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On Generalizations of ϕ-2-absorbing primary submodules

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Abstract. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. In this paper, we extend the concept of ϕ-2-absorbing primary submodules to the context of ϕ-2-absorbing semi-primary submodules. A proper submodule N of M is called a ϕ-2-absorbing semi-primary submodule, if for each $m \in M$ and $a_1, a_2 \in R$ with $a_1a_2m \in N - \phi(N)$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Those are extended from 2-absorbing primary, weakly 2-absorbing primary, almost 2-absorbing primary, ϕ_n-2-absorbing primary, ω-2-absorbing primary and ϕ-2-absorbing primary submodules, respectively. Some characterizations of 2-absorbing semi-primary, ϕ_n-2-absorbing semi-primary and ϕ-2-absorbing semi-primary submodules are obtained. Moreover, we investigate relationships between 2-absorbing semi-primary, ϕ_n-2-absorbing semi-primary and ϕ-primary submodules of modules over commutative rings. Finally, we obtain necessary and sufficient conditions of a ϕ-2-absorbing semi-primary in order to be a 2-absorbing semi-primary submodule.

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1. Introduction

Throughout this paper, we assume that all rings are commutative with a nonzero identity Suppose that R is a ring and M is an R-module The concept of φ-prime ideals, a generalization of prime ideals was introduced and investigated in [2]. Let $\varphi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function where $\mathcal{J}(R)$ is a set of ideals of R. A proper ideal I of R is said to be φ-prime if whenever $a_1, a_2 \in R$ and $a_1a_2 \in I - \varphi(I)$, then $a_1 \in I$ or $a_2 \in I$. Since $I - \varphi(I) = I - (I \cap \varphi(I))$, so without loss of generality, throughout this paper we will consider $\varphi(I) \subseteq I$. Later, Darani [7] gave a generalization of primary ideals which covers all the above mentioned definitions. He defined the a proper ideal I of R is said to be φ-primary if for $a_1, a_2 \in R$ with $a_1a_2 \in I - \varphi(I)$, either $a_1 \in I$ or $a_2^n \in I$ for some

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positive integer n. Thus a φ-prime ideal is just a φ-primary ideal. In [9], Ebrahimpour and Nekooei called a proper ideal I of a commutative ring R to be φ-2-absorbing if whenever $a_1, a_2, a_3 \in R$ and $a_1a_2a_3 \in I - \varphi(I)$, either $a_1a_2 \in I$ or $a_2a_3 \in I$ or $a_1a_3 \in I$. Badawi, et al. [1] generalized the concept of 2-absorbing primary ideals to φ-2-absorbing primary ideals. According to their definition, a proper ideal I of R is called a φ-2-absorbing primary ideal if whenever $a_1a_2a_3 \in I - \varphi(I)$ for $a_1, a_2, a_3 \in R$, then $a_1a_2 \in I$ or $a_2a_3 \in \sqrt{I}$ or $a_1a_3 \in \sqrt{I}$. Clearly a φ-2-absorbing ideal of R is also a φ-2-absorbing primary ideal of R. Other generalizations of prime ideals have recently been studied in [4, 3, 5, 6].

The notion of ϕ-prime submodule, which is a generalization of prime submodule, was introduced by Zamani in [11]. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is a set of all submodules of M. A proper submodule N of M is called ϕ-prime submodule of M if whenever $a \in R$ and $am \in N - \phi(N)$, then $m \in N$ or $a \in (N : M)$. Since $N - \phi(N) = N - (N \cap \phi(I))$, without loss of generality we may assume that $\phi(N) \subseteq N$. Recall that a proper submodule N of M is called a ϕ-primary submodule submodule of M as in [11] if whenever $am \in N - \phi(N)$ for some $a \in R, m \in M$, then $m \in N$ or $a^n \in (N : M)$ for some positive integer n. In 2017, Ebrahimpour and Mirzaee [8] generalized the concept of semiprime submodules to ϕ-semiprime submodules. According to their definition, a proper submodule N of M is called a ϕ-semiprime submodule if whenever $a^2m \in N - \phi(N)$ for $a \in R, m \in M$, then $am \in N$. In [10], the concept of ϕ-prime and ϕ-primary submodules generalized to ϕ-2-absorbing primary submodule of a module over a commutative ring. Let N be a proper submodule of M. N is said to be a ϕ-2-absorbing primary submodule of M if whenever $a_1, a_2 \in R$ and $m \in M$ with $a_1a_2m \in N - \phi(N)$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2m \in N$. Moreover, recall from [10] that a proper submodule N of M is said to be a ϕ-2-absorbing submodule of M if whenever $a_1, a_2 \in R$ and $m \in M$ with $a_1a_2m \in N - \phi(N)$ implies $a_1a_2 \in (N : M)$ or $a_1m \in N$ or $a_2m \in N$. Thus a ϕ-2-absorbing submodule is just a ϕ-2-absorbing primary submodule.

In this paper, we extend the concept of ϕ-2-absorbing primary submodule to the context of ϕ-2-absorbing semi-primary submodule. Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ is the set of all submodules of M. A proper submodule N of M is called a ϕ-2-absorbing semi-primary submodule if for each $m \in M$ and $a_1, a_2 \in R$ with $a_1a_2m \in N - \phi(N)$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Let N be a ϕ-2-absorbing semi-primary submodule of M.

- If $\phi(N) = \emptyset$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_\emptyset$ and N is called a $\phi_\emptyset$-2-absorbing semi-primary submodule of M, and hence N is a 2-absorbing semi-primary submodule of M.

- If $\phi(N) = \{0\}$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_0$ and N is called a ϕ_0-2-absorbing semi-primary submodule of M, and hence N is a weakly 2-absorbing semi-primary submodule of M.

- If $\phi(N) = N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_1$ and N is called a ϕ_1-2-absorbing semi-primary submodule of M. It is easy to see that every proper submodule is ϕ_1-2-absorbing semi-primary.

- If $\phi(N) = (N : M)N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_2$ and N is called a ϕ_2-2-absorbing semi-primary submodule of M, and hence N is an almost 2-absorbing semi-primary submodule of M.
- If $\phi(N) = (N : M)^{n-1}N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_{n\geq 2}$ and N is called a ϕ_n-2-absorbing semi-primary submodule of M, and hence N is a n-2-absorbing semi-primary submodule of M.
- If $\phi(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_\omega$ and N is called a ϕ_ω-2-absorbing semi-primary submodule of M, and hence N is a ω-2-absorbing semi-primary submodule of M.

In section 2, we give some basic properties of ϕ-2-absorbing semi-primary submodules. Among many results in this paper, it is shown that N is a ϕ-2-absorbing semi-primary submodule of M if and only if for every $a_1, a_2 \in R - (N : M)$ and $a_1a_2 \in R - \sqrt{(N : M)}$, $(N : a_1a_2) \subseteq (\phi(N) : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n.

In Section 3, we study the stability of ϕ_α-2-absorbing semi-primary submodules. Moreover, we investigate relationships between 2-absorbing semi-primary, ϕ_0-2-absorbing semi-primary, ϕ_n-2-absorbing semi-primary and ϕ-primary submodules of modules over commutative rings. Finally, we obtain necessary and sufficient conditions of a ϕ-2-absorbing semi-primary in order to be a 2-absorbing semi-primary.

2. Properties of ϕ-2-Absorbing Semi-primary Submodules

The results of the following theorems seem to play an important role to study ϕ-classical semi-primary submodules of modules over commutative rings; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 1. *Let M be an R-module and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function where $S(M)$ be a set of all submodules of M. A proper submodule N of M is called a ϕ-2-absorbing semi-primary submodule, if for each $m \in M$ and $a_1, a_2 \in R$ with $a_1a_2m \in N - \phi(N)$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n.*

Remark 1. *It is easy to see that every ϕ-2-absorbing primary submodule is ϕ-2-absorbing semi-primary.*

The following example shows that the converse of Remark 1 is not true.

Example 1. *Let $R = \boldsymbol{Z}$ and $M = \boldsymbol{Z}$. Consider the submodule $N = 12\boldsymbol{Z}$ of M. Define $\phi : S(M) \to S(M) \cup \{\emptyset\}$ by $\phi(N) = \{0\}$ for every $N \in S(M)$. It is easy to see that N is a ϕ-2-absorbing semi-primary submodule of M. Notice that $2 \cdot 2 \cdot 3 \in N - \phi(N)$, but $2 \cdot 3 \notin N$ and $(2 \cdot 2)^n \notin (N : M)$ for all positive integer n. Therefore N is not a ϕ-2-absorbing primary submodule of M.*

Theorem 1. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ and $\varphi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be two functions.*

(i) If N is a ϕ-2-absorbing semi-primary submodule of M, then $(N : m)$ is a φ-2-absorbing primary ideal of R with $m \in M - N$ and $(\phi(N) : m)) \subseteq \varphi(N : m)$.

(ii) For every $m \in M - N$ if $(N : m)$ is a φ-primary ideal of R, then N is a ϕ-2-absorbing semi-primary submodule of M with $\varphi(N : m) \subseteq (\phi(N) : m))$.

Proof. 1. Let $a_1, a_2, a_3 \in R$ such that $a_1a_2a_3 \in (N : m) - \varphi((N : m))$. By assumption, $a_1a_3(a_2m) \in N - \phi(N)$. Then by Definition 1, $a_1a_3 \in \sqrt{(N : M)} \subseteq \sqrt{(N : m)}$ or $a_1a_2m \in N$ or $a_3^n a_2 m \in N$ for some positive integer n. Therefore $a_1a_2 \in (N : m)$ or $a_2a_3 \in \sqrt{(N : m)}$ or $a_1a_3 \in \sqrt{(N : m)}$. This completes the proof.

2. Let $a_1, a_2 \in R$ such that $a_1a_2m \in N - \phi(N)$. Then $a_1a_2 \in (N : m)$ and $a_1a_2 \notin (\phi(N) : m)$. By assumption, $a_1a_2 \in (N : m) - \varphi((N : m))$. Again, by assumption, $a_1 \in (N : m)$ or $a_2^n \in (N : m)$ for some positive integer n. This completes the proof.

The following example shows that the converse of Theorem 1 is not true.

Example 2. *1. Let $M = \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$ be an $\boldsymbol{Z}$-module. Define $\varphi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ by $\varphi(I) = \{0\}$ for every $I \in \mathcal{J}(R)$. Consider the submodule $N = \{0\} \times 12\boldsymbol{Z} \times \boldsymbol{Z}$ of M. Clearly, $(N : (m_1, m_2, m_3)) = \{0\}$ is a φ-2-absorbing primary ideal of $\mathcal{J}(R)$, where $(m_1, m_2, m_3) \in M - N$. Define $\phi : S(M) \to S(M) \cup \{\emptyset\}$ by $\phi(N) = \{(0, 0, 0)\}$ for every $N \in S(M)$. Notice that $3 \cdot 4(0, 1, 1) \in N - \phi(N)$, but $(3 \cdot 4) \notin \sqrt{(N : M)}, 3(0, 1, 1) \notin N$ and $4^n(0, 1, 1) \notin N$ for all positive integer n. Hence N is not a ϕ-2-absorbing semi-primary submodule of M.*

2. Let $M = \boldsymbol{Z}_{12}$ be an $\boldsymbol{Z}_{12}$-module. Define $\phi : S(M) \to S(M) \cup \{\emptyset\}$ by $\phi(N) = \{[0]\}$ for every $N \in S(M)$. Consider the submodule $N = \{[0]\}$ of M. Clearly, N is a ϕ-2-absorbing semi-primary submodule of M. Define $\varphi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ by $\varphi(I) = \emptyset$ for every $I \in \mathcal{J}(R)$. Notice that $[4][3] \in \{[0]\} = (N : [1]) - \varphi((N : [1]))$, but $[4] \in (N : [1])$ and $[3]^n \in (N : [1])$ for all positive integer n.

Let N be a submodule of an R-module M and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Define $\phi_N : S(M/N) \to S(M/N) \cup \{\emptyset\}$ by

$$\phi_N(K/N) = \begin{cases} (\phi(K) + N)/N & ; \phi(K) \neq \emptyset \\ \emptyset & ; \phi(K) = \emptyset, \end{cases}$$

for every submodule K of M with $N \subseteq K$[11]. In, 2010 Zamani in [11] gives relations between ϕ-prime submodules of M and ϕ_N-prime submodules of M/N. This leads us to give relations between ϕ-2-absorbing semi-primary submodules of M and ϕ_N-2-absorbing semi-primary submodules of M/N.

Theorem 2. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let N, K be two submodules of M with $N \subseteq K$. If K is a ϕ-2-absorbing semi-primary submodule of M, then K/N is a ϕ_N-2-absorbing semi-primary submodule of M/N.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2(m+N) \in (K/N) - \phi_N(K/N)$. Then $a_1a_2m \in K - \phi(K)$. By Definition 1, $a_1a_2 \in \sqrt{(K:M)}$ or $a_1m \in K$ or $a_2^nm \in K$ for some positive integer n. Clearly, $a_1a_2 \in \sqrt{(K/N:M/N)}$ or $a_1(m+N) \in K/N$ or $a_2^n(m+N) \in K/N$ for some positive integer n. This completes the proof.

Theorem 3. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let N, K be two submodules of M. If $N \subseteq \phi(K)$ and K/N is a ϕ_N-2-absorbing semi-primary submodule of M/N, then K is a ϕ-2-absorbing semi-primary submodule of M.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in K - \phi(K)$. Then $a_1a_2(m+N) \in (K - \phi(K))/N$. By Definition 1, $a_1a_2 \in \sqrt{(K/N:M/N)}$ or $a_1(m+N) \in K/N$ or $a_2^n(m+N) \in K/N$ for some positive integer n. Clearly, $a_1a_2 \in \sqrt{(K:M)}$ or $a_1m \in K$ or $a_2^nm \in K$ for some positive integer n.

Now, by Theorem 2 and Theorem 3, we have the following corollary.

Corollary 1. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let N, K be two submodules of M with $N \subseteq \phi(K)$. Then K is a ϕ-2-absorbing semi-primary submodule of M if and only if K/N is a ϕ_N-2-absorbing semi-primary submodule of M/N.*

Proof. The proof follows from Theorem 2, 3.

Zamani in [11] gives relations between ϕ-prime submodules of M and ϕ_S-prime submodules of $S^{-1}M$. This leads us to give relations between ϕ-2-absorbing semi-primary submodules of M and ϕ_S-2-absorbing semi-primary submodules of $S^{-1}M$.

Theorem 4. *Let S be a multiplicative closed subset of R and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If N is a ϕ-2-absorbing semi-primary submodule of M, then $S^{-1}N$ is a ϕ_S-2-absorbing semi-primary submodule of $S^{-1}M$.*

Proof. Let $a_1, a_2 \in R, s_1, s_2, s_3 \in S$ and $m \in M$ such that $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} \in S^{-1}N - \phi_S(S^{-1}N)$. Then there exists $s \in S$ such that $sa_1a_2m \in N$. If $sa_1a_2m \in \phi(N)$, then $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} = \frac{sa_1a_2m}{ss_1s_2s_3} \in S^{-1}\phi(N) = \phi_S(S^{-1}N)$, a contradiction. Now if $sa_1a_2m \notin \phi(N)$, then $a_1a_2(sm) \in N - \phi(N)$. By Definition 1, $a_1a_2 \in \sqrt{(N:M)}$ or $a_1sm \in N$ or $a_2^nsm \in N$ for some positive integer n. If $a_1sm \in N$ or $a_2^nsm \in N$, then $\frac{a_1}{s_1}\frac{m}{s_3} = \frac{a_1sm}{s_1ss_3} \in S^{-1}N$ or $(\frac{a_2}{s_2})^n\frac{m}{s_3} = \frac{a_2^nsm}{s_2^ns_3s} \in S^{-1}N$. Now if $a_1a_2 \in \sqrt{(N:M)}$, then there exists positive integer n_1 such that $(a_1a_2)^{n_1} \in (N:M)$. Clearly, $(\frac{a_1}{s_1}\frac{a_2}{s_2})^n \in S^{-1}(N:M)$. This completes the proof.

Theorem 5. *Let S be a multiplicative closed subset of R and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. If $S^{-1}N$ is a ϕ_S-2-absorbing semi-primary submodule of $S^{-1}M$ such that $S \cap Zd(N/\phi(N)) = \emptyset$ and $S \cap Zd(M/N) = \emptyset$, then N is a ϕ-2-absorbing semi-primary submodule of M.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in N - \phi(N)$. Then $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} = \frac{abm}{1} \in S^{-1}N$. If $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \in \phi_S(S^{-1}N) = S^{-1}\phi(N)$, then there exists $s \in S$ such that $sa_1a_2m \in \phi(N)$ which is a contradiction. If $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \notin \phi_S(S^{-1}N)$, then $\frac{a_1}{1}\frac{a_2}{1}\frac{m}{1} \in S^{-1}N - \phi_S(S^{-1}N)$.

By Definition 1, $\frac{a_1}{1}\frac{a_2}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$ or $\frac{a_1}{1}\frac{m}{1} \in S^{-1}N$ or $(\frac{a_2}{1})^n\frac{m}{1} \in S^{-1}N$ for some positive integer n. If $\frac{a_1}{1}\frac{a_2}{1} \in \sqrt{(S^{-1}N : S^{-1}M)}$, then $(\frac{a_1}{1}\frac{a_2}{1})^n \in (S^{-1}N : S^{-1}M)$ for some positive integer n. Thus there exists $s \in S$ such that $s(a_1a_2)^nM \subseteq N$ for some positive integer n. By assumption, $(a_1a_2)^nM \subseteq N$ so $a_1a_2 \in \sqrt{(N : M)}$.

In view of Theorem 4 and Theorem 5, we have the following result.

Corollary 2. *Let S be a multiplicative closed subset of R and let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function with $S \cap Zd(N/\phi(N)) = \emptyset$ and $S \cap Zd(M/N) = \emptyset$. Then N is a ϕ-2-absorbing semi-primary submodule of M if and only if $S^{-1}N$ is a ϕ_S-2-absorbing semi-primary submodule of $S^{-1}M$.*

Proof. The proof follows from Theorem 4, 5.

In the following result, we give an equivalent definition of ϕ-2-absorbing semi-primary submodules.

Theorem 6. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*

(i) N is a ϕ-2-absorbing semi-primary submodule of M.

(ii) For every $a_1, a_2 \in R - (N : M)$ if $a_1a_2 \in R - \sqrt{(N : M)}$, then $(N : a_1a_2) \subseteq (\phi(N) : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n.

Proof. $(i \Rightarrow ii)$ Let $m \in (N : a_1a_2)$. Then $a_1a_2m \in N$. If $a_1a_2m \in \phi(N)$, then $m \in (\phi(N) : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n. If $a_1a_2m \notin \phi(N)$, then $a_1a_2m \in N - \phi(N)$. By Definition 1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^nm \in N$ for some positive integer n. By assumption, $m \in (N : a_1)$ or $m \in (N : a_2^n)$ for some positive integer n. Therefore $(N : a_1a_2) = (\phi(N) : a_1a_2) \cup (N : a_1) \cup (N : a_2^n)$ for some positive integer n.

$(ii \Rightarrow i)$ It is obvious.

Corollary 3. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*

(i) N is a ϕ-2-absorbing semi-primary submodule of M.

(ii) For every $a \in R - (N : M)$ and every ideal I of R such that $I \nsubseteq (N : M)$ if $aI \nsubseteq \sqrt{(N : M)}$, then $(N : aI) \subseteq (\phi(N) : aI) \cup (N : a) \cup (N : I^n)$ for some positive integer n.

(iii) For every ideals I, J of R such that $I, J \nsubseteq (N : M)$ if $IJ \nsubseteq \sqrt{(N : M)}$, then $(N : IJ) \subseteq (\phi(N) : IJ) \cup (N : I) \cup (N : J^n)$ for some positive integer n.

Proof. The proof is similar to Theorem 6.

The following theorem offers a characterization of ϕ-2-absorbing semi-primary submodules.

Theorem 7. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*

(i) N is a ϕ-2-absorbing semi-primary submodule of M.

(ii) For every $a \in R - (N : M)$ and $m \in M$ if $am \notin N$, then $(N : am) \subseteq (\phi(N) : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$.

Proof. $(i \Rightarrow ii)$ Let $a \in R - (N : M)$ and $m \in M$ such that $am \notin N$. Assume that $r \in (N : am)$. If $ram \notin \phi(N)$, then $ram \in N - \phi(N)$. By Definition 1, $ar \in \sqrt{(N : M)}$ or $am \in N$ or $r^n m \in N$ for some positive integer n. By assumption, $r \in (\sqrt{(N : M)} : a) \cup \sqrt{(N : m)} \subseteq (\phi(N) : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$. Now if $ram \in \phi(N)$, then $r \in (\phi(N) : am) \subseteq (\phi(N) : am) \cup (\sqrt{((N : M)} : a) \cup \sqrt{(N : m)}$.

$(ii \Rightarrow i)$ It is obvious.

Corollary 4. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. The following conditions are equivalent:*

(i) N is a ϕ-2-absorbing semi-primary submodule of M.

(ii) For every ideal I of R such that $I \subseteq R - (N : M)$ and $m \in M$ if $Im \nsubseteq N$, then $(N : Im) \subseteq (\phi(N) : Im) \cup (\sqrt{(N : M)} : I) \cup \sqrt{(N : m)}$.

Proof. The proof is similar to Theorem 7.

3. Properties of ϕ_α-2-Absorbing Semi-primary Submodules

We start with the following theorem that gives a relation between ϕ_α-2-absorbing semi-primary and ϕ-2-absorbing semi-primary submodule. Our starting points are the following definitions:

Definition 2. *Let M be an R-module and let $S(M)$ be the set of all submodules of M. Define the following functions $\phi_\alpha : S(M) \to S(M) \cup \{\emptyset\}$ and the corresponding ϕ_α-2-absorbing semi-primary submodules:*

- *If $\phi(N) = \emptyset$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_\emptyset$ and N is called a $\phi_\emptyset$-2-absorbing semi-primary submodule of M, and hence N is a 2-absorbing semi-primary submodule of M.*
- *If $\phi(N) = \{0\}$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_0$ and N is called a ϕ_0-2-absorbing semi-primary submodule of M, and hence N is a weakly 2-absorbing semi-primary submodule of M.*
- *If $\phi(N) = N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_1$ and N is called a ϕ_1-2-absorbing semi-primary submodule of M.*

- *If $\phi(N) = (N : M)N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_2$ and N is called a ϕ_2-2-absorbing semi-primary submodule of M, and hence N is an almost 2-absorbing semi-primary submodule of M.*

- *If $\phi(N) = (N : M)^{n-1}N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_{n\geq 2}$ and N is called a ϕ_n-2-absorbing semi-primary submodule of M, and hence N is a n-2-absorbing semi-primary submodule of M.*

- *If $\phi(N) = \bigcap_{i=1}^{\infty}(N : M)^i N$ for every $N \in \mathcal{S}(M)$, then we say that $\phi = \phi_\omega$ and N is called a ϕ_ω-2-absorbing semi-primary submodule of M, and hence N is a ω-2-absorbing semi-primary submodule of M.*

Remark 2. *Let M be an R-module and let $S(M)$ be a set of all submodules of M. For two functions $\phi_\alpha, \phi_\beta : S(M) \to S(M) \cup \{\emptyset\}$. We define $\phi_\alpha \leq \phi_\beta$, if $\phi_\alpha(N) \subseteq \phi_\beta(N)$ for all $N \in S(M)$[11]. Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \ldots \leq \phi_{n+1} \leq \phi_n \leq \ldots \leq \phi_2 \leq \phi_1$.*

Notice that for an R-module M, the zero submodule $\{0\}$ is always a ϕ_0-2-absorbing semi-primary submodule. In the following example, we give a module in which a ϕ_0-2-absorbing semi-primary submodule is not ϕ-2-absorbing semi-primary .

Example 3. *Let $R = \boldsymbol{Z}$ and $M = \boldsymbol{Z}_{30} \times \boldsymbol{Z}_{30}$. Consider the submodule $N = \{[0]\} \times \boldsymbol{Z}_{30}$ of M. Define $\phi : S(M) \to S(M) \cup \{\emptyset\}$ by $\phi(N) = \{[0]\} \times \{[0]\}$ for every $N \in S(M)$. It is easy to see that N is a ϕ_0-2-absorbing semi-primary submodule of M. Notice that $(2\cdot 3)([5],[1]) \in N - \phi(N)$, but $2 \cdot 3 \notin \sqrt{(N : M)}, 2([5],[1]) \notin \{[0]\} \times \boldsymbol{Z}_{30}$ and $3^n([5],[1]) \notin \{[0]\} \times \boldsymbol{Z}_{30}$ for all positive integer n. Therefore N is not a ϕ-2-absorbing semi-primary submodule of M.*

We are finding additional condition to show that a 2-absorbing semi-primary submodule is a ϕ-2-absorbing semi-primary submodule of an R-module.

Theorem 8. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let $\phi(N)$ is a 2-absorbing semi-primary submodule of M. Then N is a ϕ-2-absorbing semi-primary submodule of M if and only if N is a 2-absorbing semi-primary submodule of M.*

Proof. Suppose that N is a 2-absorbing semi-primary submodule of M. Clearly, N is a ϕ-2-absorbing semi-primary submodule of M. Conversely, assume that N is a ϕ-2-absorbing semi-primary submodule of M. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in N$. If $a_1a_2m \notin \phi(N)$, then $a_1a_2m \in N - \phi(N)$. By Definition 1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Now if $a_1a_2m \in \phi(N)$, then $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n.

Further, we give another characterization of ϕ_α-2-absorbing semi-primary submodule of M.

Theorem 9. *Let $\phi_2 : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let $(0 : r^k) \subseteq r^k M \neq M$, where $r \in R$. Then $r^k M$ is a ϕ_2-2-absorbing semi-primary submodule of M if and only if it is a 2-absorbing semi-primary submodule of M.*

Proof. Suppose that r^kM is a 2-absorbing semi-primary submodule of M. Clearly, r^kM is a ϕ_2-2-absorbing semi-primary submodule of M. Conversely, assume that r^kM is a ϕ_2-2-absorbing semi-primary submodule of M. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in r^kM$. If $a_1a_2m \notin \phi_2(r^kM)$, then $a_1a_2m \in r^kM - \phi_2(r^kM)$. By Definition 1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Assume that $a_1a_2m \in \phi_2(r^kM)$. Since $a_1a_2m, r^ka_2m \in r^kM$, we have $(a_1 + r^k)a_2m \in r^kM$. If $(a_1 + r^k)a_2m \notin \phi_2(r^kM)$, then $(a_1 + r^k)a_2m \in r^kM - \phi_2(r^kM)$. Then by Definition 1, $a_1a_2 \in \sqrt{(r^kM : M)}$ or $a_1m \in r^kM$ or $a_2^n m \in r^kM$ for some positive integer n. Now if $(a_1 + r^k)a_2m \in \phi_2(r^kM)$, then $r^ka_2m \in \phi_2(r^kM)$. Then there exists $m_0 \in (r^kM : M)M$ such that $r^ka_2m = r^km_0$. By assumption, $a_2m - m_0 \in r^kM$. Hence $a_2m \in r^kM$.

Theorem 10. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let N, K be two submodules of M with $N \subseteq K$. If $\phi(K) \subseteq N$ and K is a ϕ-2-absorbing semi-primary submodule of M, then K/N is a ϕ_0-2-absorbing semi-primary submodule of M/N.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m + N \in K/N - (\phi_N)_0(K/N)$. Since $\phi(K) \subseteq N$, we have $a_1a_2m \notin \phi(K)$. Clearly, $a_1a_2m \in K - \phi(K)$. By Definition 1, $a_1a_2 \in \sqrt{(K : M)}$ or $a_1m \in K$ or $a_2^n m \in K$ for some positive integer n. Thus $a_1a_2 \in \sqrt{(K/N : M/N)}$ or $a_1(m + N) \in K/N$ or $a_2^n(m + N) \in K/N$ for some positive integer n.

Theorem 11. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function, N be a ϕ-2-absorbing semi-primary submodule of M and let K be a submodule of M with $N \subseteq K$. If $\phi(N) \subseteq \phi(K)$ and K/N is a ϕ_0-2-absorbing semi-primary submodule of M/N, then K is a ϕ-2-absorbing semi-primary submodule of M.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in K - \phi(K)$. By assumption, $a_1a_2m \notin \phi(N)$. If $a_1a_2m \in N$, then $a_1a_2m \in N - \phi(N)$. By Definition 1, $a_1a_2 \in \sqrt{(N : M)} \subseteq \sqrt{(K : M)}$ or $a_1m \in N \subseteq K$ or $a_2^n m \in N \subseteq K$ for some positive integer n. If $a_1a_2m \notin N$, then $a_1a_2(m+N) \notin \phi_0(K/N)$. Therefore $a_1a_2(m+N) \in K/N - \phi_0(K/N)$. By Definition 1, $a_1a_2 \in \sqrt{(K/N : M/N)}$ or $a_1(m + N) \in K/N$ or $a_2^n(m + N) \in K/N$ for some positive integer n. This completes the proof.

As an immediate consequence of Theorem 10 and Theorem 11 we have the next corollary.

Corollary 5. *Let $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let N, K be two submodules of M with $N \subseteq K$. Then N is a ϕ-2-absorbing semi-primary submodule of M if and only if $N/\phi(N)$ is a ϕ_0-2-absorbing semi-primary submodule of $M/\phi(N)$.*

Proof. It is straightforward by Theorem 10 and Theorem 11.

Theorem 12. *Let $\phi_\alpha : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following hold.*

(i) If N is a ϕ_β-2-absorbing semi-primary submodule of M such that $\phi_\beta \leq \phi_\gamma$, then N is a ϕ_γ-2-absorbing semi-primary submodule of M.

(ii) If N is a $\phi_\emptyset$-2-absorbing semi-primary submodule of M, then N is a ϕ_0-2-absorbing semi-primary submodule of M.

(iii) If N is a ϕ_0-2-absorbing semi-primary submodule of M, then N is an ω-2-absorbing semi-primary submodule of M.

(iv) If N is a ϕ_ω-2-absorbing semi-primary submodule of M, then N is a ϕ_n-2-absorbing semi-primary submodule of M.

Proof. i. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in N - \phi_\gamma(N)$. By assumption, $\phi_\beta(N) \subseteq \phi_\gamma(N)$. Then $a_1a_2m \in N - \phi_\gamma(N) \subseteq N - \phi_\beta(N)$. Then by Definition 1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n.
ii, iii, iv. It is obvious.

From the above definitions we obtain immediately the following implication chart for the considered types of submodules:

2-absorbing semi-primary $\Rightarrow$ weakly 2-absorbing semi-primary $\Rightarrow$ ω-2-absorbing semi-primary $\Rightarrow$ $\phi_{n\geq 2}$-2-absorbing semi-primary $\Rightarrow$ almost 2-absorbing semi-primary

Theorem 13. *Let $\phi, \phi_3 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions and let N be a ϕ-2-absorbing semi-primary submodule. If $\phi_3 \not\leq \phi$, then N is a 2-absorbing semi-primary submodule of M.*

Proof. Let $a_1, a_2 \in R$ and $m \in M$ such that $a_1a_2m \in N$. If $a_1a_2m \notin \phi(N)$, then $a_1a_2m \in N - \phi(N)$. By Definition 1, $a_1a_2 \in \sqrt{(N : M)}$ or $a_1m \in N$ or $a_2^n m \in N$ for some positive integer n. Next, let $a_1a_2m \in \phi(N)$. In this case, we may assume that $a_1a_2N \subseteq \phi(N)$, because if $a_1a_2N \not\subseteq \phi(N)$ then there exists $m_0 \in N$ such that $a_1a_2m_0 \notin \phi(N)$. Clearly, $a_1m \in N$ or $a_1a_2 \in \sqrt{(N : M)}$ or $a_2^n m \in N$ for some positive integer n. Second we may assume that $(N : M)^2 m \subseteq \phi(N)$. If this is not the case, there exist $r_1, r_2 \in (N : M)$ such that $(a_1 + r_1)(a_2 + r_2)m \notin \phi(N)$. By assumption, $a_1m \in N$ or $a_1a_2 \in \sqrt{(N : M)}$ or $a_2^n \in \sqrt{(N : m)}$ for some positive integer n. Again, by assumption, $(N : M)^2 N \not\subseteq \phi(N)$. There exist $r_1, r_2 \in (N : M)$ and $m_0 \in N$ such that $r_1r_2m_0 \notin \phi(N)$. Thus by Definition 1, $a_1m \in N$ or $a_1a_2 \in \sqrt{(N : M)}$ or $a_2^n m \in N$ for some positive integer n.

Corollary 6. *Let $\phi_n : S(M) \to S(M) \cup \{\emptyset\}$ be a function and let N be a ϕ_0-2-absorbing semi-primary submodule of M. If $\phi_3 \neq \phi_0$, then N is a 2-absorbing semi-primary submodule of M.*

Proof. Similar to the proof of Theorem 13.

Theorem 14. *Let $\phi, \phi_4 : S(M) \to S(M) \cup \{\emptyset\}$ be two functions. If N is a ϕ-2-absorbing semi-primary submodule such that $\phi \leq \phi_4$, then N is a ω-2-absorbing semi-primary submodule of M.*

Proof. If N is a 2-absorbing semi-primary submodule of M, then there is nothing to prove. Assume that N is not a 2-absorbing semi-primary submodule of M. Then by Theorem 13, $(N : M)^2N = \phi_3(N) \subseteq \phi(N) \subseteq (N : M)^3N$. This implies that $\phi(N) = (N : M)^2N = (N : M)^3N$. Thus $\phi(N) = (N : M)^iN$ for all $i \geq 3$.

Theorem 15. *Let M be a multiplication R-module and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. Then the following properties hold.*

(i) If N is a ϕ-2-absorbing semi-primary submodule of M with $N^3 \nsubseteq \phi(N)$, then N is a 2-absorbing semi-primary submodule of M.

(ii) If N is a ϕ_n-2-absorbing semi-primary submodule of M with $N^3 \neq N^n$ for all $n \geq 3$, then N is a 2-absorbing semi-primary submodule of M.

Proof. 1. Suppose that N is a ϕ-2-absorbing semi-primary submodule of M that is not 2-absorbing semi-primary. Clearly, $N = (N : M)M$. Then by Theorem 13, $N^3 = (N : M)^3M = (N : M)^2((N : M)M) = (N : M)^2N = \phi_3(N) \subseteq \phi(N)$.

2. Suppose that N is a ϕ_n-2-absorbing semi-primary submodule of M that is not 2-absorbing semi-primary. Clearly, $N^n \subseteq N^3$, for all $n \geq 3$. Then by parts 1, $N^3 \subseteq \phi_n(N) = (N : M)^{n-1}N = (N : M)^nM = N^n$. Hence $N^3 = N^n$. This completes the proof.

Theorem 16. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. Then the following statements are equivalent:*

(i) $N_1 \times M_2$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2$.

(ii) (a) N_1 is a ψ_1-2-absorbing semi-primary submodule of M_1.

(b) For each $a_1, a_2 \in R$ and $m \in M_1$ such that $a_1a_2m \in \psi_1(N_1)$ if $a_1a_2 \notin \sqrt{(N_1 : M_1)}$ and $a_1m \notin N_1, a_2^nm \notin N_1$ for all positive integer n, then $a_1a_2 \in (\psi_2(M_2) : M_2)$.

Proof. $(i \Rightarrow ii)$. (a). It is obvious.

(b). Let $a_1a_2m \in \psi_1(N_1), a_1m \notin N_1$ and $a_2^nm \notin N_1$, where $a_1, a_2 \in R$ and $m \in M_1$. Suppose that $a_1a_2 \notin (\psi_2(M_2) : M_2)$. There exists $m_2 \in M_2$ such that $a_1a_2m_2 \notin \psi_2(M_2)$. Thus $a_1a_2(m, m_2) \in N_1 \times M_2 - \phi(N_1 \times M_2)$. By part (1), i.e., $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m \in N_1$ or $a_2^nm \in N_1$ which is a contradiction.

$(ii \Rightarrow i)$. Let $a_1, a_2 \in R$ and $(m_1, m_2) \in M_1 \times M_2$ such that $a_1a_2(m_1, m_2) \in N_1 \times M_2 - \phi(N_1 \times M_2)$. If $a_1a_2m_1 \notin \psi_1(N_1)$, then $a_1a_2m_1 \in N_1 - \psi_1(N_1)$. By part (a), $a_1a_2 \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $a_1(m_1, m_2) = (a_1m_1, a_1m_2) \in N_1 \times M_2$ or $a_2^n(m_1, m_2) = (a_2^nm_1, a_2^nm_2) \in N_1 \times M_2$, and thus we are done. If $a_1a_2m_1 \in \psi_1(N_1)$, then $a_1a_2m_2 \notin \psi_2(M_2)$. Therefore $a_1a_2 \notin (\psi_2(M_2) : M_2)$. By part (b), $a_1a_2 \in \sqrt{(N_1 \times M_2 : M_1 \times M_2)}$ or $a_1(m_1, m_2) \in N_1 \times M_2$ or $a_2^n(m_1, m_2) \in N_1 \times M_2$.

Corollary 7. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. Then the following conditions are equivalent:*

(i) $M_1 \times N_2$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times M_2$*.*

(ii) *(a)* N_2 *is a* ψ_2*-2-absorbing semi-primary submodule of* M_2*.*

(b) *For each* $a_1, a_2 \in R$ *and* $m \in M_2$ *such that* $a_1a_2m \in \psi_2(N_2)$*, if* $a_1a_2 \notin \sqrt{(N_2 : M_2)}, a_1m \notin N_2$ *and* $a_2^n m \notin N_2$ *for all positive integer* n*, then* $a_1a_2 \in (\psi_1(M_1) : M_1)$*.*

Proof. Similar to the proof of Theorem 16.

Theorem 17. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *is a function with* $\phi = \psi_1 \times \ldots \times \psi_k$*. Then the following conditions are equivalent:*

(i) $M_1 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times \ldots \times M_k$*.*

(ii) *(a)* N_i *is a* ψ_i*-2-absorbing semi-primary submodule of* M_i*.*

(b) *For each* $a_1, a_2 \in R$ *and* $m \in M_i$ *such that* $a_1a_2m \in \psi_i(N_i)$*, if* $a_1a_2 \notin \sqrt{(N_i : M_i)}, a_1m \notin N_i$ *and* $a_2^n m \notin N_i$ *for all positive integer* n*, then there exists* $j \in \{1, 2, \ldots, k\}$ *such that* $a_1a_2 \in (\psi_j(M_j) : M_j)$*.*

Proof. Similar to the proof of Theorem 16.

Next, let R_i be a commutative ring with identity and let M_i be an R_i-module. Then $M_1 \times M_2$ is an $R_1 \times R_2$-module and each submodule of $M_1 \times M_2$ is of the form $N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. Next we show that, if N_1 is a $(\psi_1)_0$-2-absorbing semi-primary submodule of M_1, then $N_1 \times M_2$ is a ϕ-2-absorbing semi-primary submodule if $\{0\} \times M_2 \subseteq \psi_1 \times \psi_2(N_1 \times M_2)$. First, we would like to show that, N_1 is a ψ_1-2-absorbing semi-primary submodule of M_1 if $N_1 \times M_2$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2$.

Theorem 18. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *be a function with* $\phi = \psi_1 \times \psi_2$*. If* $N_1 \times M_2$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times M_2$*, then* N_1 *is a* ψ_1*-2-absorbing semi-primary submodule of* M_1*.*

Proof. Let $a_1, a_2 \in R_1$ and $m \in M_1$ such that $a_1a_2m \in N_1 - \psi_1(N_1)$. Then $(a_1, 0)(a_2, 0)(m, 0) \in N_1 \times M_2 - \psi_1(N_1) \times \psi_2(M_2)$. By Definition 1, $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m \in N_1$ or $a_2^n m \in N_1$.

Lemma 1. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *be a function. If* N_1 *is a* $(\psi_1)_0$*-2-absorbing semi-primary submodule of* M_1 *such that* $\{0\} \times M_2 \subseteq \psi_1 \times \psi_2(N_1 \times M_2)$*, then* $N_1 \times M_2$ *is a* $\psi_1 \times \psi_2$*-2-absorbing semi-primary submodule of* $M_1 \times M_2$*.*

Proof. Let $(a_1, b_1), (a_2, b_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ such that

$$(a_1, b_1)(a_2, b_2)(m_1, m_2) \in N_1 \times M_2 - \phi(N_1 \times M_2).$$

By assumption, $(a_1a_2m_1, b_1b_2m_2) \notin \{0\} \times M_2$. Clearly, $a_1a_2m_1 \in N_1 - (\psi_1)_0(N_1)$. By Definition 1, $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m_1 \in N_1$ or $a_2^n m_1 \in N_1$ for some positive integer n. Therefore $N_1 \times M_2$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2$.

Corollary 8. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. If N_2 is a $(\psi_2)_0$-2-absorbing semi-primary submodule of M_2 such that $M_1 \times \{0\} \subseteq \psi_1 \times \psi_2(M_1 \times N_2)$, then $M_1 \times N_2$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2$.*

Proof. Similar to the proof of Lemma 1.

Theorem 19. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \ldots \times \psi_k$ and $M_1 \times \ldots \times M_{i-1} \times \{0\} \times M_{i+1} \times \ldots \times M_k \subseteq \phi(M_1 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k)$. Then N_j is a $(\psi_j)_0$-2-absorbing semi-primary submodule of M_j if and only if $M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2 \times \ldots \times M_k$.*

Proof. Similar to the proof of Theorem 18 and Lemma 1.

Theorem 20. *Let $(\psi_i)_n : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi_n = (\psi_1)_n \times (\psi_2)_n$. If N_1 is a $(\psi_1)_0$-2-absorbing semi-primary submodule of M_1 such that $(\psi_2)_3(M_2) = M_2$, then $N_1 \times M_2$ is a ϕ_3-2-absorbing semi-primary submodule of $M_1 \times M_2$.*

Proof. If N_1 is a 2-absorbing semi-primary submodule of M_1, then $N_1 \times M_2$ is a 2-absorbing semi-primary submodule of $M_1 \times M_2$. Clearly, $N_1 \times M_2$ is a ϕ_3-2-absorbing semi-primary submodule of $M_1 \times M_2$. Assume that N_1 is not 2-absorbing semi-primary. By Corollary 6, $(\psi_1)_3 \leq (\psi_1)_0$ so $(N_1 : M_1)^2 N_1 = \{0\}$. Therefore $(\psi_1)_3 \times (\psi_2)_3(N_1 \times M_2) = (\psi_1)_3(N_1) \times (\psi_2)_3(M_2) = \{0\} \times M_2$. Now, by Lemma 1, $N_1 \times M_2$ is a $(\psi_1)_3 \times (\psi_2)_3$-2-absorbing semi-primary submodule of $M_1 \times M_2$.

Corollary 9. *Let $(\psi_i)_n : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi_n = (\psi_1)_n \times (\psi_2)_n$. If N_2 is a $(\psi_2)_0$-2-absorbing semi-primary submodule of M_2 such that $(\psi_1)_3(M_1) = M_1$, then $M_1 \times N_2$ is a ϕ_3-2-absorbing semi-primary submodule of $M_1 \times M_2$.*

Proof. Similar to the proof of Theorem 20.

Theorem 21. *Let $(\psi_i)_n : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi_n = (\psi_1)_n \times \ldots \times (\psi_k)_n$. If N_j is a $(\psi_j)_0$-2-absorbing semi-primary submodule of M_j such that $(\psi_j)_3(M_j) = M_j$, then $M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k$ is a ϕ_3-2-absorbing semi-primary submodule of $M_1 \times \ldots \times M_k$.*

Proof. Similar to the proof of Theorem 20 and Corollary 9.

Theorem 22. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. Then the following conditions are equivalent:*

(i) N_1 *is a 2-absorbing semi-primary submodule of* M_1*.*

(ii) $N_1 \times M_2$ *is a 2-absorbing semi-primary submodule of* $M_1 \times M_2$*.*

(iii) $N_1 \times M_2$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times M_2$*, where* $\psi_2(M_2) \neq M_2$*.*

Proof. $(1 \Rightarrow 2)$. Let $(a_1, b_1), (a_2, b_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ such that $(a_1, b_1)(a_2, b_2)(m_1, m_2) \in N_1 \times M_2$. Clearly, $a_1a_2m_1 \in N_1$. By Definition 2, $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m_1 \in N_1$ or $a_2^n m_1 \in N_1$ for some positive integer n. This completes the proof.

$(2 \Rightarrow 3)$. It is easy to see that every 2-absorbing primary submodule is ϕ-2-absorbing semi-primary.

$(3 \Rightarrow 1)$. Let $a_1, a_2 \in R_1$ and $m \in M_1$ such that $a_1a_2m \in N_1$. By assumption, there exists $m_2 \in M_2$ such that $m_2 \notin \psi_2(M_2)$. Since $\phi(N_1 \times M_2) \subseteq M_1 \times \psi_2(M_2)$, we have $(a_1, 1)(a_2, 1)(m, m_2) \in N_1 \times M_2 - \psi_1 \times \psi_2(N_1 \times M_2)$. By Definition 1, $a_1a_2 \in \sqrt{(N_1 : M_1)}$ or $a_1m \in N_1$ or $a_2^n m \in N_1$.

Corollary 10. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *be a function* $\phi = \psi_1 \times \psi_2$*. Then the following conditions are equivalent:*

(i) N_2 *is a 2-absorbing semi-primary submodule of* M_2*.*

(ii) $M_1 \times N_2$ *is a 2-absorbing semi-primary submodule of* $M_1 \times M_2$*.*

(iii) $M_1 \times N_2$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times M_2$*, where* $\psi_1(M_1) \neq M_1$*.*

Proof. Similar to the proof of Theorem 22.

Theorem 23. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *be a function with* $\phi = \psi_1 \times \ldots \times \psi_k$*. Then the following conditions are equivalent:*

(i) N_i *is a 2-absorbing semi-primary submodule of* M_i*.*

(ii) $M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k$ *is a 2-absorbing semi-primary submodule of* $M_1 \times \ldots \times M_2$*.*

(iii) $M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times \ldots \times M_2$ *with* $\psi_j(M_j) \neq M_j$*.*

Proof. Similar to the proof of Theorem 22 and Corollary 10.

Theorem 24. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *be a function with* $\psi_2(M_2) = M_2$ *and* $\phi = \psi_1 \times \psi_2$*. Then* $N_1 \times M_2$ *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times M_2$ *if and only if* N_1 *is a* ψ_1*-2-absorbing semi-primary submodule of* M_1*.*

Proof. The proof is clear.

Corollary 11. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\psi_1(M_1) = M_1$ and $\phi = \psi_1 \times \psi_2$. Then $M_1 \times N_2$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2$ if and only if N_2 is a ψ_2-2-absorbing semi-primary submodule of M_2.*

Proof. Similar to the proof of Theorem 24.

Theorem 25. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\psi_j(M_j) = M_j$ and $\phi = \psi_1 \times \ldots \times \psi_k$. Then $M_1 \times M_2 \times \ldots \times M_{i-1} \times N_i \times M_{i+1} \times \ldots \times M_k$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times \ldots \times M_k$ if and only if N_i is a ψ_i-2-absorbing semi-primary submodule of M_i.*

Proof. Similar to the proof of Theorem 24 and Corollary 11.

Theorem 26. *Let N_i be a proper submodule of M_i and let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\phi = \psi_1 \times \psi_2$. If $N_1 \times N_2$ is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2$, then*

(i) N_1 is a ψ_1-2-absorbing semi-primary submodule of M_1,

(ii) N_2 is a ψ_2-2-absorbing semi-primary submodule of M_2.

Proof. The proof is clear.

The next theorem gives conditions for a ϕ-2-absorbing semi-primary to be 2-absorbing semi-primary.

Theorem 27. *Let $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ be a function with $\psi_i(M_i) \neq M_i, \phi = \psi_1 \times \psi_2 \times \psi_3$. If N is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$, then $N = \phi(N)$ or N is a 2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$.*

Proof. Suppose that N is a ϕ-2-absorbing semi-primary submodule of $M_1 \times M_2 \times M_3$ that is not 2-absorbing semi-primary. Now suppose that $N_1 \times N_2 \times N_3 = N \neq \psi_1 \times \psi_2 \times \psi_3(N)$. Thus $N_i \neq \psi_i(N_i)$ for some $i = 1, 2, 3$. We may assume that $N_1 \neq \psi_1(N_1)$. There exists $m_1 \in N_1$ such that $m_1 \notin \psi_1(N_1)$. Assume that $N_2 \neq M_2$ and $N_3 \neq M_3$. Thus there exist $m_2 \in M_2$ and $m_3 \in M_3$ such that $m_2 \notin N_2$ and $m_3 \notin N_3$. Since $(1,0,1)(1,1,0)(m_1, m_2, m_3) \notin \psi_1 \times \psi_2 \times \psi_3(N_1 \times N_2 \times N_3)$, we have $(1,0,1)(1,1,0)(m_1, m_2, m_3) \in N - \phi(N)$. By Definition 1, $m_2 \in N_2$ or $m_3 \in N_3$, a contradiction. Therefore $N = N_1 \times M_2 \times N_3$ or $N = N_1 \times N_2 \times M_3$. If $N = N_1 \times M_2 \times N_3$, then $(0,1,0) \in (N : M_1 \times M_2 \times M_3)$. By Theorem 13, $\{0\} \times M_2 \times \{0\} = (0,1,0)^2 N \subseteq (N : N_1 \times M_2 \times N_3)^2 N = (\psi_1)_3 \times (\psi_2)_3 \times (\psi_3)_3(N) \subseteq \psi_1 \times \psi_2 \times \psi_3(N) = \psi_1(N_1) \times \psi_2(M_2) \times \psi_3(N_3)$, which is a contradiction. This completes the proof.

The above theorem shows the relationship between 2-absorbing semi-primary and ϕ-2-absorbing semi-primary submodules in $R_1 \times R_2 \times R_3$-modules. From the above theorem, we have the following corollary.

Corollary 12. *Let* $\psi_i : S(M_i) \to S(M_i) \cup \{\emptyset\}$ *be a function with* $\psi_i(M_i) \neq M_i, \phi = \psi_1 \times \psi_2 \times \psi_3$ *and* $N \neq \phi(N)$. *Then* N *is a* ϕ*-2-absorbing semi-primary submodule of* $M_1 \times M_2 \times M_3$ *if and only if* N *is a 2-absorbing semi-primary submodule of* $M_1 \times M_2 \times M_3$.

Proof. This follows from Theorem 27.

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