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On Finsler s-manifolds

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Abstract. Finsler $s-$manifolds are a generalization of Riemannian $s-$manifolds. An important property of such manifolds is the homogeneity. In this paper we study Finsler $s-$manifolds. We first construct some example of Finsler $s-$manifolds which are neither Riemannian nor symmetric. Then we consider symmetric preserving diffeomorphism of Finsler $s-$manifolds. Finally we give some algebraic and existence theorem of these spaces.

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Key Words and Phrases: Finsler $s-$manifold, Symmetric Finsler space, Generalized symmetric space, Homogeneous Finsler space.

1. Introduction

Finsler manifold is a generalization of the Riemannian one, in the same as Riemannian manifold is for the Euclidean. A metric depends on the point and the direction. A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces.

Let (M, F) be a Finsler space, where F is positively homogeneous of degree one. Then we have two ways to define the notion of an isometry of (M, F). On the one hand, we call a diffeomorphism σ of M onto itself an isometry if $F(d\sigma_x(y)) = F(y)$, for any $x \in M$ and $y \in T_xM$. On the other hand, we can also define an isometry of (M, F) to be a one-to-one mapping of M onto itself which preserves the distance of each pair of points of M. It is well known that the two definitions are equivalent if the metric F is Riemannian. The equivalence of these two definitions in the general Finsler case is a result of S. Deng and Z. Hou [2]. Using these result, they proved that the group of isometries $I(M, F)$ of a Finsler space (M, F) is a Lie transformation group of M and for any point $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$. These results are important to study homogenous and symmetric Finsler spaces, for example [3, 4, 5, 12, 13, 14].

Symmetric spaces and generalized symmetric spaces have appeared to be very rich in content, stimulating the research in Lie groups, Mechanics, Physics, Gravity etc.

The definition of symmetric Finsler space is a naturall generalization of E. Cartan's definition of Riemannian symmetric spaces [8]. We call a Finsler space (M, F) a symmetric

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Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p, [6, 5, 9, 12].
Affine and Riemannian $s-$manifold were first defined in [18] following the introduction of generalized Riemannian symmetric spaces in [19]. They form a more general class than the symmetric spaces [11]. An isometry of (M, F) with an isolated fixed point $x \in M$ is called a symmetry of (M, F) at x. A family $\{s_x | x \in M\}$ of symmetries of a connected Finsler space (M, F) is called an $s-$structure of (M, F), [7]. $\Sigma-$spaces and reduced $\Sigma-$spaces were first introduced by Loos as a generalization of reflection spaces and symmetric spaces [20]. He then proved that any $\Sigma-$space with compact Σ is a fibre bundle over a reduced $\Sigma-$space. Basic properties of any reduced $\Sigma-$space M and affine and Riemannian $\Sigma-$space and Finsler $\Sigma-$space was given in [15, 21].
In this paper we are concerned with properties of Finsler spaces admitting such an $s-$structure. We construct some example of Finsler $s-$manifolds which are neither Riemannian nor symmetric. Then we study symmetry preserving diffeomorphism of Finsler $s-$manifolds and show that the group of symmetry preserving diffeomorphism is a transitive group. We then study some existence theorems and consider some geometric properties of Finsler $s-$manifolds.

2. Preliminaries

Let M be an $n-$dimensional smooth manifold without boundary and TM denote its tangent bundle. A Finsler structure on M is a map $F : TM \longrightarrow [0, \infty)$ which has the following properties [1]:

(i) F is smooth on $\widetilde{TM} := TM \backslash \{0\}$.

(ii) $F(x, \lambda y) = \lambda F(x, y)$, for any $x \in M, y \in T_x M$ and $\lambda > 0$.

(iii) F^2 is strongly convex, i.e.,

$$\mathrm{g}_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$$

is positive definite for all $(x, y) \in \widetilde{TM}$.

Let $V = v^i \partial / \partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U} \subset M$. One can introduce a Riemannian metric g_V and a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as following [1] :

$$g_V(X, Y) = X^i Y^j g_{ij}(x, v), \qquad \forall X = X^i \frac{\partial}{\partial x^i}, Y = Y^i \frac{\partial}{\partial x^i},$$

$$\nabla^V_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x, v) \frac{\partial}{\partial x^k}.$$

From the torsion freeness and $g-$compatibility of Chern connection we have

$$\nabla^V_X Y - \nabla^V_Y X = [X, Y],$$

$$Xg_V(Y,Z) = g_V(\nabla^V_X Y, Z) + g_V(Y, \nabla^V_X Z) + 2C_V(\nabla^V_X V, Y, Z),$$

where C_V is the Cartan tensor defined by

$$C_V(X,Y,Z) = X^i Y^j Z^k C_{ijk}(x,v), \qquad C_{ijk}(x,v) = \frac{1}{4}\frac{\partial^3 F^2(x,v)}{\partial y^i \partial y^j \partial y^k},$$

and it satisfies

$$C_V(V,X,Y) = 0.$$

Given a nonzero vector field V on a Finsler manifold (M,F) with connection ∇^V, one can consider the curvature tensor R^V defined by

$$R^V(X,Y)Z = \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.$$

For a flag (V,σ) consisting of a nonzero tangent vector $V \in T_xM$ and a plane $\sigma \subset T_xM$ spanned by the tangent vectors V, W, the flag curvature is defined as

$$K(V,\sigma) = K(V,W) = \frac{g_V(R^V(V,W)W,V)}{g_V(V,V)g_V(W,W) - g_V(V,W)^2}.$$

In the Riemannian case the flag curvature is the sectional curvature of the plane σ and does not depend on V.
A diffeomorphism, $\varphi : M \longrightarrow M$, is an isometry on a Finsler manifold (M,F) if it preserves the Finsler function:

$$F(\varphi(x), d\varphi_x(X)) = F(x,X) \qquad \forall x \in M, X \in T_xM.$$

By the classical Dantzing-van der Waereden Theorem ([10]vol I, chapter I, Theorem 4.7) and the Montgomery- Zippin Theorem ([10],vol I, chapter I, Theorem 4.6), the group of isometries on a connected Finsler manifold form a Lie group. Strictly speaking, these theorems prove the statement for absolute homogeneous Finsler functions. For positive homogeneous Finsler functions consider the metric, d^*, defined by the function

$$F^*(X) = F(X) + F(-X)$$

Then the G is a closed subgroup of G^* defined for d^*. Thus both groups are Lie groups [22].

3. Finsler s-manifolds

Affine and Riemmannian $s-$manifolds were first defined in [18] following the introduction of generalized Riemannian spaces in [19]. They form a more general class than the symmetric spaces of E. Cartan.
Let (M,g) be a connected Riemannian manifold. A symmetry at $x \in M$ is a isometry of (M,g) for which x is an isolated fixed point. A $s-$structure on (M,g) is a family $\{s_x\}_{x\in M}$

such that s_x is a symmetry at $x \in M$, for each $x \in M$. An $s-$structure is called regular if for any two points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

If $\{s_x\}_{x \in M}$ is regular, then the map $s : M \longrightarrow I(M, g)$, $x \longrightarrow s_x$ is always C^∞, here $I(M, g)$ denotes the group of isometries of (M, g). An $s-$structure $\{s_x\}_{x \in M}$ is called of order k if $(s_x)^k = id_M$ for all $x \in M$ and k is the minimal number with this property. It is well known that if (M, g) admits an $s-$structure, then it always admits an $s-$structure of finite order. Further if (M, g) admits a regular $s-$structure, then (M, g) admits a regular $s-$structure of finite order [11]. In particular if (M, g) admits an $s-$structure of order two then it is a usual Riemannian symmetric space.
Let (M, F) be a Finsler space, where F is positively homogeneous but not necessarily absolutely homogeneous. We introduce isometries of (M, F) which form a Lie transformation group of M as a result of S. Deng and Z. Hou [2] and moreover for any point $x \in M$, the isotropic subgroup $I_x(M, F)$ is a compact subgroup of $I(M, F)$, the group of isometries, which can be used to study homogeneous and symmetric Finsler spaces. The definition of symmetric Finsler space is a natural generalization of E. Cartan's definition of Riemannian symmetric space [5], [6], [12]. We call a Finsler space (M, F) a symmetric Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p.
If we drop the involution property in the definition of symmetric Finsler space keeping the property

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y),$$

we get a bigger class of Finsler manifolds as symmetric Finsler spaces.

The definition of Finsler $s-$manifolds is a natural generalization of definition of Riemannian $s-$manifolds [7, 16].

Definition 1. *Let (M, F) be a connected Finsler space. An isometry on (M, F) with an isolated fixed point x will be called a symmetry at x, and will usually be written as s_x.*

Definition 2. *A family $\{s_x | x \in M\}$ of symmetries on a connected Finsler manifold (M, F) is called an $s-$structure on (M, F)*

An $s-$structure $\{s_x | x \in M\}$ is called of order k $(k \geq 2)$ if $(s_x)^k = id$ for all $x \in M$ and k is the least integer of this property. Obviously a Finsler space is symmetric if and only if it admits an $s-$structure of order 2. An $s-$structure $\{s_x | x \in M\}$ on (M, F) is called regular if for every pair of points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

Definition 3. *A Finsler $s-$manifold is a connected Finsler manifold (M, F) admitting a regular $s-$structure and a Finsler space (M, F) is said to be $k-$symmetric $(k \geq 2)$ if it admits a regular $s-$structure of order k.*

Here we construct some Finsler $s-$manifolds which are non-Riemannian and non-symmetric.

Example 1. *Let k be a constant $|k| < \frac{1}{\sqrt{3}}$. Consider the following Randers metric on R^3,*

$$F(p_1, p_2, p_3, y_1, y_2, y_3) = \sqrt{y_1^2 + y_2^2 + y_3^2} + k(y_1 + y_2 + y_3)$$

where $p = (p_1, p_2, p_3) \in R^3$ and $(y_1, y_2, y_3) \in T_pR^3$.
Define

$$s_p(x_1, x_2, x_3) = (x_3 - p_3 + p_1, x_1 - p_1 + p_2, x_2 - p_2 + p_3),$$

for any $p \in R^3$ we clearly see that s_p is an isometry of F such that p is an isolated fixed point of s_p.
It is evident that $s_p^3 = id$ and $s_p^2 \neq id$ and $s_p \neq id$ and

$$s_p \circ s_q(x) = s_z \circ s_p(x), \quad z = s_p(q).$$

So $\{s_p\}$ is a regular 3-structure on (R^3, F) $\square$

Example 2. *Let G be a compact connected Lie group. Consider the coset space $(G \times G)/G^*$, where G^* is the diagonal of $G \times G$. $(G \times G)/G^*$ is diffeomorphic to G via the map*

$$(g_1, g_g)G^* \longrightarrow g_1 g_2^{-1}$$

$G \times G$ acts on G by

$$(g_1, g_2)y = g_1 y g_2^{-1}.$$

The isotropy group at the origin $e \in G$ is G^*. Now define $\sigma : G \times G \longrightarrow G \times G$ by

$$\sigma(g_1, g_2) = (g_2, g_1),$$

which is an involute automorphism. The fixed point set is $(G \times G)^\sigma = G^*$ and σ induces the map $s : G \longrightarrow G$, $s(g) = g^{-1}$. Take a bi-invariant absolutely homogeneous Finsler metric F on G. Then F is invariant with respect to the action of $G \times G$ on G. It is also invariant with respect to s. Then (G, F) is a symmetric Finsler space [17]. We now consider the more general case of G^{k+1}/G^* where G^{k+1} is the direct product of G with itself (k+1) times, and G^* is the diagonal of G^{k+1}. We have

$$G^{k+1}/G^* \cong G^k$$

via $\pi : G^{k+1} \longrightarrow G^k$, where

$$\pi(g_1, ..., g_{k+1}) = (g_1 g_{k+1}^{-1}, ..., g_k g_{k+1}^{-1}).$$

Further define $\sigma : G^{k+1} \longrightarrow G^{k+1}$ by

$$\sigma(g_1, ..., g_{k+1}) = (g_{k+1}, g_1, ..., g_k).$$

Then σ is an automorphism of order $k+1$. It induces a map $s : G^k \longrightarrow G^k$ defined by

$$s(g_1, ..., g_k) = (g_k^{-1}, g_1 g_k^{-1}, ..., g_{k-1} g_k^{-1}).$$

Let F be a bi-invariant Finsler metric on G. Then F generates a bi-invariant Finsler metric F^{k+1} on G^{k+1} such that

$$(G^{k+1}, F^{k+1}) \cong (G, F) \times ... \times (G, F)$$

Then F^{k+1} induces a $G^{k+1}-$invariant Finsler metric $F[k]$ on G^k. The Finsler space $(G^k, F^{[k]})$ is a (k+1)-symmetric Finsler space. Similar to the Riemannian case $(G^k, F^{[k]})$ is not a symmetric space.□

Example 3. *Let $(G_1/H_1, g_1)$, $(G_2/H_2, g_2)$ be two Riemannian $s-$manifolds with H_1 and H_2 compact and $\{\tau_p\}$, $\{\sigma_q\}$ be $s-$structures on G_1/H_1 , G_2/H_2, respectively of order k. Let $M = G_1/H_1 \times G_2/H_2$ and o_1 , o_2 be the origins of G_1/H_1, G_2/H_2 respectively, and denote the origin of M by $o = (o_1, o_2)$. Now for*

$$y = y_1 + y_2 \in T_o M = T_{o_1}(G_1/H_1) + T_{o_2}(G_2/H_2),$$

we define

$$F(y) = \sqrt{g_1(y_1, y_1) + g_2(y_2, y_2) + \sqrt[s]{g_1(y_1, y_1)^s + g_2(y_2, y_2)^s}}$$

where s is any integer ≥ 2. Then $F(y)$ is a Minkowski norm on $T_o M$ which is invariant under $H_1 \times H_2$. Hence it defines a $G-$invariant Finsler metric on M. It is easy to see that Finsler manifold (M, F) is non-Riemannian $s-$manifold with regular $s-$structure $\{\tau_p \times \sigma_q\}$.□

Given an $s-$structure $\{s_x | x \in M\}$ on (M, F) we shall always denote by S the tensor field of type $(1, 1)$ defined by $S_x = (s_x)_*$ for all $x \in M$. Suppose there exists a nonzero vector $X \in T_x M$ such that $S_x X = X$. Since s_x is isometry, $s_x(\exp_x(tX))$, $|t| < \epsilon$ is a geodesic. Now $\exp_x(tX)$ and $s_x(\exp_x(tX))$ are two geodesics through x with the same initial vector X. Therefore, for any $|t| < \epsilon$ we have

$$s_x(\exp_x(tX)) = \exp_x(tX).$$

But this contradicts to assumption that x is an isolated fixed point of s_x. Therefore S_x has no non-zero invariant vector.

Theorem 1. *Let (M, F) be a Finsler $s-$manifold. Then we have*

(a) *For any $x \in M$, $S_x = (ds_x)_x$ has no invariant vector,*

(b) *(M, F) is homogeneous. That is, the group of isometries of (M, F), $I(M, F)$, acts transitively on M.*

(c) *(M, F) is forward complete.*

Proof: see [7].$\square$

Theorem 2. *Let (M, F) be a Finsler $s-$manifol with regular $s-$structure $\{s_x\}$. Then there is a unique connection $\widetilde{\nabla}$ on M such that*

(i) *$\widetilde{\nabla}$ is invariant under all s_x*

(ii) $\widetilde{\nabla} S = 0$

Proof: The proof is similar to the Riemannian case [11].$\square$
If the Finsler space (M, F) is of Berwald type, then $\widetilde{\nabla}$ is given by the formula

$$\widetilde{\nabla}_X Y = \nabla_X Y - (\nabla_{(I-S)^{-1}X} S)(S^{-1}Y)$$

where ∇ is the Chern connection of (M, F).

Definition 4. *Let (M, F) be a generalized symmetric Finsler space, and let $\{s_x\}$ be the regular $s-$structure of (M, F). Then a diffeomorphism $\phi : M \longrightarrow M$ is called symmetry preserving if $\phi(s_x(y)) = s_{\phi(x)}\phi(y)$ for all $x, y \in M$.*

Obviously, all symmetries s_x are symmetry preserving due to $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$. We denote the group of symmetry preserving diffeomorphism by $Aut(\{s_x\})$. Let us denote by $A(M)$ the Lie group of all affine transformations of M with respect to the connection $\widetilde{\nabla}$. Each symmetry preserving diffeomorphism is an affine transformation of $(M, \widetilde{\nabla})$, i.e.

$$Aut(M, \{s_x\}) \subset A(M).$$

Lemma 1. *An affine transformation $\phi \in A(M)$ is symmetry preserving if and only if it preserves the tensor field S. Consequently, $Aut(\{s_x\})$ is a closed subgroup of $A(M)$ and hence a Lie transformation group of M.*

Proof: Let $\phi \in A(M)$ be symmetry preserving transformation then for each $x \in M$, maps $\phi \circ s_x$, $s_{\phi(x)} \circ \phi$ coincide, so $(\phi \circ s_x)_{*x} = (s_{\phi(x)} \circ \phi)_{*x}$. Then ϕ preserves the tensor field S. On the other hand if $\phi \in A(M)$ preserves the tensor field S then for each $x \in M$, $(\phi \circ s_x)_{*x} = (s_{\phi(x)} \circ \phi)_{*x}$. Because $\phi \circ s_x$ and $s_{\phi(x)} \circ \phi$ are affine transformations, so $\phi \circ s_x = s_{\phi(x)} \circ \phi$ that is ϕ is symmetry preserving map.$\square$

In the following we show that the group $Aut(\{s_x\})$ of all symmetry preserving diffeomorphisms of (M, F) is a transitive Lie transformation group.

Theorem 3. *The Lie transformation group $Aut(\{s_x\})$ act transitively on M.*

Proof: Let $K \subset Aut(\{s_x\})$ be the transformation group of M generated algebraicaly by all the symmetries s_x, $x \in M$. Choose an origin $o \in M$. Let $K(o)$ be the orbit of o with respect to K. Consider the map $f(x) = s_x(p)$ where $p \in K(o)$ and $x \in M$. Clearly $f(p) = p$. For $v \in T_pM$ we have $f_{*p}(v) = (I_p - S_p)v$. Hence $f_{*p} = (I_p - S_p)$ is a non-singular transformation and f maps a neighborhood U of p diffeomorphically onto a neighborhood V of p. We get $V \subset K(o)$ and the orbit $K(o)$ is open. The union of all other orbits of K must be also open and hence $K(o)$ is closed. Consequently $K(o) = M$. □

Let V be a finite dimensional vector space and $T : V \longrightarrow V$ an endomorphism. Then there is a unique decomposition $V = V_{0T} + V_{1T}$ of V into $T-$invariant subspaces such that the restriction of T to V_{0T} is nilpotent and the restriction of T to V_{1T} is an automorphism.

Definition 5. *A regular homogeneous $s-$manifold is a triplet (G, H, σ), where G is a connected Lie group, H its closed subgroup and σ an automorphism of G such that*

(i) $G^{\circ}_{\sigma} \subset H \subset G_{\sigma}$ *where G_{σ} is the subgroup consisting of the fixed points of σ in G and G°_{σ} denotes the identity component of G_{σ}.*

(ii) *If T denotes the linear endomorphism $Id - \sigma_*$, then $\mathfrak{g}_{0T} = \mathfrak{h}$.*

Clearly if (G, H, σ) is a regular homogeneous $s-$manifold, then $\mathfrak{g}_{0T} = \mathfrak{h} = ker\ T$ and $\mathfrak{g}_{1T} = Im(T)$.

Let G be a connected Lie group and H its closed subgroup. Consider the homogeneous manifold G/H. Here $\pi : G \longrightarrow G/H$ will denote the canonical projection, and $o = \pi(H)$ the origin of G/H. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. Suppose that there is a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum of vector spaces) and $Ad(h)\mathfrak{m} = \mathfrak{m}$ for every $h \in H$. Then the homogeneous space G/H is said to be reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

Lemma 2. *Let (G, H, σ) be a regular homogeneous $s-$manifold, then the homogeneous space G/H is reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_{1T}$*

Proof: Let (G, H, σ) be a regular homogeneous $s-$manifold, since $H \subset G_{\sigma}$ we obtain $Ad(h) \circ \sigma_* = \sigma_* \circ Ad(h)$ for each $h \in H$. Hence $Ad(h)$ commutes with T on $\mathfrak{g}$ and

$$\begin{aligned} Ad(h)(\mathfrak{g}_{1T}) &= (Ad(h) \circ T)(\mathfrak{g}) \\ &= T(Ad(h)\mathfrak{g}) \\ &= T(\mathfrak{g}) = \mathfrak{g}_{1T}. \end{aligned}$$

□

Theorem 4. *Let G be a connected Lie group, H its closed subgroup and σ an automorphism of G such that*

(i) $(G_{\sigma})^{\circ} \subset H \subset G_{\sigma}$,

(ii) $\sigma^k = id$, *where k is the minimum number with this property,*

Then the triple (G, H, σ) is a regular homogeneous $s-$manifold of order k.

Proof: Let $T = id - \sigma_*$. We have to show that $\mathfrak{g}_{0T} = \mathfrak{h}$. Clearly $\mathfrak{h} = KerT$ and hence $\mathfrak{h} \subset \mathfrak{g}_{0T}$. Suppose now that there is $X \in \mathfrak{g}_{0T}$ such that X is not in $\mathfrak{h}$. Without loss of generality we assume that $TX \neq 0$, $T^2X = 0$. Then we get

$$\begin{aligned} \sigma_* X &= TX - \sigma_*^2 X \\ &= X - \sigma_*(X - \sigma_* X). \end{aligned}$$

Let $Z = \sigma_*(X - \sigma_* X) = TX$. So we have $\sigma_* X = X - Z$ and $\sigma_* Z = Z$. Hence by the induction we get

$$\begin{aligned} \sigma_*^2 X &= X - 2Z \\ &. \\ &. \\ &. \\ \sigma_*^k X &= X - kZ \end{aligned}$$

Now Since $\sigma_*^k X = X$, we get $Z = 0$, a contradiction. This completes the proof. □

Theorem 5. *Let (G, H, σ) be a regular homogeneous $s-$manifold, $\pi : G \longrightarrow G/H$ the canonical projection and let F be a $G-$invariant Finsler metric on G/H such that the transformation s of G/H determined by σ, i.e. $s \circ \pi = \pi \circ \sigma$ is metric preserving at the origin eH of G/H. Then G/H is a Finslerian $s-$manifold and the symmetry s_x is given by*

$$s_x = g \circ s \circ g^{-1} \qquad g \in G, x = \pi(g)$$

Proof: We will identify the elements of G with the corresponding transformations of $M = G/H$. Choose $g \in G$ and $x \in M$ then $x = \pi(g')$ for some $g' \in G$. Now,

$$\begin{aligned} (s \circ g \circ s^{-1})(x) &= (s \circ g \circ s^{-1} \circ \pi)(g') \\ &= (s \circ g \circ \pi)(\sigma^{-1}(g')) \\ &= (s \circ \pi)(g\sigma^{-1}(g')) \\ &= (\pi \circ \sigma)(g\sigma^{-1}(g')) \\ &= \pi(\sigma(g)g') = \sigma(g)[\pi(g')] = \sigma(g)(x). \end{aligned}$$

Hence we get

$$s \circ g \circ s^{-1} = \sigma(g) \qquad g \in G \tag{1}$$

So for $h \in H$ we obtain $s \circ h \circ s^{-1} = h$ and hence $h \circ s \circ h^{-1} = s$. Consequently the transformation $g \circ s \circ g^{-1}$ always depends only on $\pi(g)$ and

$$s_{\pi(g)} = g \circ s \circ g^{-1} \qquad g \in G$$

defines a family $\{s_x | x \in M\}$ of diffeomorphisms of M. We can also easily that $(x, y) \longrightarrow s_x(y)$ is differentiable. Further for $x \in M$, $x = \pi(g)$ we have $x = g(o)$ and hence

$$s_x(x) = (g \circ s \circ g^{-1})(x) = x,$$

because $s(o) = o$.
Now for $x, y \in M$ put $s_x = g \circ s \circ g^{-1}$, $s_y = g' \circ s \circ (g')^{-1}$, where $x = g(o)$ and $y = g'(o)$. Then

$$(g \circ s \circ g^{-1} \circ g' \circ s^{-1})(o) = s_x(g'(o)) = s_x(y),$$

on the other hand, (1) yields $g \circ s \circ g^{-1} \circ g' \circ s^{-1} = g\sigma(g^{-1}g')$. Thus, the map $g \circ s \circ g^{-1} \circ g' \circ s^{-1}$ coincides with the action of an element $g'' \in G$, $g''(o) = s_x(y)$. Now

$$\begin{aligned} s_x \circ s_y &= g \circ s \circ g^{-1} \circ g' \circ s \circ (g')^{-1} \\ &= g'' \circ s \circ (g'')^{-1} \circ g \circ s \circ g^{-1} \\ &= s_{s_x(y)} \circ s_x. \end{aligned}$$

It remains to prove that s_{x*} has no fixed vector except the null vector. If we identify $\mathfrak{g}$ with T_eG, then the projection $\pi_{*e} : T_eG \longrightarrow T_oM$ induces an isomorphism of $\mathfrak{g}_{1T}$ onto T_oM. From the relation $\pi_* \circ \sigma_* = s_* \circ \pi_*$ we can see that $\pi_* \circ T = (I_o - s_{*o}) \circ \pi_*$. Because T is an automorphism on $\mathfrak{g}_{1T}$, $I_o - s_{*o}$ is an automorphism of T_oM. From

$$s_{\pi(g)} = g \circ s \circ g^{-1}, \quad g \in G, x = \pi(g),$$

we obtain easily that $I_p - S_p$ is an automorphism of T_pM for each $p \in M$. Thus $\{s_x | x \in M\}$ is a regular $s-$structure on (M, F).□

Corollary 1. *Let (G, H, σ) be a regular homogeneous $s-$manifold of order k, with the $G-$invariant Finsler metric F on G/H such that the transformation s of G/H determined by $\pi \circ \sigma = s \circ \pi$ is metric preserving at the origin eH of G/H. Then G/H is a Finsler $s-$manifold of order k.*

Proof: It is a consequence of Theorem 4 and Theorem 5.□

Theorem 6. *Let (M, F) be a Finsler $s-$manifold and $o \in M$ a fixed point. Let G be the identity component of the symmetry preserving group $Aut(\{M, s_x\})$ and G_o the isotropy subgroup of G at o. Define a map $\sigma : G \longrightarrow Aut(M, \{s_x\})$ by the formula*

$$\sigma(g) = s_o \circ g \circ s_o^{-1} \quad g \in G.$$

Then σ is an automorphism of G, and (G, G_o, σ) is a regular homogeneous $s-$manifold. The symmetries s_x are given by the formula

$$s_{\pi(g)} = g \circ s_o \circ g^{-1}, \quad x = \pi(g)$$

and $M \simeq G/G_o$.

Proof: By Theorem 3 $Aut(M, \{s_x\})$ is transitive on M and M is connected. So G is also transitive on M. Obviously, the map σ given by

$$\sigma(g) = s_o \circ g \circ s_o^{-1} \quad g \in G$$

is an isomorphism and $\sigma(G) = G$. Let G_o be the isotropy group of G at o, then $M \simeq G/G_o$. Let $\pi : G \longrightarrow G/G_o \simeq M$ be the canonical projection. Then for any $g \in G$ we have

$$\begin{aligned}(\pi \circ \sigma)(g) &= \sigma(g)(o) \\ &= (s_o \circ g \circ s_o^{-1})(o) \\ &= s_o(g(o)) \\ &= (s_o \circ \pi)(g).\end{aligned}$$

Hence on G we have

$$\pi \circ \sigma = s_o \circ \pi. \tag{2}$$

Because $g \in G$ is a symmetry preserving diffeomorphism, we have

$$g \circ s_x = s_{g(x)} \circ g.$$

In particular, for each $h \in G$ we have

$$\begin{aligned}\sigma(h) &= s_o \circ h \circ s_o^{-1} \\ &= s_{h(o)} \circ h \circ s_o^{-1} \\ &= h,\end{aligned}$$

so $G_o \subset G^\sigma$.
Let $\mathfrak{g}, \mathfrak{g}_o, \mathfrak{g}^\sigma$ denote the Lie algebra of G, G_o, G^σ respectively from (1) we have $\pi_* \circ \sigma_* = s_o \circ \pi_*$ on $\mathfrak{g} = T_eG$. Let $T = I - \sigma_*$, hence on $\mathfrak{g}$ we have

$$\pi_* \circ T = (I - S_o) \circ \pi_*. \tag{3}$$

Consider the decomposition $\mathfrak{g} = \mathfrak{g}_{0T} + \mathfrak{g}_{1T}$. Clearly $\mathfrak{g}_o \subseteq \mathfrak{g}^\sigma \subseteq \mathfrak{g}_{0T}$. Now we show that $\mathfrak{g}_o = \mathfrak{g}^\sigma = \mathfrak{g}_{0T}$. Suppose that there is a vector $X \in \mathfrak{g}_{0T} - \mathfrak{g}_o$. Then $X' = \pi_*(X)$ is a non-zero vector of $T_o(M)$. On the other hand, $T^i(X) = 0$ for some i, and from (2) we obtain $(I - S_o)^i X' = 0$ for some i. Because $(I - S_0)$ is invertible, we get $X' = 0$, a contradiction. Hence $\mathfrak{g}_o = \mathfrak{g}^\sigma = \mathfrak{g}_{0T}$. Consequently (G, G_o, σ) is a regular homogeneous $s-$manifold. Because $g \in G$ is a symmetry preserving diffeomorphism we get

$$s_{\pi(g)} = s_{g(o)} = g \circ s_o \circ g^{-1}$$

□

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