleright v\big)}$.

(iv) $R_{s\triangleleft uv} = R_{(s\triangleleft u)\triangleleft v}$.

(v) $R_{f(p,s)f(p*s,t)} = R_{\left(p\triangleright f(s,t)\right)f\left(p\triangleleft f(s,t),s*t\right)}$.

(vi) $R_{\left(p\triangleleft f(s,t)\right)*(s*t)} = R_{(p*s)*t}$.

Proof. The associativity of X implies that $R_{(st)u} = R_{s(tu)}$ which is used to prove relations (i) and (ii) as follows:

$$R_{(st)u} = R_{f(s,t)(s*t)u} = R_{f(s,t)\left((s*t)\triangleright u\right)\left((s*t)\triangleleft u\right)}.$$

On the other hand,

$$\begin{aligned} R_{s(tu)} &= R_{s(t\triangleright u)(t\triangleleft u)} \\ &= R_{\left(s\triangleright(t\triangleright u)\right)\left(s\triangleleft(t\triangleright u)\right)(t\triangleleft u)} \\ &= R_{\left(s\triangleright(t\triangleright u)\right)f\left(s\triangleleft(t\triangleright u),t\triangleleft u\right)\left(s\triangleleft(t\triangleright u)*(t\triangleleft u)\right)}. \end{aligned}$$

Thus,

$$R_{f(s,t)\left((s*t)\triangleright u\right)\left((s*t)\triangleleft u\right)} = R_{\left(s\triangleright(t\triangleright u)\right)f\left(s\triangleleft(t\triangleright u),t\triangleleft u\right)\left(s\triangleleft(t\triangleright u)*(t\triangleleft u)\right)}.$$

As the factorization is unique, we get:

$$R_{\left(s\triangleright(t\triangleright u)\right)f\left(s\triangleleft(t\triangleright u),t\triangleleft u\right)} = R_{f(s,t)\left((s*t)\triangleright u\right)},$$

or equivalently,

$$R_{s\triangleright(t\triangleright u)} = R_{f(s,t)\left((s*t)\triangleright u\right)f\left(s\triangleleft(t\triangleright u),t\triangleleft u\right)^{-1}}.$$

Also,

$$R_{\left((s*t)\triangleleft u\right)} = R_{\left(s\triangleleft(t\triangleright u)*(t\triangleleft u)\right)}.$$

Next, to prove relations (iii) and (iv), we consider $R_{s(uv)} = R_{(su)v}$, which is true by the associativity of X, as follows:

$$R_{s(uv)} = R_{(s\triangleright uv)(s\triangleleft uv)}.$$

On the other hand,

$$R_{(su)v} = R_{(s\triangleright u)(s\triangleleft u)v} = R_{(s\triangleright u)\left((s\triangleleft u)\triangleright v\right)\left((s\triangleleft u)\triangleleft v\right)}.$$

Thus,

$$R_{(s\triangleright uv)(s\triangleleft uv)} = R_{(s\triangleright u)\left((s\triangleleft u)\triangleright v\right)\left((s\triangleleft u)\triangleleft v\right)}.$$

Again, by the uniqueness of factorization, we get:

$$R_{(s\triangleright uv)} = R_{(s\triangleright u)\left((s\triangleleft u)\triangleright v\right)},$$

and,

$$R_{(s\triangleleft uv)} = R_{(s\triangleleft u)\triangleleft v}.$$

Finally, as before, the associativity of X yields $R_{p(st)} = R_{(ps)t}$ which is used to prove relations (v) and (vi) as follows:

$$\begin{aligned} R_{p(st)} &= R_{pf(s,t)(s*t)} \\ &= R_{\left(p\triangleright f(s,t)\right)\left(p\triangleleft f(s,t)\right)(s*t)} \\ &= R_{\left(p\triangleright f(s,t)\right)f\left(p\triangleleft f(s,t),(s*t)\right)\left((p\triangleleft f(s,t))*(s*t)\right)}. \end{aligned}$$

On the other hand,

$$R_{(ps)t} = R_{f(p,s)(p*s)t} = R_{f(p,s)f\left((p*s),t\right)\left((p*s)*t\right)}.$$

Thus,

$$R_{f(p,s)f\left((p*s),t\right)\left((p*s)*t\right)} = R_{\left(p\triangleright f(s,t)\right)f\left(p\triangleleft f(s,t),(s*t)\right)\left((p\triangleleft f(s,t))*(s*t)\right)}.$$

The uniqueness of factorization yields:

$$R_{f(p,s)f\left((p*s),t\right)} = R_{\left(p\triangleright f(s,t)\right)f\left(p\triangleleft f(s,t),(s*t)\right)},$$

and

$$R_{\left((p*s)*t\right)} = R_{\left(p\triangleleft f(s,t)\right)*(s*t)}.$$

Proposition 4. *Let X be a group, H be a subgroup of X, $\mathfrak{G} \subset X$ be a set of left coset representatives and R be a $\mathfrak{G}$-weak graded ring. Then for any $t \in \mathfrak{G}$ and $v \in H$, the following properties are satisfied:*

(i) $R_{e_{\mathfrak{G}}\triangleleft v} = R_{e_{\mathfrak{G}}}$, *and* $R_{e_{\mathfrak{G}}\triangleright v} = R_{e_{\mathfrak{G}}ve_{\mathfrak{G}}^{-1}}$.

(ii) $R_{t\triangleright e} = R_e$, *and* $R_{t\triangleleft e} = R_t$.

(iii) $R_{f(e_{\mathfrak{G}},t)} = R_{e_{\mathfrak{G}}}$.

(iv) $R_{t\triangleright e_{\mathfrak{G}}^{-1}} = R_{f\left(t\triangleleft e_{\mathfrak{G}}^{-1},e_{\mathfrak{G}}\right)^{-1}}$, *and* $R_{\left(t\triangleleft e_{\mathfrak{G}}^{-1}\right)*e_{\mathfrak{G}}} = R_t$.

Proof. To prove (i) we consider $R_{e_{\mathfrak{G}}v} = R_{(e_{\mathfrak{G}}v)e_{\mathfrak{G}}^{-1}e_{\mathfrak{G}}} = R_{(e_{\mathfrak{G}}ve_{\mathfrak{G}}^{-1})e_{\mathfrak{G}}}$, by the associativity of X, where $e_{\mathfrak{G}} \in \mathfrak{G}$ and $e_{\mathfrak{G}}ve_{\mathfrak{G}}^{-1} \in H$. But $R_{e_{\mathfrak{G}}v} = R_{(e_{\mathfrak{G}}\triangleright v)(e_{\mathfrak{G}}\triangleleft v)}$. As the factorization is unique we get:

$$R_{e_{\mathfrak{G}}\triangleleft v} = R_{e_{\mathfrak{G}}}, \qquad \text{and} \qquad R_{e_{\mathfrak{G}}\triangleright v} = R_{e_{\mathfrak{G}}ve_{\mathfrak{G}}^{-1}}.$$

Now for (ii), as X is a group, we have $R_{te} = R_{et}$ where $e \in H$ is the identity and $t \in \mathfrak{G}$. Also, $R_t = R_{te} = R_{(t\triangleright e)(t\triangleleft e)}$ which can be written as $R_{et} = R_{(t\triangleright e)(t\triangleleft e)}$. The uniqueness of factorization implies

$$R_e = R_{(t\triangleright e)}, \qquad \text{and} \qquad R_t = R_{(t\triangleleft e)}.$$

Next, for (iii) consider $R_{e_{\mathfrak{G}}t} = R_{f(e_{\mathfrak{G}},t)(e_{\mathfrak{G}}*t)} = R_{f(e_{\mathfrak{G}},t)t}$ which is true by the definitions of $*$ and f. Hence, the uniqueness of factorization gives

$$R_{e_{\mathfrak{G}}} = R_{f(e_{\mathfrak{G}},t)}.$$

Finally, to prove (iv) we consider:

$$R_{et} = R_t = R_{te_{\mathfrak{G}}^{-1}e_{\mathfrak{G}}} = R_{(t \triangleright e_{\mathfrak{G}}^{-1})(t \triangleleft e_{\mathfrak{G}}^{-1})e_{\mathfrak{G}}} = R_{(t \triangleright e_{\mathfrak{G}}^{-1})f\big((t \triangleleft e_{\mathfrak{G}}^{-1}),e_{\mathfrak{G}}\big)\big((t \triangleleft e_{\mathfrak{G}}^{-1})*e_{\mathfrak{G}}\big)},$$

which implies, $R_e = R_{(t \triangleright e_{\mathfrak{G}}^{-1})f\big((t \triangleleft e_{\mathfrak{G}}^{-1}),e_{\mathfrak{G}}\big)}$, or equivalently,

$$R_{(t \triangleright e_{\mathfrak{G}}^{-1})} = R_{f\big((t \triangleleft e_{\mathfrak{G}}^{-1}),e_{\mathfrak{G}}\big)^{-1}},$$

and,

$$R_t = R_{(t \triangleleft e_{\mathfrak{G}}^{-1})*e_{\mathfrak{G}}}.$$

Example 2. *Consider the Morita ring* $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ *mentioned in example 1 with* $X = \mathbb{Z}_2 \times \mathbb{Z}_3$, $H = \{(0,0),(1,0)\}$ *and a set of left coset representatives* $\mathfrak{G} = \{(1,0),(0,1),(1,2)\}$. *As before* $T = T_{(1,0)} \oplus T_{(0,1)} \oplus T_{(1,2)}$ *is a* $\mathfrak{G}$*-weak graded ring where,*

$$T_{(1,0)} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} : r \in R, \text{and } s \in S \right\},$$

$$T_{(0,1)} = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : m \in M \right\}$$

and

$$T_{(1,2)} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} : n \in N \right\}.$$

Now put $s = (0,1),\ t = (1,2),\ p = (1,0)$ *in* $\mathfrak{G}$ *and* $u = (0,0),\ v = (1,0)$ *in* H, *then the above properties can be illustrated one by one as follows:*

(i) $T_{(s*t)\triangleleft u} = T_{\big(s\triangleleft(t\triangleright u)\big)*(t\triangleleft u)}$.

We start with the left hand side as follows :

$$T_{(s*t)\triangleleft u} = T_{\big((0,1)*(1,2)\big)\triangleleft(0,0)} = T_{(1,0)\triangleleft(0,0)} = T_{(1,0)}.$$

On the other hand,

$$\begin{aligned} T_{\big(s\triangleleft(t\triangleright u)\big)*(t\triangleleft u)} &= T_{\big((0,1)\triangleleft\big((1,2)\triangleright(0,0)\big)\big)*\big((1,2)\triangleleft(0,0)\big)} \\ &= T_{\big((0,1)\triangleleft(0,0)\big)*(12)} \\ &= T_{(0,1)*(1,2)} = T_{(1,0)}. \end{aligned}$$

Hence, the equality is satisfied.

(ii) $T_{s\triangleright(t\triangleright u)} = T_{f(s,t)\big((s*t)\triangleright u\big)f\big(s\triangleleft(t\triangleright u),t\triangleleft u\big)^{-1}}$.
We start with the left hand side as follows :

$$T_{s\triangleright(t\triangleright u)} = T_{(0,1)\triangleright\big((1,2)\triangleright(0,0)\big)} = T_{(0,1)\triangleright(0,0)} = T_{(0,0)}.$$

On the other hand,

$$\begin{aligned} T_{f(s,t)\big((s*t)\triangleright u\big)f\big(s\triangleleft(t\triangleright u),t\triangleleft u\big)^{-1}} &= T_{f\big((0,1),(1,2)\big)\Big(\big((0,1)*(1,2)\big)\triangleright(0,0)\Big)f\Big((0,1)\triangleleft\big((1,2)\triangleright(0,0)\big)\ ,\ (1,2)\triangleleft(0,0)\Big)^{-1}} \\ &= T_{(0,0)\big((1,0)\triangleright(0,0)\big)f\big((0,1)\triangleleft(0,0)\ \ (1,2)\big)^{-1}} \\ &= T_{(0,0)f\big((0,1)\ \ (1,2)\big)^{-1}} \\ &= T_{(0,0)\big((1,0)\triangleright(0,0)\big)f\big((0,1)\triangleleft(0,0)\ \ (1,2)\big)^{-1}} \\ &= T_{f\big((0,1)\ ,\ (1,2)\big)^{-1}} = T_{(0,0)^{-1}} = T_{(0,0)}. \end{aligned}$$

Hence, the equality is satisfied.

(iii) $T_{s\triangleright uv} = T_{(s\triangleright u)\big((s\triangleleft u)\triangleright v\big)}$.
We start with the left hand side as follows :

$$T_{s\triangleright uv} = T_{(0,1)\triangleright\big((0,0)+(1,0)\big)} = T_{(0,1)\triangleright(1,0)} = T_{(1,0)}.$$

On the other hand,

$$\begin{aligned} T_{(s\triangleright u)\big((s\triangleleft u)\triangleright v\big)} &= T_{\big((0,1)\triangleright(0,0)\big)\Big(\big((0,1)\triangleleft(0,0)\big)\triangleright(1,0)\Big)} \\ &= T_{(0,0)\big((0,1)\triangleleft(1,0)\big)} = T_{(0,0)+(1,0)} = T_{(1,0)}. \end{aligned}$$

Hence, the equality is satisfied.

(iv) $T_{s\triangleleft uv} = T_{(s\triangleleft u)\triangleleft v}$.
We start with the left hand side as follows :

$$T_{s\triangleleft uv} \ = \ T_{(0,1)\triangleleft\big((0,0)+(1,0)\big)} \ = \ T_{(0,1)\triangleleft(1,0)} \ = \ T_{(0,1)}.$$

On the other hand,

$$T_{(s\triangleleft u)\triangleleft v} \ = \ T_{\big((0,1)\triangleleft(0,0)\big)\triangleleft(1,0)} \ = \ T_{(0,1)\triangleleft(1,0)} \ = \ T_{(0,1)}.$$

Hence, the equality is satisfied.

(v) $T_{f(p,s)f(p*s,t)} = T_{\big(p\triangleright f(s,t)\big)f\big(p\triangleleft f(s,t),s*t\big)}$.
We start with the left hand side as follows :

$$T_{f(p,s)f(p*s,t)} \ = \ T_{f\big((1,0),(0,1)\big)f\big((1,0)*(0,1),(1,2)\big)} \ = \ T_{(1,0)f\big((0,1),(1,2)\big)} \ = \ T_{(1,0)(0,0)} \ = \ T_{(1,0)}.$$

On the other hand,

$$\begin{aligned} T_{\left(p\triangleright f(s,t)\right)f\left(p\triangleleft f(s,t),s*t\right)} &= T_{\left((1,0)\triangleright f\left((0,1),(1,2)\right)\right)f\left((1,0)\triangleleft f\left((0,1),(1,2)\right),(0,1)*(1,2)\right)} \\ &= T_{\left((1,0)\triangleright(0,0)\right)f\left((1,0)\triangleleft(0,0),(1,0)\right)} \\ &= T_{(0,0)f\left((1,0),(1,0)\right)} = T_{(1,0)}. \end{aligned}$$

Hence, the equality is satisfied.

(vi) $T_{\left(p\triangleleft f(s,t)\right)*(s*t)} = T_{(p*s)*t}$.
We start with the left hand side as follows :

$$\begin{aligned} T_{\left(p\triangleleft f(s,t)\right)*(s*t)} &= T_{\left((1,0)\triangleleft f\left((0,1),(1,2)\right)\right)*\left((0,1)*(1,2)\right)} \\ &= T_{\left((1,0)\triangleleft(0,0)\right)*(1,0)} = T_{(1,0)*(1,0)} = T_{(1,0)}. \end{aligned}$$

On the other hand,

$$T_{(p*s)*t} = T_{\left((1,0)*(0,1)\right)*(1,2)} = T_{(0,1)*(1,2)} = T_{(1,0)}.$$

Hence, the equality is satisfied.

In addition, to illustrate the proposition in 4, let $t = (1,2) \in \mathfrak{G}$ *and* $v = (0,0) \in H$ *where* $e_{\mathfrak{G}} = (1,0) \in H \cap \mathfrak{G}$*. Then*

(i) $T_{e_{\mathfrak{G}}\triangleleft v} = T_{(1,0)\triangleleft(0,0)} = T_{(1,0)} = T_{e_{\mathfrak{G}}}$*, as required.*

(ii) $T_{t\triangleleft e} = T_{(1,2)\triangleleft e} = T_{(1,2)} = T_t$*, as required.*

(iii) $T_{(t\triangleleft e_{\mathfrak{G}}^{-1})*e_{\mathfrak{G}}} = T_{\left((1,2)\triangleleft(1,0)^{-1}\right)*(1,0)} = T_{(1,2)*(1,0)} = T_{(1,2)} = T_t$*, as required.*

5. Grading by $H \times \mathfrak{G}$

Definition 4. *Let X be a group, H be a subgroup of X and $(\mathfrak{G}, *)$ be a fixed set of left coset representatives for the subgroup H with the binary operation $*$ which is defined as in 2. A ring R is called a $H \times \mathfrak{G}$-graded ring if*

$$R = \bigoplus_{(u,s)\in H\times\mathfrak{G}} R_{(u,s)} \tag{4}$$

and

$$R_{(u,s)}R_{(v,t)} \subseteq R_{(u,s)(v,t)} \qquad \text{for all } (u,s),(v,t) \in H\times\mathfrak{G}, \text{ where } u,v\in H \text{ and } s,t\in\mathfrak{G}, \tag{5}$$

where $R_{(u,s)}$ is an additive subgroup for each $(u,s) \in H \times \mathfrak{G}$. If (5) is replaced by

$$R_{(u,s)}R_{(v,t)} = R_{(u,s)(v,t)} \qquad \text{for all } (u,s),(v,t) \in H\times\mathfrak{G}, \text{ where } u,v\in H \text{ and } s,t\in\mathfrak{G}, \tag{6}$$

then R is called a fully (or strongly) $H \times \mathfrak{G}$-graded ring.

Theorem 1. *Let X be a group, H be a subgroup of X and $\mathfrak{G} \subset X$ be a set of left coset representatives. Then the equality $R_{(u,s)(v,t)} = R_{\big(u(s\triangleright v)f((s\triangleleft v),t),(s\triangleleft v)*t\big)}$ for $s,t \in \mathfrak{G}$ and $u,v \in H$, makes R into a $H \times \mathfrak{G}$-graded ring where the functions $\triangleright : \mathfrak{G} \times H \to H$, $\triangleleft : \mathfrak{G} \times H \to \mathfrak{G}$ and $f : \mathfrak{G} \times \mathfrak{G} \to H$ are supposed to satisfy the identities in proposition 1.*

Proof. The proof of the theorem follows from the next three lemmas.

Lemma 1. *Let X be a group, H be a subgroup of X, $\mathfrak{G} \subset X$ be a set of left coset representatives and $R_{(u,s)(v,t)} = R_{\big(u(s\triangleright v)f((s\triangleleft v),t),(s\triangleleft v)*t\big)}$. Then the equality $R_{(u,s)\big((v,t)(w,p)\big)} = R_{\big((u,s)(v,t)\big)(w,p)}$ is satisfied for all $(u,s),(v,t)$ and (w,p) in $H \times \mathfrak{G}$ where the functions $\triangleright : \mathfrak{G} \times H \to H$, $\triangleleft : \mathfrak{G} \times H \to \mathfrak{G}$ and $f : \mathfrak{G} \times \mathfrak{G} \to H$ are supposed to satisfy the identities in proposition 1.*

Proof. For $s,t,p \in \mathfrak{G}$ and $u,v,w \in H$, we start with the left hand side as follows:

$$\begin{aligned} R_{(u,s)\big((v,t)(w,p)\big)} &= R_{(u,s)\big(v(t\triangleright w)f\big((t\triangleleft w),p\big),(t\triangleleft w)*p\big)} \\ &= R_{\Big(\Big(u\big(s\triangleright(v(t\triangleright w)f((t\triangleleft w),p))\big)f\big(s\triangleleft(v(t\triangleright w)f((t\triangleleft w),p)),(t\triangleleft w)*p\big)\Big),\Big((s\triangleleft(v(t\triangleright w)f((t\triangleleft w),p)))*((t\triangleleft w)*p)\Big)\Big)}. \end{aligned} \tag{7}$$

Now, we simplify (7) using the proposition in 1 as follows:

$$u\big(s \triangleright (v(t \triangleright w)f\left((t \triangleleft w), p)\right)\big)f\big(s \triangleleft \big(v(t \triangleright w)f\left((t \triangleleft w), p\right)\big), (t \triangleleft w) * p\big)$$

$$\begin{aligned}
&= u\big(s \triangleright (v(t \triangleright w))\big)\big(s \triangleleft (v(t \triangleright w))\big) \triangleright f\big((t \triangleleft w), p\big)f\big(s \triangleleft \big(v(t \triangleright w)f\left((t \triangleleft w), p\right)\big), (t \triangleleft w) * p\big) \\
&= u\big(s \triangleright (v(t \triangleright w))\big)\big(s \triangleleft (v(t \triangleright w))\big) \triangleright f\big((t \triangleleft w), p\big)f\Big(\big(s \triangleleft (v(t \triangleright w))\big) \triangleleft f\big((t \triangleleft w), p\big), (t \triangleleft w) * p\Big) \\
&= u\big(s \triangleright (v(t \triangleright w))\big)f\big(s \triangleleft (v(t \triangleright w)), (t \triangleleft w)\big)f\big((s \triangleleft (v(t \triangleright w))) * (t \triangleleft w), p\big) \\
&= u\big(s \triangleright (v(t \triangleright w))\big)f\Big(\big((s \triangleleft v) \triangleleft (t \triangleright w)\big), (t \triangleleft w)\Big)f\Big(\big((s \triangleleft v) \triangleleft (t \triangleright w)\big) * (t \triangleleft w), p\Big) \\
&= u\big(s \triangleright (v(t \triangleright w))\big)f\Big(\big((s \triangleleft v) \triangleleft (t \triangleright w)\big), (t \triangleleft w)\Big)f\big(((s \triangleleft v) * t) \triangleleft w, p\big) \\
&= u\Big((s \triangleright v)\big((s \triangleleft v) \triangleright (t \triangleright w)\big)\Big)f\Big(\big((s \triangleleft v) \triangleleft (t \triangleright w)\big), (t \triangleleft w)\Big)f\big(((s \triangleleft v) * t) \triangleleft w, p\big) \\
&= u\Big((s \triangleright v)f((s \triangleleft v), t)\big(((s \triangleleft v) * t) \triangleright w\big)\Big)f\big(((s \triangleleft v) * t) \triangleleft w, p\big).
\end{aligned}$$

Also, we have:

$$\begin{aligned}
\big(s \triangleleft \big(v(t \triangleright w)f\left((t \triangleleft w), p\right)\big)\big) * \big((t \triangleleft w) * p\big) &= \Big(\big(s \triangleleft (v(t \triangleright w))\big) \triangleleft f\big((t \triangleleft w), p\big)\Big) * \big((t \triangleleft w) * p\big) \\
&= \Big(\big(s \triangleleft (v(t \triangleright w))\big) * (t \triangleleft w)\Big) * p \\
&= \Big(\big((s \triangleleft v) \triangleleft (t \triangleright w)\big) * (t \triangleleft w)\Big) * p \\
&= \Big(\big((s \triangleleft v) * t\big) \triangleleft w\Big) * p.
\end{aligned}$$

Hence, equation (7) can be rewritten as

$$\begin{aligned} R_{(u,s)\big((v,t)(w,p)\big)} &= R_{(u,s)\big(v(t\triangleright w)f\big((t\triangleleft w),p\big),(t\triangleleft w)*p\big)} \\ &= R_{\Big(\big(u(s\triangleright v)f((s\triangleleft v),t)\big)\big(((s\triangleleft v)*t)\triangleright w\big)f\big(((s\triangleleft v)*t)\triangleleft w,p\big),\big(((s\triangleleft v)*t)\triangleleft w\big)*p\Big)}. \end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned} R_{\big((u,s)(v,t)\big)(w,p)} &= R_{\big(u(s\triangleright v)f((s\triangleleft v),t),(s\triangleleft v)*t\big)(w,p)} \\ &= R_{\Big(\big(u(s\triangleright v)f((s\triangleleft v),t)\big)\big(((s\triangleleft v)*t)\triangleright w\big)f\big(((s\triangleleft v)*t)\triangleleft w,p\big),\big(((s\triangleleft v)*t)\triangleleft w\big)*p\Big)}. \end{aligned} \tag{9}$$

Thus, equations (8) and (9) show that the equality is satisfied.

Lemma 2. *Let X be a group, H be a subgroup of X, $\mathfrak{G} \subset X$ be a set of left coset representatives and $R_{(u,s)(v,t)} = R_{\big(u(s\triangleright v)f((s\triangleleft v),t),(s\triangleleft v)*t\big)}$ for all (u,s) and (v,t) in $H\times\mathfrak{G}$. Suppose that there is an element $e_H \in H$ such that for all $s\in\mathfrak{G}$ and $u\in H$, we have*

$$e_{\mathfrak{G}} \triangleleft u = e_{\mathfrak{G}}\,, \qquad e_{\mathfrak{G}} \triangleright u = e_H u e_H^{-1}\,, \qquad s\triangleright e = e\,, \qquad s\triangleleft e = s\,,$$

$$f(e_{\mathfrak{G}}, s) = e_H\,, \qquad s\triangleright e_H^{-1} = f(s\triangleleft e_H^{-1}, e_{\mathfrak{G}})^{-1}\,, \qquad (s\triangleleft e_H^{-1}) * e_{\mathfrak{G}} = s.$$

Then $R_{(e_H^{-1},e_{\mathfrak{G}})}$ is the identity component for $R_{H\times\mathfrak{G}}$, where $e_{\mathfrak{G}}$ is the left identity in $\mathfrak{G}$ and the functions $\triangleright : \mathfrak{G}\times H \to H$, $\triangleleft : \mathfrak{G}\times H\to\mathfrak{G}$ and $f : \mathfrak{G}\times\mathfrak{G}\to H$ are supposed to satisfy the identities in proposition 1.

Proof. We start with

$$\begin{aligned} R_{(u,s)(e_H^{-1},e_{\mathfrak{G}})} &= R_{\big(u(s\triangleright e_H^{-1})f(s\triangleleft e_H^{-1},e_{\mathfrak{G}}),\,(s\triangleleft e_H^{-1})*e_{\mathfrak{G}}\big)} \\ &= R_{\big(uf(s\triangleleft e_H^{-1},e_{\mathfrak{G}})^{-1}f(s\triangleleft e_H^{-1},e_{\mathfrak{G}}),\,(s\triangleleft e_H^{-1})*e_{\mathfrak{G}}\big)} \\ &= R_{(u,\,s)}. \end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned} R_{(e_H^{-1},e_{\mathfrak{G}})(u,s)} &= R_{\big(e_H^{-1}(e_{\mathfrak{G}}\triangleright u)\,f(e_{\mathfrak{G}}\triangleleft u,s),(e_{\mathfrak{G}}\triangleleft u)*s\big)} \\ &= R_{\big(e_H^{-1}(e_{\mathfrak{G}}\triangleright u)\,f(e_{\mathfrak{G}},\,s),\,e_{\mathfrak{G}}*s\big)} \\ &= R_{\big(e_H^{-1}(e_{\mathfrak{G}}\triangleright u)e_H,\,s\big)} \\ &= R_{\big(e_H^{-1}(e_H u e_H^{-1})\,e_H,\,s\big)} \\ &= R_{(u,\,s)}. \end{aligned} \tag{11}$$

Thus, equations (10) and (11) show that $R_{(e_H^{-1},e_{\mathfrak{G}})}$ is the identity component for $R_{H\times\mathfrak{G}}$. Equivalently, we can say that $(e_H^{-1}, e_{\mathfrak{G}})$ is a 2-sided identity of $H\times\mathfrak{G}$.

It can be noted that since $(\mathfrak{G}, *)$ has the right division property, i.e., for all $s,t\in\mathfrak{G}$ there is a unique solution $p\in\mathfrak{G}$ to the equation $p*s=t$, then there is a unique left inverse s^L for all $s\in\mathfrak{G}$ by putting $t=e_{\mathfrak{G}}$, the left identity in $\mathfrak{G}$. Consequently, we can define a left inverse for all (u,s) in $H\times\mathfrak{G}$ as we see in the next lemma.

Lemma 3. *Let X be a group, H be a subgroup of X, $\mathfrak{G} \subset X$ be a set of left coset representatives and $R_{(u,s)(v,t)} = R_{\left(u(s\triangleright v)f((s\triangleleft v),t),(s\triangleleft v)*t\right)}$ for all $(u,s),(v,t)$ in $H \times \mathfrak{G}$. Suppose that the properties in lemmas 1 and 2 are satisfied, then $H \times \mathfrak{G}$ has a left inverse satisfying the following equality:*

$$R_{(v,t)^L} = R_{\left(e_H^{-1}f(t^L,t)^{-1}(t^L\triangleright v^{-1}),\, t^L\triangleleft v^{-1}\right)}.$$

Proof. To show that $R_{(v,t)^L(v,t)} = R_{(e_H^{-1},e_{\mathfrak{G}})}$, we start with the left hand side as follows:

$$\begin{aligned}
R_{(v,t)^L(v,t)} &= R_{\left(e_H^{-1}f(t^L,t)^{-1}(t^L\triangleright v^{-1}),\, t^L\triangleleft v^{-1}\right)(v,t)}\\
&= R_{\left(e_H^{-1}f(t^L,t)^{-1}(t^L\triangleright v^{-1})\left((t^L\triangleleft v^{-1})\triangleright v\right)f\left((t^L\triangleleft v^{-1})\triangleleft v,t\right),\, \left((t^L\triangleleft v^{-1})\triangleleft v\right)*t\right)}\\
&= R_{\left(e_H^{-1}f(t^L,t)^{-1}(t^L\triangleright v^{-1}v)f\left(t^L\triangleleft v^{-1}v,t\right),\, (t^L\triangleleft v^{-1}v)*t\right)}\\
&= R_{\left(e_H^{-1}f(t^L,t)^{-1}(t^L\triangleright e)f\left(t^L\triangleleft e,t\right),\, (t^L\triangleleft e)*t\right)}\\
&= R_{\left(e_H^{-1}f(t^L,t)^{-1}f(t^L,t)\right),\, t^L*t\right)}\\
&= R_{\left(e_H^{-1},\, e_G\right)}.
\end{aligned}$$

As required.

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