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# Between closed and $\mathcal{I}_g$-closed sets

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**Abstract.** The concept of closed sets is a central object in general topology. In order to extend many of important properties of closed sets to a larger families, Norman Levine initiated the study of generalized closed sets. In this paper we introduce, via ideals, new generalizations of closed subsets, which are strong forms of the $\mathcal{I}_g$-closed sets, called $\rho\mathcal{I}_g$-closed sets and closed-$\mathcal{I}$ sets. We present some properties and applications of these new sets and compare the $\rho\mathcal{I}_g$-closed sets and the closed-$\mathcal{I}$ sets with the g-closed sets introduced by Levine. We show that $\mathcal{I}$-closed and closed-$\mathcal{I}$ are independent concepts, as well as $\mathcal{I}^*$-closed sets and closed-$\mathcal{I}$ concepts.

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## 1. Introduction and preliminaries

The g-closed sets, which is a extension of closed sets, was introduced by Levine and the $\mathcal{I}_g$-closed sets, which is a generalization of g-closed sets, was defined by Jafari-Rajesh, in terms of ideals. In this paper we introduce and study new intermediate concepts between closed and $\mathcal{I}_g$-closed sets, via ideals. We also present some applications of these new sets, related to compactness and normality.

An ideal $\mathcal{I}$ in a set $X$ is a subset of $\mathcal{P}(X)$, the power set of $X$, such that:

($i$) if $A \subseteq B \subseteq X$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$, and

($ii$) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.

Some simple and useful ideals in $X$ are:

($i$) $\mathcal{P}(A)$, where $A \subseteq X$,

($ii$) $\mathcal{I}_f(X)$, the ideal of all finite subsets of $X$, and

($iii$) $\mathcal{I}_c(X)$, the ideal of all countable subsets of $X$.

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If $(X, \tau)$ is a topological space and $\mathcal{I}$ is an ideal in $X$, then $(X, \tau, \mathcal{I})$ is called an *ideal space*. If $(X, \tau)$ is a topological space and $A \subseteq X$ then the closure and the interior of $A$ are denoted by $\overline{A}$ (or $adh_\tau(A)$) and $\overset{0}{A}$ (or $int_\tau(A)$), respectively. If $A$ and $B$ are subsets of the space $(X, \tau)$ and $A \cap \overline{B} = \varnothing = \overline{A} \cap B$ then $A$ and $B$ are called *separated*. If $A \subseteq \overline{\overset{0}{A}}$ then $A$ is said to be *pre-open* [6]. If $\overset{0}{\overline{A}} \subseteq A$ then $A$ is defined to be *pre-closed* [6]. It is clear that $A$ is pre-open if and only if $X \backslash A$ is pre-closed.

If $(X, \tau)$ is a topological space and $A \subseteq X$ then $A$ is said to be *g-closed* [5] if, for each $U \in \tau$, $A \subseteq U$ implies $\overline{A} \subseteq U$. An ideal space $(X, \tau, \mathcal{I})$ is defined to be *$\mathcal{I}$-normal* [1] if for every pair of disjoint closed subsets $F$ and $G$, there exist disjoint open sets $U$ and $V$ such that $F \backslash U \in \mathcal{I}$ and $G \backslash V \in \mathcal{I}$.

The symbol □ is used to indicate the end of a proof.

## 2. $\rho\mathcal{I}_g$-closed sets

The generalized closed sets via ideals, that we consider, are due to Jafari-Rajesh and these are extensions of the g-closed sets of Levine. In this section we define the $\rho\mathcal{I}_g$-closed sets, which is a new intermediate concept between closed and $\mathcal{I}_g$-closed sets. Some properties, characterizations and applications are presented.

If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$ then $A$ is defined to be *$\mathcal{I}_g$-closed* [2] if, for all $U \in \tau$, $A \subseteq U$ implies $\overline{A} \backslash U \in \mathcal{I}$. It is noted that closed $\to$ g-closed $\to$ $\mathcal{I}_g$-closed.

**Definition 2.1.** *If $(X, \tau, \mathcal{I})$ is an ideal topological space and $A \subseteq X$ then $A$ is said to be* $\rho\mathcal{I}_g$-closed *if for each $U \in \tau$, if $A \backslash U \in \mathcal{I}$ then $\overline{A} \backslash U \in \mathcal{I}$.*

It is clear that

$$\text{closed} \to \rho\mathcal{I}_g\text{-closed} \to \mathcal{I}_g\text{-closed}$$

The converse are not true, as we can see in the next example.

**Example 2.2.** (1) *If $\mathcal{U}$ is the usual topology in the set $\mathbb{R}$, then all $A \subseteq \mathbb{R}$ is $\rho\mathcal{I}_g$-closed in the ideal space $(\mathbb{R}, \mathcal{U}, \mathcal{I} = \mathcal{P}(\mathbb{R}))$, but $(0, 1)$ is not g-closed. Then $\rho\mathcal{I}_g$-closed$\nrightarrow$g-closed and so $\rho\mathcal{I}_g$-closed$\nrightarrow$closed.*

(2) *If $\mathcal{C} = \{\varnothing, \mathbb{R}\} \cup \{(r, \infty) : r \in \mathbb{R}\}$ then $\mathbb{Z}$ is not $\rho\mathcal{I}_g$-closed in the space $(\mathbb{R}, \mathcal{C}, \mathcal{I} = \mathcal{I}_c(\mathbb{R}))$, because $\mathbb{Z} \backslash (0, \infty) \in \mathcal{I}$ but $\overline{\mathbb{Z}} \backslash (0, \infty) = (-\infty, 0] \notin \mathcal{I}$. However $\mathbb{Z}$ is g-closed, and then $\mathbb{Z}$ is $\mathcal{I}_g$-closed. Thus g-closed$\nrightarrow$ $\rho\mathcal{I}_g$-closed and $\mathcal{I}_g$-closed $\nrightarrow$ $\rho\mathcal{I}_g$-closed.*

Observe that g-closed and $\rho\mathcal{I}_g$-closed are independent concepts.

An application of $\rho\mathcal{I}_g$-closed sets is shown in the next theorem.

If $(X, \tau, \mathcal{I})$ is an ideal space then a subset $A$ is said to be:

(1) *$\mathcal{I}$-compact* [8] if for each open cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of $A$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $A \backslash \bigcup_{\alpha \in \Lambda_0} V_\alpha \in \mathcal{I}$, and

(2) *$\rho\mathcal{I}$-compact* [9] if for each family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of $X$, if $A\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $A\setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$. The space $(X,\tau,\mathcal{I})$ is *$\mathcal{I}$-compact* if $X$ is $\mathcal{I}$-compact, and $(X,\tau,\mathcal{I})$ is *$\rho\mathcal{I}$-compact* if $X$ is $\rho\mathcal{I}$-compact.

**Theorem 2.3.** *If the ideal space $(X,\tau,\mathcal{I})$ is $\rho\mathcal{I}$-compact and $A \subseteq X$ we have that:*

(1) *If $A$ is closed then $A$ is $\rho\mathcal{I}$-compact.*

(2) *If $A$ is $\rho\mathcal{I}_g$-closed then $A$ is $\rho\mathcal{I}$-compact.*

(3) *If $A$ is $\mathcal{I}_g$-closed then $A$ is $\mathcal{I}$-compact.*

*Proof.*

(1) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets of $X$ such that $A\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, this is, $X\setminus \left[(X\setminus A) \cup \bigcup_{\alpha\in\Lambda} V_\alpha\right] \in \mathcal{I}$. There exists $\Lambda_0 \subseteq \Lambda$, finite, with $X\setminus \left[(X\setminus A) \cup \bigcup_{\alpha\in\Lambda_0} V_\alpha\right] \in \mathcal{I}$, this is, $A\setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

(2) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets of $X$ such that $A\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Since $A$ is $\rho\mathcal{I}_g$-closed we have that $\overline{A}\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Given that $\overline{A}$ is $\rho\mathcal{I}$-compact, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\overline{A}\setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$. Hence $A\setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

(3) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets of $X$ such that $A \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$. Given that $A$ is $\mathcal{I}_g$-closed we have that $\overline{A}\setminus \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. But $\overline{A}$ is $\rho\mathcal{I}$-compact, and so there exists $\Lambda_0 \subseteq \Lambda$, finite, with $\overline{A}\setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$. Thus $A\setminus \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$. □

We recall that an ideal space $(X,\tau,\mathcal{I})$ is said to be *$\mathcal{I}_g$-normal* if for every pair of disjoint g-closed subsets $F$ and $G$ of $X$, there exist disjoint open sets $U$ and $V$ such that $F\setminus U \in \mathcal{I}$ and $G\setminus V \in \mathcal{I}$.

Renukadevi-Sivaraj have shown that if $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-compact and $(X,\tau)$ is $T_2$ then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-normal. In contrast we have the following result.

**Theorem 2.4.** *If $(X,\tau,\mathcal{I})$ is $\rho\mathcal{I}$-compact and $(X,\tau)$ is $T_2$ then $(X,\tau,\mathcal{I})$ is $\mathcal{I}_g$-normal.*

*Proof.* Suppose that $F$ and $G$ are disjoint g-closed sets. It is noted that, by Theorem 2.3, $F$ and $G$ are $\mathcal{I}$-compact subsets of $X$. Let $g \in G$, arbitrary. For each $f \in F$ there are disjoint $U_f \in \tau$ and $V_f \in \tau$ such that $f \in U_f$ and $g \in V_f$. Given that $F \subseteq \bigcup_{f\in F} U_f$ and

$F$ is $\mathcal{I}$-compact, there exists $F_0 \subseteq F$, finite, with $F \backslash \bigcup_{f \in F_0} U_f \in \mathcal{I}$. Let $T_g = \bigcup_{f \in F_0} U_f$ and $W_g = \bigcap_{f \in F_0} V_f$.

It is noted that $T_g \cap W_g = \varnothing$ and $F \backslash T_g \in \mathcal{I}$.

Now, since $G \subseteq \bigcup_{g \in G} W_g$ and $G$ is $\mathcal{I}$-compact, there exists $G_0 \subseteq G$, finite, with $G \backslash \bigcup_{g \in G_0} W_g \in \mathcal{I}$.

If $V = \bigcup_{g \in G_0} W_g$ and $U = \bigcap_{g \in G_0} T_g$ then $U$ and $V$ are disjoint, $G \backslash V \in \mathcal{I}$ and $F \backslash U = \bigcup_{g \in G_0} (X \backslash T_g) \in \mathcal{I}$. □

If $\mathcal{I}$ is an ideal in $X$ and $B \subseteq X$, it is easy to see that the set $\mathcal{I}_B = \{I \cap B : I \in \mathcal{I}\}$ is an ideal in $B$.

**Theorem 2.5.** *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A \subseteq X$ and $B \subseteq X$ then:*

(1) *If $A$ and $B$ are $\rho\mathcal{I}_g$-closed then $A \cup B$ is $\rho\mathcal{I}_g$-closed.*

(2) *$A$ is $\rho\mathcal{I}_g$-closed if and only if, for each closed set $F$, if $F \backslash \left(\overline{A} \backslash A\right) \in \mathcal{I}$ then $F \in \mathcal{I}$.*

(3) *If $A \backslash B \in \mathcal{I}$, $\overline{B} \backslash \overline{A} \in \mathcal{I}$ and $A$ is $\rho\mathcal{I}_g$-closed then $B$ is $\rho\mathcal{I}_g$-closed.*

(4) *If $A \subseteq B \subseteq \overline{A}$ and $A$ is $\rho\mathcal{I}_g$-closed, then $B$ is $\rho\mathcal{I}_g$-closed.*

(5) *If $A$ is $\rho\mathcal{I}_g$-closed and $B$ is closed, then $A \cap B$ is $\rho\mathcal{I}_g$-closed.*

(6) *If $A \subseteq B$ and $A$ is $\rho\mathcal{I}_g$-closed in the space $(X, \tau, \mathcal{I})$, then $A$ is $\rho(\mathcal{I}_B)_g$-closed in the space $(B, \tau_B, \mathcal{I}_B)$, where $\tau_B = \{U \cap B : U \in \tau\}$.*

*Proof.*

(1) Suppose that $U \in \tau$ and $(A \cup B) \backslash U \in \mathcal{I}$. Then $A \backslash U \in \mathcal{I}$ and $B \backslash U \in \mathcal{I}$, and so $\overline{A} \backslash U \in \mathcal{I}$ and $\overline{B} \backslash U \in \mathcal{I}$. This implies that $\overline{A \cup B} \backslash U \in \mathcal{I}$.

(2) $(\rightarrow)$ Suppose that $A$ is $\rho\mathcal{I}_g$-closed, $F \subseteq X$ is closed and that $F \backslash \left(\overline{A} \backslash A\right) \in \mathcal{I}$, this is, $F \cap \left[\left(X \backslash \overline{A}\right) \cup A\right] \in \mathcal{I}$. Then $F \cap \left(X \backslash \overline{A}\right) \in \mathcal{I}$ and $A \backslash (X \backslash F) = F \cap A \in \mathcal{I}$. Since $A$ is $\rho\mathcal{I}_g$-closed we have that $\overline{A} \backslash (X \backslash F) \in \mathcal{I}$, this is $F \cap \overline{A} \in \mathcal{I}$. Thus $F = \left(F \cap \overline{A}\right) \cup \left[F \cap \left(X \backslash \overline{A}\right)\right] \in \mathcal{I}$.

$(\leftarrow)$ Let $U \in \tau$ with $A \backslash U \in \mathcal{I}$. Given that $A \backslash U = \left(\overline{A} \backslash U\right) \backslash \left(\overline{A} \backslash A\right)$ and $\overline{A} \backslash U$ is closed, the hypothesis implies that $\overline{A} \backslash U \in \mathcal{I}$.

(3) Suppose that $V \in \tau$ and $B \backslash V \in \mathcal{I}$. Since $A \backslash V \subseteq (A \backslash B) \cup (B \backslash V) \in \mathcal{I}$ then $A \backslash V \in \mathcal{I}$. Given that $A$ is $\rho\mathcal{I}_g$-closed we have that $\overline{A} \backslash V \in \mathcal{I}$. Hence $\left(\overline{A} \backslash V\right) \cup \left(\overline{B} \backslash \overline{A}\right) \in \mathcal{I}$. But $\overline{B} \backslash V \subseteq \left(\overline{A} \backslash V\right) \cup \left(\overline{B} \backslash \overline{A}\right)$ and so $\overline{B} \backslash V \in \mathcal{I}$.

(4) It is a consequence of (3).

(5) If $U \in \tau$ and $(A \cap B) \backslash U \in \mathcal{I}$, this is, $A \backslash [U \cup (X \backslash B)] \in \mathcal{I}$, then $\overline{A} \backslash [U \cup (X \backslash B)] \in \mathcal{I}$ because $A$ is $\rho\mathcal{I}_g$-closed. Thus $\left(\overline{A} \cap B\right) \backslash U \in \mathcal{I}$. Now, $\overline{A \cap B} \backslash U \subseteq \left(\overline{A} \cap \overline{B}\right) \backslash U = \left(\overline{A} \cap B\right) \backslash U$ and so $\overline{A \cap B} \backslash U \in \mathcal{I}$.

(6) Suppose that $V \in \tau_B$ and $A \backslash V = I_0 \in \mathcal{I}_B$. There are $U \in \tau$ and $I \in \mathcal{I}$ with $V = B \cap U$ and $I_0 = I \cap B$. Then $A \backslash V = A \backslash (B \cap U) = B \cap I$ and this implies that $A \backslash U \subseteq A \backslash (B \cap U) = B \cap I \subseteq I$. Thus $A \backslash U \in \mathcal{I}$. Since A is $\rho\mathcal{I}_g$-closed we have that $\overline{A} \backslash U \in \mathcal{I}$. This implies that $\left(\overline{A} \backslash U\right) \cap B \in \mathcal{I}_B$, this is, $\left(\overline{A} \cap B\right) \backslash U \in \mathcal{I}_B$, and finally $adh_{\tau_B}(A) \backslash V = adh_{\tau_B}(A) \backslash (U \cap B) = \left(B \cap \overline{A}\right) \backslash (U \cap B) = \left(B \cap \overline{A}\right) \backslash U \in \mathcal{I}_B$. □

**Example 2.6.** *Let $\mathcal{C} = \{\varnothing, \mathbb{R}\} \cup \{(r, \infty) : r \in \mathbb{R}\}, \mathcal{I} = \mathcal{I}_f(\mathbb{R})$, $A = 2\mathbb{Z}$ and $B = \{p \in \mathbb{Z} : |p| \text{ is a prime number}\}$. We have that, in the space $(\mathbb{R}, \mathcal{C}, \mathcal{I})$, $A$ and $B$ are $\rho\mathcal{I}_g$-closed sets, because if $U \in \mathcal{C}$ and $A \backslash U \in \mathcal{I}\,(\text{or } B \backslash U \in \mathcal{I})$ then $U = \mathbb{R}$ and so $\overline{A} \backslash U \in \mathcal{I}\left(\text{or } \overline{B} \backslash U \in \mathcal{I}\right)$. However $A \cap B$ is not $\rho\mathcal{I}_g$-closed since $A \cap B = \{-2, 2\}$, $(A \cap B) \backslash (0, \infty) \in \mathcal{I}$, but $\overline{A \cap B} \backslash (0, \infty) = (-\infty, 0] \notin \mathcal{I}$.*

The following result is due to Newcomb.

**Lemma 2.7.** *If $f : X \to Y$ is a function we have that:*

(1) *If $\mathcal{I}$ is an ideal in $X$, then $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal in $Y$.*

(2) *If $f$ is inyective and $\mathcal{J}$ is an ideal in $Y$, then the set $f^{-1}(\mathcal{J}) = \left\{f^{-1}(J) : J \in \mathcal{J}\right\}$ is an ideal in $X$.*

**Theorem 2.8.** (1) *If $f : (X, \tau) \to (Y, \beta)$ is a continuous, closed and inyective function, $\mathcal{I}$ is an ideal on $X$, $\mathcal{J} = f(\mathcal{I})$ and if $A \subseteq X$ is $\rho\mathcal{I}_g$-closed, then $f(A)$ is $\rho\mathcal{J}_g$-closed.*

(2) *If $f : (X, \tau) \to (Y, \beta)$ is a continuous, closed and inyective function, $\mathcal{I}$ is an ideal on $X$, $\mathcal{J} = \left\{V \subseteq Y : f^{-1}(V) \in \mathcal{I}\right\}$ and if $A \subseteq X$ is $\rho\mathcal{I}_g$-closed, then $f(A)$ is $\rho\mathcal{J}_g$-closed.*

(3) *If $f : (X, \tau) \to (Y, \beta)$ is a continuous, open and inyective function, $\mathcal{I}$ is an ideal on $X$, $\mathcal{J} = \left\{V \subseteq Y : f^{-1}(V) \in \mathcal{I}\right\}$ and if $B \subseteq Y$ is $\rho\mathcal{J}_g$-closed, then $f^{-1}(B)$ is $\rho\mathcal{I}_g$-closed.*

(4) *If $f : (X, \tau) \to (Y, \beta)$ is an inyective, continuous and closed function, $\mathcal{J}$ is an ideal in $Y$ and if $A$ is $\rho\left(f^{-1}(\mathcal{J})\right)_g$-closed, then $f(A)$ is $\rho\mathcal{J}_g$-closed.*

*Proof.*

(1) If $V \in \beta$ and $f(A) \backslash V \in \mathcal{J}$ then $A \backslash f^{-1}(V) \in \mathcal{I}$, because $f$ is inyective. Given that $A$ is $\rho\mathcal{I}_g$-closed we have that $\overline{A} \backslash f^{-1}(V) \in \mathcal{I}$, and so $f\left[\overline{A} \backslash f^{-1}(V)\right] \in f(\mathcal{I})$. But $f\left(\overline{A}\right) \backslash V \subseteq f\left(\overline{A}\right) \backslash f(f^{-1}(V)) \subseteq f[\overline{A} \backslash f^{-1}(V)]$. Moreover $\overline{f(A)} \subseteq f(\overline{A})$, since $f$ is closed. In consequence $\overline{f(A)} \backslash V \in f(\mathcal{I})$.

(2) It is similar to (1).

(3) If $U \in \tau$ and $f^{-1}(B)\backslash U \in \mathcal{I}$, this is, $f^{-1}\left[B\backslash f(U)\right] \in \mathcal{I}$, then $B\backslash f(U) \in \mathcal{J}$. Given that $B$ is $\rho\mathcal{J}_g$-closed we have that $\overline{B}\backslash f(U) \in \mathcal{J}$. In consequence $f^{-1}\left[\overline{B}\backslash f(U)\right] \in \mathcal{I}$, this is $f^{-1}\left(\overline{B}\right)\backslash U \in \mathcal{I}$. Since $f$ is continuous we have that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$, and so $\overline{f^{-1}(B)}\backslash U \in \mathcal{I}$.

(4) If $W \in \beta$ and $f(A)\backslash W \in \mathcal{J}$ then $A\backslash f^{-1}(W) = f^{-1}\left[f(A)\backslash W\right] \in f^{-1}(\mathcal{J})$. Since $A$ is $\rho\left(f^{-1}(\mathcal{J})\right)_g$-closed there is $J \in \mathcal{J}$ with $\overline{A}\backslash f^{-1}(W) = f^{-1}(J)$. But $\overline{f(A)}\backslash W \subseteq f\left(\overline{A}\right)\backslash W \subseteq f\left(\overline{A}\right)\backslash f\left[f^{-1}(W)\right] \subseteq f\left[\overline{A}\backslash f^{-1}(W)\right] = f\left(f^{-1}(J)\right) \subseteq J$, and so $\overline{f(A)}\backslash W \in \mathcal{J}$. □

**Definition 2.9.** *If $(X,\tau,\mathcal{I})$ is an ideal topological space and $A \subseteq X$ then $A$ is said to be* $\rho\mathcal{I}_g$-open *if $X\backslash A$ is $\rho\mathcal{I}_g$-closed.*

The following result is a consequence of Theorem 2.5.

**Theorem 2.10.** *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $A \subseteq X$ and $B \subseteq X$ then:*

(1) *If $A$ and $B$ are $\rho\mathcal{I}_g$-open then $A \cap B$ is $\rho\mathcal{I}_g$-open.*

(2) *$A$ is $\rho\mathcal{I}_g$-open if and only if, for each closed set $F$, if $F\backslash\left(A\backslash\overset{0}{A}\right) \in \mathcal{I}$ then $F \in \mathcal{I}$.*

(3) *If $B\backslash A \in \mathcal{I}$, $\overset{0}{A}\backslash\overset{0}{B} \in \mathcal{I}$ and $A$ is $\rho\mathcal{I}_g$-open then $B$ is $\rho\mathcal{I}_g$-open.*

(4) *If $\overset{0}{A} \subseteq B \subseteq A$ and $A$ is $\rho\mathcal{I}_g$-open, then $B$ is $\rho\mathcal{I}_g$-open.*

(5) *If $A$ is $\rho\mathcal{I}_g$-open and $B$ is open, then $A \cup B$ is $\rho\mathcal{I}_g$-open.*

Next we present other useful properties of $\rho\mathcal{I}_g$-open sets.

**Theorem 2.11.** *If $(X,\tau,\mathcal{I})$ is an ideal space then $A \subseteq X$ is $\rho\mathcal{I}_g$-open if and only if, for each $F \subseteq X$, closed, if $F\backslash A \in \mathcal{I}$ then $F\backslash\overset{0}{A} \in \mathcal{I}$.*

*Proof.* $(\rightarrow)$ Suppose that $F \subseteq X$ is closed and that $F\backslash A \in \mathcal{I}$, this is, $(X\backslash A)\backslash(X\backslash F) \in \mathcal{I}$. Given that $X\backslash A$ is $\rho\mathcal{I}_g$-closed we have that $\overline{X\backslash A}\backslash(X\backslash F) \in \mathcal{I}$ or, equivalently, $F\backslash\overset{0}{A} \in \mathcal{I}$.

$(\leftarrow)$ Suppose that $V \in \tau$ and $(X\backslash A)\backslash V \in \mathcal{I}$, this is, $(X\backslash V)\backslash A \in \mathcal{I}$. The hypothesis implies that $(X\backslash V)\backslash\overset{0}{A} \in \mathcal{I}$, or equivalently, $\left(X\backslash\overset{0}{A}\right)\backslash V \in \mathcal{I}$. Hence $\overline{X\backslash A}\backslash V \in \mathcal{I}$ and so $X\backslash A$ is $\rho\mathcal{I}_g$-closed. □

**Theorem 2.12.** *If $(X,\tau,\mathcal{I})$ is an ideal space, then $A \subseteq X$ is $\rho\mathcal{I}_g$-closed if and only if $\overline{A}\backslash A$ is $\rho\mathcal{I}_g$-open.*

*Proof.* $(\rightarrow)$ Suppose that $F \subseteq X$ is closed and that $F \backslash \left(\overline{A} \backslash A\right) \in \mathcal{I}$. By the Theorem 2.5 we have that $F \in \mathcal{I}$, and so $F \backslash int\left(\overline{A} \backslash A\right) \in \mathcal{I}$, because $int\left(\overline{A} \backslash A\right) = \varnothing$. Thus $\overline{A} \backslash A$ is $\rho\mathcal{I}_g$-open.

$(\leftarrow)$ Suppose that $U \in \tau$ and that $A \backslash U \in \mathcal{I}$. Given that $\left(\overline{A} \backslash U\right) \backslash \left(\overline{A} \backslash A\right) = A \backslash U \in \mathcal{I}$ and $\overline{A} \backslash A$ is $\rho\mathcal{I}_g$-open, the Theorem 2.11 implies $\left(\overline{A} \backslash U\right) \backslash int\left(\overline{A} \backslash A\right) \in \mathcal{I}$, this is, $\overline{A} \backslash U \in \mathcal{I}$. □

**Theorem 2.13.** *If $A$ and $B$ are $\rho\mathcal{I}_g$-open subsets of an ideal space $(X, \tau, \mathcal{I})$, such that $\overline{A} \cap B \in \mathcal{I}$ and $A \cap \overline{B} \in \mathcal{I}$, then $A \cup B$ is $\rho\mathcal{I}_g$-open.*

*Proof.* Suppose that $F \subseteq X$ is closed and that $F \backslash (A \cup B) \in \mathcal{I}$. We have that:

(a) $F \backslash \overline{A \cup B} \in \mathcal{I}$.

(b) $\left(F \cap \overline{A}\right) \backslash A \in \mathcal{I}$, because $\left(F \cap \overline{A}\right) \backslash A \subseteq \left(\overline{A} \cap B\right) \cup [F \backslash (A \cup B)] \in \mathcal{I}$.

(c) $\left(F \cap \overline{B}\right) \backslash B \in \mathcal{I}$.

(d) $\left(F \cap \overline{A}\right) \backslash \overset{0}{A} \in \mathcal{I}$, because $F \cap \overline{A}$ is closed and $A$ is $\rho\mathcal{I}_g$-open.

(e) $\left(F \cap \overline{B}\right) \backslash \overset{0}{B} \in \mathcal{I}$.

(f) $\left[F \cap \overline{A \cup B}\right] \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right) \in \mathcal{I}$. In fact, given that

$\left[\left(F \cap \overline{A}\right) \backslash \overset{0}{A}\right] \cup \left[\left(F \cap \overline{B}\right) \backslash \overset{0}{B}\right] \in \mathcal{I}$ and $\left(\overline{A} \cup \overline{B}\right) \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right) \subseteq \left(\overline{A} \backslash \overset{0}{A}\right) \cup \left(\overline{B} \backslash \overset{0}{B}\right)$, we have that

$\left[F \cap \overline{A \cup B}\right] \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right) = F \cap \left[\left(\overline{A} \cup \overline{B}\right) \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right)\right] \subseteq F \cap \left[\left(\overline{A} \backslash \overset{0}{A}\right) \cup \left(\overline{B} \backslash \overset{0}{B}\right)\right] =$

$\left[\left(F \cap \overline{A}\right) \backslash \overset{0}{A}\right] \cup \left[\left(F \cap \overline{B}\right) \backslash \overset{0}{B}\right]$, and so $\left[F \cap \overline{A \cup B}\right] \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right) \in \mathcal{I}$.

(g) $F \backslash (A \overset{0}{\cup} B) \in \mathcal{I}$, because

$F \backslash (A \overset{0}{\cup} B) \subseteq F \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right) \subseteq \left[\left(F \cap \overline{A \cup B}\right) \backslash \left(\overset{0}{A} \cup \overset{0}{B}\right)\right] \cup \left(F \backslash \overline{A \cup B}\right) \in \mathcal{I}$.

Therefore $A \cup B$ is $\rho\mathcal{I}_g$-open. □

**Corollary 2.14.** (1) *If $A$ and $B$ are separated $\rho\mathcal{I}_g$-open subsets of an ideal space $(X, \tau, \mathcal{I})$ then $A \cup B$ is $\rho\mathcal{I}_g$-open.*

(2) *If $A$ and $B$ are $\rho\mathcal{I}_g$-closed subsets of an ideal space $(X, \tau, \mathcal{I})$, such that $X \backslash \left(\overset{0}{A} \cup B\right) \in \mathcal{I}$ and $X \backslash \left(A \cup \overset{0}{B}\right) \in \mathcal{I}$, then $A \cap B$ is $\rho\mathcal{I}_g$-closed.*

We end this section with an application to $\rho\mathcal{I}_g$-open sets to $\mathcal{I}$-normality.

**Theorem 2.15.** *The ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-normal if and only if, for each pair of disjoint closed sets $F$ and $G$, there are disjoint $\rho\mathcal{I}_g$-open sets $A$ and $B$ such that $F \backslash A \in \mathcal{I}$ and $G \backslash B \in \mathcal{I}$.*

*Proof.* $(\rightarrow)$ This is simple because open $\rightarrow$ $\rho\mathcal{I}_g$-open.

$(\leftarrow)$ If $F$ and $G$ are disjoint closed sets, then there exist disjoint $\rho\mathcal{I}_g$-open sets $A$ and $B$ with $F \backslash A \in \mathcal{I}$ and $G \backslash B \in \mathcal{I}$. The Theorem 2.11 implies that $F \backslash \overset{0}{A} \in \mathcal{I}$ and $G \backslash \overset{0}{B} \in \mathcal{I}$. Moreover $\overset{0}{A}$ and $\overset{0}{B}$ are disjoint open sets. □

## 3. Closed-$\mathcal{I}$ sets

In this section we introduce the closed-$\mathcal{I}$ sets, an intermediate concept between closed sets and $\rho\mathcal{I}_g$-closed sets. We also consider some applications of these sets.

Given an ideal space $(X, \tau, \mathcal{I})$ and a set $A \subseteq X$, we denote by

$$A^* (\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}, \text{ for every } U \in \tau \text{ with } x \in U\},$$

written simply as $A^*$ when there is no chance for confusion. It is clear that $A^* \subseteq \overline{A}$. A Kuratowski closure operator for a topology $\tau^* (\mathcal{I})$, finer than $\tau$, is defined by $Cl^* (A) = A \cup A^*$, for all $A \subseteq X$. When there is no chance for confusion $\tau^* (\mathcal{I})$ is denoted by $\tau^*$. The topology $\tau^*$ has as a base $\beta (\tau, \mathcal{I}) = \{V \backslash I : V \in \tau \text{ and } I \in \mathcal{I}\}$ [12] . In 1990, D. Jancovic and T. R. Hamlett introduced the notion of $\mathcal{I}$-open sets. If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, $A$ is said to be *$\mathcal{I}$-open* [3] if $A \subseteq int (A^*)$. $A$ is said to be *$\mathcal{I}$-closed* if $X \backslash A$ is $\mathcal{I}$-open. In 1992, D. Jancovic and T. R. Hamlett introduced the notion of $\mathcal{I}^*$-open sets. If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, $A$ is said to be *$\mathcal{I}^*$-closed* [4] if $A^* \subseteq A$ or, equivalently, if A is closed in $(X, \tau^*)$. $A$ is said to be *$\mathcal{I}^*$-open* if $X \backslash A$ is $\mathcal{I}^*$-closed.

**Definition 3.1.** *If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, then $A$ is said to be* closed-$\mathcal{I}$ *if $\overline{A} \backslash A \in \mathcal{I}$. A subset $B$ is defined to be* open-$\mathcal{I}$ *if $X \backslash B$ is closed-$\mathcal{I}$.*

It is observed that:

(1) closed $\rightarrow$ closed-$\mathcal{I}$.

(2) $A$ is open-$\mathcal{I}$ if and only if $A \backslash \overset{0}{A} \in \mathcal{I}$.

(3) $A$ is closed-$\mathcal{I}$ and open-$\mathcal{I}$ if and only if $Fr(A) \in \mathcal{I}$, where $Fr(A)$ is the frontier of $A$.

(4) $A$ is closed-$\mathcal{I}$ if and only if $\overline{A} \backslash A$ is open-$\mathcal{I}$.

(5) Each $I \in \mathcal{I}$ is open-$\mathcal{I}$.

(6) If $A$ is open then $A$ is open-$\mathcal{I}$.

**Example 3.2.** (1) *If* $\mathcal{U}$ *is the usual topology in* $\mathbb{R}$ *and if* $\mathcal{I} = \mathcal{I}_f(\mathbb{R})$*, then* $[0,1)$ *is closed-*$\mathcal{I}$ *but* $[0,1)$ *is not closed. Since* $\overline{\mathbb{Q}} \backslash \mathbb{Q} \notin \mathcal{I}$ *then* $\mathbb{Q}$ *is not closed-*$\mathcal{I}$*.*

(2) *If* $X = \{a,b,c,d\}$*,* $\tau = \{\varnothing, X, \{c\}, \{a,b\}, \{a,b,c\}\}$ *and* $\mathcal{I} = \{\varnothing, \{a\}\}$*, then the set* $A = \{b,c,d\}$ *is* $\mathcal{I}$*-open [7]. Now, since* $A \backslash \overset{0}{A} = \{b,d\} \notin \mathcal{I}$ *then* $A$ *is not open-*$\mathcal{I}$*. It is noted that* $\{a,c\} \notin \tau$ *but* $\{a,c\}$ *is open-*$\mathcal{I}$*. Moreover, since* $\overline{A} \backslash A = \{a\} \in \mathcal{I}$ *and* $A^* = \{a,b,d\} \nsubseteq A$ *then* $A$ *is closed-*$\mathcal{I}$ *but* $A$ *is not* $\mathcal{I}^*$*-closed. Now, if* $B = \{a\}$ *then* $\overline{B} \backslash B = \{b,d\} \notin \mathcal{I}$ *and* $B^* = \varnothing \subseteq B$*. Then* $B$ *is* $\mathcal{I}^*$*-closed but* $B$ *is not closed-*$\mathcal{I}$*.*

(3) *If* $\mathcal{I} = \{\varnothing, \{c\}, \{d\}, \{c,d\}\}$ *and* $\tau = \{\varnothing, X, \{d\}, \{a,c\}, \{a,c,d\}\}$*, where* $X = \{a,b,c,d\}$*, then the set* $A = \{a,c,d\}$ *is open-*$\mathcal{I}$*. However* $A$ *is not* $\mathcal{I}$*-open [7]. Hence, in general, open* $\nrightarrow$ $\mathcal{I}$*-open.*

In consequence the $\mathcal{I}$-closed and closed-$\mathcal{I}$ are independent concepts, as well as $\mathcal{I}^*$-closed and closed-$\mathcal{I}$ concepts.

**Theorem 3.3.** *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. If* $A \subseteq X$ *and* $B \subseteq X$ *then:*

(1) *If* $A$ *is closed-*$\mathcal{I}$ *then* $A$ *is* $\rho\mathcal{I}_g$*-closed.*

(2) *If* $A$ *is closed-*$\mathcal{I}$ *then* $\left(\overline{A}\right)^* \backslash A \in \mathcal{I}$*, and so* $\left(\overset{0}{A}\right)^* \backslash A \in \mathcal{I}$*.*

(3) *If* $A$ *and* $B$ *are closed-*$\mathcal{I}$ *then* $A \cup B$ *and* $A \cap B$ *are closed-*$\mathcal{I}$*.*

(4) *If* $A \backslash B \in \mathcal{I}$*,* $\overline{B} \backslash \overline{A} \in \mathcal{I}$ *and* $A$ *is closed-*$\mathcal{I}$ *then* $B$ *is closed-*$\mathcal{I}$*.*

(5) *If* $A \subseteq B \subseteq \overline{A}$ *and* $A$ *is closed-*$\mathcal{I}$*, then* $B$ *is closed-*$\mathcal{I}$*.*

(6) *If* $A$ *is pre-open and closed-*$\mathcal{I}$ *then* $\overline{A}$ *is open-*$\mathcal{I}$*.*

(7) *If* $A \subseteq B$ *and* $A$ *is closed-*$\mathcal{I}$ *in* $(X, \tau, \mathcal{I})$*, then* $A$ *is closed-*$\mathcal{I}_B$ *in* $(B, \tau_B, \mathcal{I}_B)$*.*

(8) *If* $A \subseteq B$*,* $A$ *is closed-*$\mathcal{I}_B$ *in* $(B, \tau_B, \mathcal{I}_B)$*,* $B$ *is closed-*$\mathcal{I}$ *in* $(X, \tau, \mathcal{I})$*, then* $A$ *is closed-*$\mathcal{I}$ *in* $(X, \tau, \mathcal{I})$*.*

*Proof.*

(1) Suppose that $U \in \tau$ and $A \backslash U \in \mathcal{I}$. Given that $\overline{A} \backslash U \subseteq \left(\overline{A} \backslash A\right) \cup (A \backslash U) \in \mathcal{I}$, we have that $\overline{A} \backslash U \in \mathcal{I}$.

(2) Since $\overline{A}$ is closed then $\left(\overline{A}\right)^* \subseteq \overline{A}$, and so $\left(\overset{0}{A}\right)^* \backslash A \subseteq \left(\overline{A}\right)^* \backslash A \subseteq \overline{A} \backslash A \in \mathcal{I}$.

(3) It is enough to note that $\overline{A \cup B} \backslash (A \cup B) = \left(\overline{A} \cup \overline{B}\right) \backslash (A \cup B) \subseteq \left(\overline{A} \backslash A\right) \cup \left(\overline{B} \backslash B\right) \in \mathcal{I}$, and that $\overline{A \cap B} \backslash (A \cap B) \subseteq \left(\overline{A} \cap \overline{B}\right) \backslash (A \cap B) \subseteq \left(\overline{A} \backslash A\right) \cup \left(\overline{B} \backslash B\right) \in \mathcal{I}$.

(4) Since $\overline{B} \backslash B \subseteq \left(\overline{A} \backslash A\right) \cup \left(\overline{B} \backslash \overline{A}\right) \cup (A \backslash B) \in \mathcal{I}$, we have that $\overline{B} \backslash B \in \mathcal{I}$.

(5) It is a consequence of (4).

(6) By hypothesis, $\overline{A}\backslash\overset{0}{\overline{A}} \subseteq \overline{A}\backslash A \in \mathcal{I}$.

(7) Given that $adh_{\tau_B}(A)\backslash A = \left(\overline{A}\cap B\right)\backslash A = \left(\overline{A}\backslash A\right)\cap B \in \mathcal{I}_B$, then $adh_{\tau_B}(A)\backslash A \in \mathcal{I}_B$.

(8) We have that $\overline{B}\backslash B \in \mathcal{I}$ and $adh_{\tau_B}(A)\backslash A \in \mathcal{I}_B \subseteq \mathcal{I}$. Now, $adh_{\tau_B}(A) = \overline{A}\cap B$ and $\overline{A}\backslash A \subseteq \left[\left(\overline{A}\cap B\right)\backslash A\right] \cup \left(\overline{B}\backslash B\right) \in \mathcal{I}$. □

**Example 3.4.** (1) *In the space* $(\mathbb{R}, \mathcal{C}, \mathcal{I})$ *of Example 2.2,* $\mathbb{Z}$ *is g-closed but* $\mathbb{Z}$ *is not closed-*$\mathcal{I}$*, because* $\mathbb{Z}$ *is not* $\rho\mathcal{I}_g$*-closed.*

(2) *If* $\mathcal{C} = \{\varnothing, \mathbb{R}\} \cup \{(r,\infty) : r \in \mathbb{R}\}$ *and* $\mathcal{I} = \mathcal{P}((0,\infty))$*, then the set* $A = (-\infty, 0)$ *is* $\rho\mathcal{I}_g$*-closed in the space* $(\mathbb{R}, \mathcal{C}, \mathcal{I})$ *because if* $U \in \mathcal{C}$ *and* $A\backslash U \in \mathcal{I}$ *then* $U = \mathbb{R}$*, and so* $\overline{A}\backslash U \in \mathcal{I}$*. However, since* $\overline{A}\backslash A = \{0\} \notin \mathcal{I}$*, we have that* $A$ *is not closed-*$\mathcal{I}$*.*

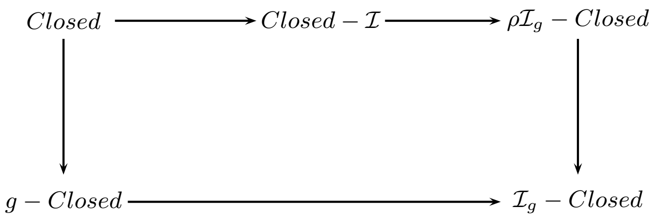
*Thus, in general,* $\rho\mathcal{I}_g$*-closed* $\nrightarrow$*closed-*$\mathcal{I}$*.*

(3) *If* $\mathcal{U}$ *is the usual topology in* $\mathbb{R}$ *and* $\mathcal{I} = \mathcal{P}(\{0,1\})$*, then the set* $A = (0,1)$ *is not g-closed. However, given that* $\overline{A}\backslash A = \{0,1\} \in \mathcal{I}$*, we conclude that* $(0,1)$ *is closed-*$\mathcal{I}$*.*

*So, in general, closed-*$\mathcal{I}$ $\nrightarrow$*g-closed.*

*Thus, closed-*$\mathcal{I}$ *and g-closed are independent concepts.*

We have the following diagram.




In the Theorem 3.5 we review the behavior of closed-$\mathcal{I}$ sets under continuous or closed functions.

**Theorem 3.5.** (1) *If* $(Y, \beta, \mathcal{J})$ *is an ideal space,* $f : (X,\tau) \to (Y,\beta)$ *is a continuous and inyective function and* $B$ *is closed-*$\mathcal{J}$*, then* $f^{-1}(B)$ *is closed-*$f^{-1}(\mathcal{J})$*.*

(2) *If* $(X, \tau, \mathcal{I})$ *is an ideal space,* $f : (X,\tau) \to (Y,\beta)$ *is a closed function and* $A$ *is closed-*$\mathcal{I}$*, then* $f(A)$ *is closed-*$f(\mathcal{I})$*.*

(3) *If* $(X, \tau, \mathcal{I})$ *is an ideal space,* $f : (X,\tau) \to (Y,\beta)$ *is a continuous function,* $\mathcal{J} = \left\{D \subseteq Y : f^{-1}(D) \in \mathcal{I}\right\}$ *and* $B$ *is closed-*$\mathcal{J}$*, then* $f^{-1}(B)$ *is closed-*$\mathcal{I}$*.*

(4) *If* $f : (X,\tau) \to (Y,\beta)$ *is an inyective and closed function,* $\mathcal{J}$ *is an ideal in* $Y$ *and if* $A$ *is closed-*$f^{-1}(\mathcal{J})$*, then* $f(A)$ *is closed-*$\mathcal{J}$*.*

*Proof.*

(1) We have that $\overline{f^{-1}(B)}\backslash f^{-1}(B) \subseteq f^{-1}\left(\overline{B}\right)\backslash f^{-1}(B) = f^{-1}\left(\overline{B}\backslash B\right) \in f^{-1}(\mathcal{J})$, given that $\overline{B}\backslash B \in \mathcal{J}$.

(2) Since $\overline{f(A)}\backslash f(A) \subseteq f\left(\overline{A}\right)\backslash f(A) \subseteq f\left(\overline{A}\backslash A\right) \in f(\mathcal{I})$, then $\overline{f(A)}\backslash f(A) \in f(\mathcal{I})$.

(3) Given that $\overline{B}\backslash B \in \mathcal{J}$ then $f^{-1}\left(\overline{B}\right)\backslash f^{-1}(B) = f^{-1}\left(\overline{B}\backslash B\right) \in \mathcal{I}$. But $\overline{f^{-1}(B)}\backslash f^{-1}(B) \subseteq f^{-1}\left(\overline{B}\right)\backslash f^{-1}(B)$ and so $\overline{f^{-1}(B)}\backslash f^{-1}(B) \in \mathcal{I}$.

(4) There is $J \in \mathcal{J}$ such that $\overline{A}\backslash A = f^{-1}(J)$, and so $\overline{f(A)}\backslash f(A) \subseteq f\left(\overline{A}\right)\backslash f(A) \subseteq f\left(\overline{A}\backslash A\right) = f\left(f^{-1}(J)\right) \subseteq J$. Hence $\overline{f(A)}\backslash f(A) \in \mathcal{J}$. □

The following theorem is a consequence of Theorem 3.3.

**Theorem 3.6.** *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $A \subseteq X$ and $B \subseteq X$ then:*

(1) *If $A$ is open-$\mathcal{I}$ then $A$ is $\rho\mathcal{I}_g$-open.*

(2) *If $A$ and $B$ are open-$\mathcal{I}$ then $A \cup B$ and $A \cap B$ are open-$\mathcal{I}$.*

(3) *If $B\backslash A \in \mathcal{I}$, $\overset{0}{A}\backslash \overset{0}{B} \in \mathcal{I}$ and $A$ is open-$\mathcal{I}$ then $B$ is open-$\mathcal{I}$.*

(4) *If $\overset{0}{A} \subseteq B \subseteq A$ and $A$ is open-$\mathcal{I}$, then $B$ is open-$\mathcal{I}$.*

(5) *If $A$ is pre-closed and open-$\mathcal{I}$ then $\overset{0}{A}$ is closed-$\mathcal{I}$.*

(6) *If $A \subseteq B$ and $A$ is open-$\mathcal{I}$ in $(X, \tau, \mathcal{I})$, then $A$ is open-$\mathcal{I}_B$ in $(B, \tau_B, \mathcal{I}_B)$.*

Some applications of the closed-$\mathcal{I}$ and open-$\mathcal{I}$ sets are shown now.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be *$\sigma\mathcal{I}$-compact* [9] if for each nonempty collection $\{V_\alpha\}_{\alpha\in\Lambda}$ of nonempty open sets, if $A\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ then there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. The space $(X, \tau, \mathcal{I})$ is *$\sigma\mathcal{I}$-compact* if $X$ is $\sigma\mathcal{I}$-compact.

It is simple to see that if $(X, \tau, \mathcal{I})$ is $\sigma\mathcal{I}$-compact and if $A \subseteq X$ is closed then $A$ is $\sigma\mathcal{I}$-compact.

**Theorem 3.7.** *If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$ is closed-$\mathcal{I}$ we have that:*

(1) *If $(X, \tau, \mathcal{I})$ is $\rho\mathcal{I}$-compact then $A$ is $\rho\mathcal{I}$-compact.*

(2) *If $(X, \tau, \mathcal{I})$ is $\sigma\mathcal{I}$-compact then $A$ is $\sigma\mathcal{I}$-compact.*

*Proof.*

(1) It is a consequence of Theorem 2.3, because closed-$\mathcal{I} \to \rho\mathcal{I}_g$-closed.

(2) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a nonempty collection of nonempty open sets with $A\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. Since $\overline{A}\backslash A \in \mathcal{I}$ and $\overline{A}\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \subseteq \left(A\backslash \bigcup_{\alpha\in\Lambda} V_\alpha\right) \cup \left(\overline{A}\backslash A\right) \in \mathcal{I}$ then $\overline{A}\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$. But $\overline{A}$ is $\sigma\mathcal{I}$-compact and so there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $A \subseteq \overline{A} \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. □

**Theorem 3.8.** *The ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-normal if and only if, for each pair of disjoint closed sets $F$ and $G$, there are disjoint open-$\mathcal{I}$ sets $A$ and $B$ such that $F\backslash A \in \mathcal{I}$ and $G\backslash B \in \mathcal{I}$.*

*Proof.* $(\rightarrow)$ It is clear because open$\rightarrow$open-$\mathcal{I}$.
$(\leftarrow)$ It is a consequence of Theorem 2.15 since open-$\mathcal{I} \rightarrow \rho\mathcal{I}_g$-open. □

**Remark 3.9.** *If $(X, \tau, \mathcal{I})$ is an ideal space and $A \subseteq X$, then:*

(1) *$\tau \oplus \mathcal{I}$ is the topology generated for the base $\tau \cup \mathcal{I}$.*

*It is noted that*

$$\tau \oplus \mathcal{I} = \left\{V \cup \bigcup \mathcal{C} : V \in \tau \text{ and } \mathcal{C} \subseteq \mathcal{P}(\mathcal{I})\right\}.$$

(2) *$\mathcal{I}(A)$ is the set $\bigcup_{I\in\mathcal{I},\ I\subseteq A} I$.*

In the next Theorem 3.10 we show that $\tau \oplus \mathcal{I}$ is the smallest topology in $X$, that contains $\tau$, such that all open-$\mathcal{I}$ set is an open set.

**Theorem 3.10.** *If $(X, \tau, \mathcal{I})$ is an ideal space we have that:*

(1) *If $A \subseteq X$ then $int_{\tau\oplus\mathcal{I}}(A) = int_\tau(A) \cup \mathcal{I}(A)$.*

(2) *A set $F \subseteq X$ is closed in the space $(X, \tau \oplus \mathcal{I})$ if and only if there exists $G \subseteq X$, closed in $(X, \tau)$, and a collection $\mathcal{C} \subseteq \mathcal{P}(\mathcal{I})$, such that $F = G\backslash \bigcup \mathcal{C}$.*

(3) *If $A \subseteq X$ then $adh_{\tau\oplus\mathcal{I}}(A) = adh_\tau(A) \backslash \mathcal{I}(X\backslash A)$.*

(4) *$\tau \oplus \mathcal{I}$ is the smallest topology $\beta$ in $X$ such that:*

   *(a) $\tau \subseteq \beta$ and*

   *(b) In the space $(X, \beta, \mathcal{I})$, for each $A \subseteq X$, $A$ is open-$\mathcal{I}$ if and only if $A$ is open.*

*Proof.*

(1) It is clear that $int_\tau(A) \cup \mathcal{I}(A) \in \tau \oplus \mathcal{I}$ and that $int_\tau(A) \cup \mathcal{I}(A) \subseteq A$, and so $int_\tau(A) \cup \mathcal{I}(A) \subseteq int_{\tau\oplus\mathcal{I}}(A)$.

Now, suppose that $W \in \tau \oplus \mathcal{I}$ and that $W \subseteq A$. There exist $V \in \tau$ and a collection $\{I_\alpha\}_{\alpha\in\Lambda}$ of elements in $\mathcal{I}$, such that $W = V \cup \bigcup_{\alpha\in\Lambda} I_\alpha$. Since $V \subseteq A$ then $V \subseteq int_\tau(A)$. Given that, for all $\alpha \in \Lambda$, $I_\alpha \subseteq A$ then $\bigcup_{\alpha\in\Lambda} I_\alpha \subseteq \mathcal{I}(A)$, and so $W \subseteq int_\tau(A) \cup \mathcal{I}(A)$. In particular $int_{\tau\oplus\mathcal{I}}(A) \subseteq int_\tau(A) \cup \mathcal{I}(A)$.

(2) It is obvious.

(3) Given that $A \subseteq adh_{\tau}(A) \setminus \mathcal{I}(X \setminus A)$ and $adh_{\tau}(A) \setminus \mathcal{I}(X \setminus A)$ is closed in $(X, \tau \oplus \mathcal{I})$ then $adh_{\tau \oplus \mathcal{I}}(A) \subseteq adh_{\tau}(A) \setminus \mathcal{I}(X \setminus A)$. Now, suppose that $F$ is closed in $(X, \tau \oplus \mathcal{I})$ and that $A \subseteq F$. There exists $G \subseteq X$, closed in $(X, \tau)$, and a collection $\mathcal{C} \subseteq \mathcal{P}(\mathcal{I})$, such that $F = G \setminus \bigcup \mathcal{C}$. Since $adh_{\tau}(A) \subseteq G$ and $\bigcup \mathcal{C} \subseteq \mathcal{I}(X \setminus A)$ we have that $adh_{\tau}(A) \setminus \mathcal{I}(X \setminus A) \subseteq G \setminus \bigcup \mathcal{C} = F$. In particular, $adh_{\tau}(A) \setminus \mathcal{I}(X \setminus A) \subseteq adh_{\tau \oplus \mathcal{I}}(A)$.

(4) (*i*) Suppose that $B \subseteq X$ is open-$\mathcal{I}$ in the space $(X, \tau \oplus \mathcal{I}, \mathcal{I})$, this is $B \setminus int_{\tau \oplus \mathcal{I}}(B) \in \mathcal{I} \subseteq \tau \oplus \mathcal{I}$. Since $B = [B \setminus int_{\tau \oplus \mathcal{I}}(B)] \cup int_{\tau \oplus \mathcal{I}}(B)$ then $B \in \tau \oplus \mathcal{I}$.

(*ii*) Suppose that $\beta$ is a topology in $X$ such that $\tau \subseteq \beta$ and that in the space $(X, \beta, \mathcal{I})$, for each $A \subseteq X$, $A$ is open-$\mathcal{I}$ if and only if $A$ is open.

Given that all $I \in \mathcal{I}$ is open-$\mathcal{I}$ in $(X, \beta, \mathcal{I})$ then, by hypothesis, $\mathcal{I} \subseteq \beta$. Hence $\tau \oplus \mathcal{I} \subseteq \beta$. □

**Remark 3.11.** *If $\mathcal{I}$ is an ideal in $X$ and $\mathcal{J}$ is an ideal in $Y$, then $\mathcal{I} \otimes \mathcal{J}$ is the set of all $D \subseteq X \times Y$ such that there exist $I \in \mathcal{I}$, $A \subseteq X$, $J \in \mathcal{J}$ and $B \subseteq Y$, with $D \subseteq (A \times J) \cup (I \times B)$.*

**Theorem 3.12.** (1) *If $\mathcal{I}$ is an ideal in $X$ and $\mathcal{J}$ is an ideal in $Y$, then $\mathcal{I} \otimes \mathcal{J}$ is an ideal in $X \times Y$.*

(2) *If $A$ is open-$\mathcal{I}$ in the space $(X, \tau, \mathcal{I})$ and $B$ is open-$\mathcal{J}$ in the space $(Y, \beta, \mathcal{J})$, then $A \times B$ is open-$\mathcal{I} \otimes \mathcal{J}$ in the space $(X \times Y, \tau \times \beta, \mathcal{I} \otimes \mathcal{J})$.*

*Proof.*

(1) It is clear that if $V \subseteq W \subseteq X \times Y$ and $W \in \mathcal{I} \otimes \mathcal{J}$, then $V \in \mathcal{I} \otimes \mathcal{J}$. Suppose that $\{D_1, D_2\} \subseteq \mathcal{I} \otimes \mathcal{J}$. There are $\{I_1, I_2\} \subseteq \mathcal{I}$, $\{J_1, J_2\} \subseteq \mathcal{J}$, $\{A_1, A_2\} \subseteq \mathcal{P}(X)$ and $\{B_1, B_2\} \subseteq \mathcal{P}(Y)$ such that $D_1 \subseteq (A_1 \times J_1) \cup (I_1 \times B_1)$ and $D_2 \subseteq (A_2 \times J_2) \cup (I_2 \times B_2)$. Hence $D_1 \cup D_2 \subseteq (A_1 \times J_1) \cup (A_2 \times J_2) \cup (I_1 \times B_1) \cup (I_2 \times B_2) \subseteq [(A_1 \cup A_2) \times (J_1 \cup J_2)] \cup [(I_1 \cup I_2) \times (B_1 \cup B_2)]$. This implies that $D_1 \cup D_2 \in \mathcal{I} \otimes \mathcal{J}$.

(2) Since $A \setminus \overset{0}{A} \in \mathcal{I}$ and $B \setminus \overset{0}{B} \in \mathcal{J}$, we have that $(A \times B) \setminus int(A \times B) = (A \times B) \setminus \left(\overset{0}{A} \times \overset{0}{B}\right) = \left[\left(A \setminus \overset{0}{A}\right) \times B\right] \cup \left[A \times \left(B \setminus \overset{0}{B}\right)\right] \in \mathcal{I} \otimes \mathcal{J}$. □

## 4. Other characteristics of the topology $\tau \oplus \mathcal{I}$

In this section we present some properties of the topology $\tau \oplus \mathcal{I}$, related to normality, compactness and C-compactness.

**Remark 4.1.** *If $(X,\tau,\mathcal{I})$ is an ideal space then*

$$\mathcal{I}^{\circledast} = \left\{ \bigcup \mathcal{C} : \mathcal{C} \subseteq \mathcal{P}(\mathcal{I}) \right\} \text{ and } \overline{\mathcal{I}} = \left\{ J : J \subseteq \overline{I}, \text{ for some } I \in \mathcal{I} \right\}$$

It is clear that $\mathcal{I}^{\circledast} = \mathcal{P}(U_{\mathcal{I}})$, where $U_{\mathcal{I}} = \bigcup_{I\in\mathcal{I}} I$.

It is easy to see that $\overline{\mathcal{I}}$ is an ideal in $X$, that $\mathcal{I} \subseteq \mathcal{I}^{\circledast}$, $\mathcal{I} \subseteq \overline{\mathcal{I}}$, and that if $I \in \overline{\mathcal{I}}$ then $\overline{I} \in \overline{\mathcal{I}}$. Moreover, if $\tau$ is a topology in $X$, it is clear that $\tau \oplus \mathcal{I} = \tau \oplus \mathcal{I}^{\circledast}$.

**Theorem 4.2.** *If $\mathcal{I}$ is an ideal in $X$, $\tau$ is a topology in $X$ and $(X,\tau\oplus\mathcal{I})$ is a normal space, then $(X,\tau,\mathcal{I}^{\circledast})$ is $\mathcal{I}^{\circledast}$-normal.*

*Proof.* Suppose that $F$ and $G$ are disjoint closed sets in $(X,\tau)$. Since $F$ and $G$ are closed sets in $(X,\tau\oplus\mathcal{I})$, there exists disjoint sets $U \cup \bigcup_{\alpha\in\Lambda_1} I_\alpha \in \tau\oplus\mathcal{I}$ and $V \cup \bigcup_{\alpha\in\Lambda_2} I_\alpha \in \tau\oplus\mathcal{I}$ such that $F \subseteq U \cup \bigcup_{\alpha\in\Lambda_1} I_\alpha$ and $G \subseteq V \cup \bigcup_{\alpha\in\Lambda_2} I_\alpha$. Thus $F\backslash U \subseteq \bigcup_{\alpha\in\Lambda_1} I_\alpha \in \mathcal{I}^{\circledast}$ and $G\backslash V \subseteq \bigcup_{\alpha\in\Lambda_2} I_\alpha \in \mathcal{I}^{\circledast}$. Moreover $U$ and $V$ are disjoint open sets in $(X,\tau)$. □

A space $(X,\tau)$ is said to be:

(1) *QHC* [11] if for each open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of $X$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X = \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

(2) *C-compact* [13] if for each closed set $F$ and each open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of $F$, there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $F \subseteq \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

An ideal space $(X,\tau,\mathcal{I})$ is defined to be:

(1) *$\rho\mathcal{I}$-QHC* [10] if for each collection $\{V_\alpha\}_{\alpha\in\Lambda}$ of open sets, if $X\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $X\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

(2) *$\rho C(\mathcal{I})$-compact* [10] if for each closed set $F$ and each collection $\{V_\alpha\}_{\alpha\in\Lambda}$ of open sets, if $F\backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$ there exists $\Lambda_0 \subseteq \Lambda$, finite, such that $F\backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \mathcal{I}$.

**Theorem 4.3.** (1) *If the space $(X,\tau,\mathcal{I}^{\circledast})$ is $\rho\mathcal{I}^{\circledast}$-compact then the space $(X,\tau\oplus\mathcal{I},\mathcal{I}^{\circledast})$ is $\mathcal{I}^{\circledast}$-compact.*

(2) *If the space $(X,\tau,\mathcal{I}^{\circledast})$ is $\sigma\mathcal{I}^{\circledast}$-compact then $(X,\tau\oplus\mathcal{I})$ is compact.*

(3) *If the space $(X,\tau\oplus\mathcal{I})$ is compact then the space $(X,\tau,\mathcal{I})$ is $\rho\mathcal{I}$-compact.*

(4) *If the space $\left(X,\tau\oplus\overline{\mathcal{I}}\right)$ is C-compact then the space $\left(X,\tau,\overline{\mathcal{I}}\right)$ is $\rho C(\overline{\mathcal{I}})$-compact.*

(5) *If $\left(X,\tau\oplus\overline{\mathcal{I}}\right)$ is QHC then the space $\left(X,\tau,\overline{\mathcal{I}}\right)$ is $\rho\overline{\mathcal{I}}$-QHC.*

*Proof.*

(1) Suppose that $X = \bigcup_{\alpha\in\Lambda} W_\alpha$, where $W_\alpha \in \tau \oplus \mathcal{I}$ for each $\alpha \in \Lambda$. For all $\alpha \in \Lambda$, there exist $V_\alpha \in \tau$ and a collection $\{I_j\}_{j\in\Lambda_\alpha}$ of elements in $\mathcal{I}$, such that $W_\alpha = V_\alpha \cup \bigcup_{j\in\Lambda_\alpha} I_j$. Hence $X = \bigcup_{\alpha\in\Lambda} V_\alpha \cup \bigcup_{\alpha\in\Lambda}\bigcup_{j\in\Lambda_\alpha} I_j$. Then $X \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}^{\circledast}$ and since $(X, \tau, \mathcal{I}^{\circledast})$ is $\rho\mathcal{I}^{\circledast}$-compact, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}^{\circledast}$. This implies that $X \backslash \bigcup_{\alpha\in\Lambda_0} W_\alpha \in \mathcal{I}^{\circledast}$.

(3) Suppose that $X \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \mathcal{I}$, where $\{V_\alpha\}_{\alpha\in\Lambda}$ is a collection of elements in $\tau$. There exists $I \in \mathcal{I}$ such that $X \backslash \bigcup_{\alpha\in\Lambda} V_\alpha = I$, and so $X = I \cup \bigcup_{\alpha\in\Lambda} V_\alpha$. Given that $(X, \tau \oplus \mathcal{I})$ is compact there exists $\Lambda_0 \subseteq \Lambda$, finite, with $X = I \cup \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Hence $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \subseteq I \in \mathcal{I}$ and $X \backslash \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{I}$.

(4) Suppose that $F \backslash \bigcup_{\alpha\in\Lambda} V_\alpha \in \overline{\mathcal{I}}$, where $\{V_\alpha\}_{\alpha\in\Lambda}$ is a collection of elements in $\tau$ and $F$ is closed in $(X, \tau)$. There exists $J \in \overline{\mathcal{I}}$ with $F \backslash \bigcup_{\alpha\in\Lambda} V_\alpha = J$, and so $F \subseteq J \cup \bigcup_{\alpha\in\Lambda} V_\alpha$. Given that $\left(X, \tau \oplus \overline{\mathcal{I}}\right)$ is C-compact and $F$ is closed in $\left(X, \tau \oplus \overline{\mathcal{I}}\right)$, there exists $\Lambda_0 \subseteq \Lambda$, finite, with $F \subseteq adh_{\tau\oplus\overline{\mathcal{I}}}(J) \cup \bigcup_{\alpha\in\Lambda_0} adh_{\tau\oplus\overline{\mathcal{I}}}(V_\alpha) \subseteq \overline{J} \cup \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha}$.

Hence $F \backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \subseteq \overline{J} \in \overline{\mathcal{I}}$ and $F \backslash \bigcup_{\alpha\in\Lambda_0} \overline{V_\alpha} \in \overline{\mathcal{I}}$.

Parts (2) and (5) have similar demonstrations. □

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