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Some new Hermite-Hadamard-Fejér type inequalities via k-fractional integrals concerning differentiable generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings

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Abstract. In this article, we first presented a new identity concerning differentiable mappings defined on m-invex set via k-fractional integrals. By using the notion of generalized relative semi-$(r; m, h_1, h_2)$-preinvexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite-Hadamard-Fejér type inequalities via k-fractional integrals are established. It is pointed out that some new special cases can be deduced from main results of the article.

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1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a n-dimensional vector space. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

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The most well-known inequalities related to the integral mean of a convex function f are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

Definition 1. *[28] A function $w : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be symmetric with respect to $\frac{a+b}{2}$, if $w(x) = w(a+b-x)$ holds for all $x \in [a,b]$.*

Example 1. *Assume that $w_1, w_2 : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, $w_1(x) = c$ for $c \in \mathbb{R}$, $w_2(x) = \left(x - \frac{a+b}{2}\right)^2$, then w_1, w_2 are symmetric functions with respect to $\frac{a+b}{2}$.*

In [12], Fejér established the following Hermite-Hadamard-Fejér inequality which is the weighted generalization of the Hermite-Hadamard inequality (1).

Theorem 2. *[12] Let $f : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{2}$$

holds, where $w : [a,b] \longrightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results which generalize, improve and extend the inequalities (1) and (2) through various classes of convex functions interested readers are referred to (see [[1]-[31],[33],[36],[39]-[44],[48],[49]]).

Let us recall some special functions and evoke some basic definitions as follows.

Definition 2. *The Euler beta function is defined for $a, b > 0$ as*

$$\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{3}$$

Definition 3. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x. \tag{4}$$

Here $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Definition 4. *For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by*

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n! k^n n k^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{5}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{6}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha). \tag{7}$$

For $k = 1$, (6) gives integral representation of gamma function.

Definition 5. *[35] Let $f \in L_1[a, b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as*

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad x > a$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad b > x. \tag{8}$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals.

Definition 6. *[47] A set $M_\varphi \subseteq \mathbb{R}^n$ is named as a relative convex (φ-convex) set, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,*

$$t\varphi(x) + (1-t)\varphi(y) \in M_\varphi, \ \forall \ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]. \tag{9}$$

Definition 7. *[47] A function f is named as a relative convex (φ-convex) function on a relative convex (φ-convex) set M_φ, if and only if, there exists a function $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that,*

$$f(t\varphi(x) + (1-t)\varphi(y)) \leq t f(\varphi(x)) + (1-t) f(\varphi(y)), \tag{10}$$

$\forall \ x, y \in \mathbb{R}^n : \varphi(x), \varphi(y) \in M_\varphi, t \in [0, 1]$.

Definition 8. *[8] A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow [0, +\infty)$ is said to be P-function, if*

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \ t \in [0, 1].$$

Definition 9. *[34] Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if*

$$f\big(x + t\eta(y, x)\big) \leq h(1-t) f(x) + h(t) f(y) \tag{11}$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 9, f becomes a preinvex function [38]. If the mapping $\eta(y, x) = y - x$ in Definition 9, then the non-negative function f reduces to h-convex mappings [46].

Definition 10. *[45] A non-negative function $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a tgs-convex function on K if the inequality*

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{12}$$

grips for all $x, y \in K$ and $t \in (0, 1)$.

Definition 11. *[32] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to MT-convex functions, if it is non-negative and $\forall\ x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality:*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{13}$$

Definition 12. *[36] A function: $I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be m-MT-convex, if f is positive and for $\forall\ x, y{\in}I$, and $t \in (0, 1)$, among $m \in [0, 1]$, satisfies the following inequality*

$$f\big(tx + m(1-t)y\big) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{14}$$

Definition 13. *[37] Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$. A function $f : K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, if*

$$f\big(mx + t\eta(y, x, m)\big) \leq m h_1(t) f(x) + h_2(t) f(y) \tag{15}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, together $m \in (0, 1]$, is said to be generalized (m, h_1, h_2)-preinvex functions with respect to η.

Motivated by the above literatures, the main objective of this article is to establish some new estimates on Hermite-Hadamard-Fejér type inequalities via k-fractional integrals associated with generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings. It is pointed out that some new special cases will be deduced from main results of the article. Our results also generalize Theorem 2.2 and Theorem 2.5 shown in [10].

2. Main results

The following definitions will be used in this section.

Definition 14. *[9] A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ grips for each $x, y \in K$ and any $t \in [0, 1]$.*

Remark 1. *In Definition 14, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.*

We next introduce a new class called generalized relative semi-$(r; m, h_1, h_2)$-preinvex mappings.

Definition 15. *Let $K \subseteq \mathbb{R}$ be an open nonempty m-invex set with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous. A function $f : K \longrightarrow (0,+\infty)$ is said to be generalized relative semi-$(r; m, h_1, h_2)$-preinvex, if*

$$f\big(m\varphi(x) + t\eta(\varphi(y), \varphi(x), m)\big) \le M_r(h_1(t), h_2(t); f(x), f(y), m) \tag{16}$$

holds for all $x, y \in I$ and $t \in [0,1]$, with some fixed $m \in (0,1]$, where

$$M_r(h_1(t), h_2(t); f(x), f(y), m) := \begin{cases} \Big[mh_1(t)f^r(x) + h_2(t)f^r(y)\Big]^{\frac{1}{r}}, & \text{if } r \neq 0; \\ \big[f(x)\big]^{mh_1(t)}\big[f(y)\big]^{h_2(t)}, & \text{if } r = 0, \end{cases}$$

is the weighted power mean of order r for positive numbers $f(x)$ and $f(y)$.

Remark 2. *In Definition 15, if we choose $r = 1$ and $\varphi(x) = x$, then we get Definition 13.*

Remark 3. *For $r = 1$, let us discuss some special cases in Definition 15 as follows.*

(I) If taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0,1]$, then we get generalized relative semi-$(m,s)-$ Breckner-preinvex mappings.
(II) If taking $h_1(t) = h_2(t) = 1$, then we get generalized relative semi-(m,P)-preinvex mappings.
(III) If taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0,1]$, then we get generalized relative semi-(m,s)-Godunova-Levin-Dragomir-preinvex mappings.
(IV) If taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-(m,h)-preinvex mappings.
(V) If taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-(m,tgs)-preinvex mappings.
(VI) If taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-m-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

For establishing our main results regarding some new Hermite-Hadamard-Fejér type integral inequalities associated with generalized relative semi-$(r; m, h_1, h_2)$-preinvexity via k-fractional integrals, we need the following crucial lemma.

Lemma 1. *Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), \varphi(a), m)] \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, where*

$\eta(\varphi(b),\varphi(a),m)>0$. Also, let $\varphi: I\longrightarrow K$ and $g: K\longrightarrow\mathbb{R}$ are continuous. Assume that $f: K\longrightarrow\mathbb{R}$ be a differentiable mapping on K° such that $f'\in L_1(K)$. Then for $\alpha,k>0$, the following equality holds for k-fractional integrals:

$$\left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}}\left[f(m\varphi(a))+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\right]$$

$$-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{m\varphi(a)}^{t} g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$

$$-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$

$$=\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{m\varphi(a)}^{t} g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt$$

$$-\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt. \tag{17}$$

Proof. Let denote

$$I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)=\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{m\varphi(a)}^{t} g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt$$

$$-\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}} f'(t)dt. \tag{18}$$

Integrating by parts, we get

$$I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)=\left(\int_{m\varphi(a)}^{t} g(s)ds\right)^{\frac{\alpha}{k}} f(t)\Bigg|_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{m\varphi(a)}^{t} g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$

$$-\left(\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}} f(t)\Bigg|_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}$$

$$-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)}\left(\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$

$$= \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}} \Big[f(m\varphi(a)) + f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))\Big]$$

$$-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left(\int_{m\varphi(a)}^{t} g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt$$

$$-\frac{\alpha}{k}\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left(\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right)^{\frac{\alpha}{k}-1} g(t)f(t)dt.$$

This completes the proof of the lemma.

Remark 4. *For $\eta(\varphi(b),\varphi(a),m) = \varphi(b) - m\varphi(a)$, where $\varphi(x) = x$, $\forall x \in I$ and $m = 1$, we get ([10], Lemma 2.1).*

Using Lemma 1, we now state the following theorems for the corresponding version for power of first derivative.

Theorem 3. *Let $\alpha, k > 0$ and $0 < r \leq 1$. Suppose $K = [m\varphi(a), m\varphi(a)+\eta(\varphi(b),\varphi(a),m)] \subseteq \mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, where $\eta(\varphi(b),\varphi(a),m) > 0$. Also, let $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$, $\varphi : I \longrightarrow K$ and $g : K \longrightarrow \mathbb{R}$ are continuous. Assume that $f : K \longrightarrow (0,+\infty)$ be a differentiable mapping on K° such that $f' \in L_1(K)$. If $(f'(x))^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex mapping, $q > 1$, $p^{-1}+q^{-1} = 1$ and $\|g\|_{\infty} = \sup|g(t)|$, then the following inequality for k-fractional integrals holds:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}$$

$$\times\Big[m(f'(a))^{rq}I^{r}(h_1(t);r) + (f'(b))^{rq}I^{r}(h_2(t);r)\Big]^{\frac{1}{rq}}, \tag{19}$$

where

$$I(h_i(t);r) := \int_0^1 h_i^{\frac{1}{r}}(t)dt, \quad \forall\, i = 1,2.$$

Proof. Suppose that $q > 1$, $p^{-1}+q^{-1} = 1$ and $0 < r \leq 1$. From Lemma 1, generalized relative semi-$(r;m,h_1,h_2)$-preinvexity of $(f'(x))^q$, Hölder inequality, Minkowski inequality, properties of the modulus, the fact $g(t) \leq \|g\|_{\infty}$ and changing the variable $t = m\varphi(a) + x\eta(\varphi(b),\varphi(a),m)$, we have

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{m\varphi(a)}^{t} g(s)ds\right|^{\frac{\alpha}{k}} |f'(t)|dt$$

$$+\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right|^{\frac{\alpha}{k}} |f'(t)|dt$$

$$\leq \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{m\varphi(a)}^{t} g(s)ds\right|^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}} \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (f'(t))^q dt\right)^{\frac{1}{q}}$$

$$+ \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right|^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (f'(t))^q dt\right)^{\frac{1}{q}}$$

$$\leq \|g\|_\infty^{\frac{\alpha}{k}} \times \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (f'(t))^q dt\right)^{\frac{1}{q}}$$

$$\times \left\{ \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (t - m\varphi(a))^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}}\right.$$

$$\left. + \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} (m\varphi(a) + \eta(\varphi(b),\varphi(a),m) - t)^{\frac{p\alpha}{k}} dt\right)^{\frac{1}{p}} \right\}$$

$$= \frac{2\|g\|_\infty^{\frac{\alpha}{k}} \eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}} \times \left(\int_0^1 \left(f'(m\varphi(a) + t\eta(\varphi(b),\varphi(a),m))\right)^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{2\|g\|_\infty^{\frac{\alpha}{k}} \eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}} \times \left(\int_0^1 \left[mh_1(t)(f'(a))^{rq} + h_2(t)(f'(b))^{rq}\right]^{\frac{1}{r}} dt\right)^{\frac{1}{q}}$$

$$\leq \frac{2\|g\|_\infty^{\frac{\alpha}{k}} \eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}$$

$$\times \left[\left(\int_0^1 m^{\frac{1}{r}} (f'(a))^q h_1^{\frac{1}{r}}(t) dt\right)^r + \left(\int_0^1 (f'(b))^q h_2^{\frac{1}{r}}(t) dt\right)^r \right]^{\frac{1}{rq}}$$

$$= \frac{2\|g\|_\infty^{\frac{\alpha}{k}} \eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}$$

$$\times \left[m(f'(a))^{rq} I^r(h_1(t);r) + (f'(b))^{rq} I^r(h_2(t);r)\right]^{\frac{1}{rq}}.$$

So, the proof of this theorem is complete.

Remark 5. *For $h_1(t) = 1-t$, $h_2(t) = t$, $r = m = 1$, and $\eta(\varphi(b),\varphi(a),m) = \varphi(b) - m\varphi(a)$, where $\varphi(x) = x$, $\forall x \in I$, we get ([10], Theorem 2.5).*

We point out some special cases of Theorem 3.

Corollary 1. *In Theorem 3 for $p = q = 2$, we have the following Hermite-Hadamard type inequality for generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\sqrt{\frac{2\alpha}{k}+1}}$$

$$\times\left[m(f'(a))^{2r}I^r(h_1(t);r)+(f'(b))^{2r}I^r(h_2(t);r)\right]^{\frac{1}{2r}}. \tag{20}$$

Corollary 2. *In Theorem 3 for $g(s) \equiv 1$, we have the following Hermite-Hadamard type inequality for generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings via k-fractional integrals:*

$$\left|\frac{f(m\varphi(a))+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))}{2}-\frac{\Gamma_k(\alpha+k)}{2\eta^{\frac{\alpha}{k}}(\varphi(b),\varphi(a),m)}\right.$$

$$\left.\times\left[I^{\alpha,k}_{(m\varphi(a))+}f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))+I^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-}f(m\varphi(a))\right]\right|$$

$$\leq\frac{\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}\times\left[m(f'(a))^{rq}I^r(h_1(t);r)+(f'(b))^{rq}I^r(h_2(t);r)\right]^{\frac{1}{rq}}. \tag{21}$$

Corollary 3. *In Theorem 3 for $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ and $f'(x) \leq K$, $\forall x \in I$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-$(r;m,h)$-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2K(m+1)^{\frac{1}{rq}}\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}I^{\frac{1}{q}}(h(t);r). \tag{22}$$

Corollary 4. *In Corollary 3 for $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-$(r;m,s)$-Breckner-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2K(m+1)^{\frac{1}{rq}}\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}\left(\frac{r}{r+s}\right)^{\frac{1}{q}}. \tag{23}$$

Corollary 5. *In Corollary 3 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $0 < s < r$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-$(r;m,s)$-Godunova-Levin-Dragomir-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2K(m+1)^{\frac{1}{rq}}\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}\left(\frac{r}{r-s}\right)^{\frac{1}{q}}. \tag{24}$$

Corollary 6. *In Theorem 3 for $h_1(t)=h_2(t)=t(1-t)$ and $f'(x)\leq K$, $\forall x\in I$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-(m,tgs)-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2K(m+1)^{\frac{1}{rq}}\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}\beta^{\frac{1}{q}}\left(1+\frac{1}{r},1+\frac{1}{r}\right). \tag{25}$$

Corollary 7. *In Theorem 3 for $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $f'(x)\leq K$, $\forall x\in I$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-$(r;m)$-MT-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2^{1-\frac{1}{rq}}K(m+1)^{\frac{1}{rq}}\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{p\alpha}{k}+1\right)^{1/p}}\beta^{\frac{1}{q}}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right). \tag{26}$$

Theorem 4. *Let $\alpha,k>0$ and $0<r\leq 1$. Suppose $K=[m\varphi(a),m\varphi(a)+\eta(\varphi(b),\varphi(a),m)]\subseteq\mathbb{R}$ be an open nonempty m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$, where $\eta(\varphi(b),\varphi(a),m)>0$. Also, let $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$, $\varphi: I\longrightarrow K$ and $g: K\longrightarrow\mathbb{R}$ are continuous. Assume that $f: K\longrightarrow(0,+\infty)$ be a differentiable mapping on K° such that $f'\in L_1(K)$. If $(f'(x))^q$ is generalized relative semi-$(r;m,h_1,h_2)$-preinvex mapping, $q\geq 1$ and $\|g\|_{\infty}=\sup|g(t)|$, then the following inequality for k-fractional integrals holds:*

$$\begin{aligned}\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| &\leq \frac{\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}\\ &\times\Bigg\{\left[m(f'(a))^{rq}I^{r}(h_1(t);\alpha,k,r)+(f'(b))^{rq}I^{r}(h_2(t);\alpha,k,r)\right]^{\frac{1}{rq}}\\ &+\left[m(f'(a))^{rq}\overline{I}^{r}(h_1(t);\alpha,k,r)+(f'(b))^{rq}\overline{I}^{r}(h_2(t);\alpha,k,r)\right]^{\frac{1}{rq}}\Bigg\},\end{aligned} \tag{27}$$

where

$$I(h_i(t);\alpha,k,r):=\int_0^1 t^{\frac{\alpha}{k}}h_i^{\frac{1}{r}}(t)dt,\quad \overline{I}(h_i(t);\alpha,k,r):=\int_0^1(1-t)^{\frac{\alpha}{k}}h_i^{\frac{1}{r}}(t)dt,\quad \forall\, i=1,2.$$

Proof. Suppose that $q \geq 1$ and $0 < r \leq 1$. From Lemma 1, generalized relative semi-$(r; m, h_1, h_2)$-preinvexity of $(f'(x))^q$, the well-known power mean inequality, Minkowski inequality, properties of the modulus, the fact $g(t) \leq \|g\|_\infty$ and changing the variable $t = m\varphi(a) + x\eta(\varphi(b), \varphi(a), m)$, we have

$$\left|I_{f,g,\eta,\varphi}(\alpha, k, m, a, b)\right| \leq \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{m\varphi(a)}^{t} g(s)ds\right|^{\frac{\alpha}{k}} |f'(t)|dt$$

$$+ \int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right|^{\frac{\alpha}{k}} |f'(t)|dt$$

$$\leq \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{m\varphi(a)}^{t} g(s)ds\right|^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{m\varphi(a)}^{t} g(s)ds\right|^{\frac{\alpha}{k}} (f'(t))^q dt\right)^{\frac{1}{q}}$$

$$+ \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right|^{\frac{\alpha}{k}} dt\right)^{1-\frac{1}{q}}$$

$$\times \left(\int_{m\varphi(a)}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} \left|\int_{t}^{m\varphi(a)+\eta(\varphi(b),\varphi(a),m)} g(s)ds\right|^{\frac{\alpha}{k}} (f'(t))^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\|g\|_\infty^{\frac{\alpha}{k}} \eta^{\frac{\alpha}{k}+1}(\varphi(b), \varphi(a), m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times \left\{ \left[\int_0^1 t^{\frac{\alpha}{k}} \left(f'(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))\right)^q dt\right]^{\frac{1}{q}} \right.$$

$$\left. + \left[\int_0^1 (1-t)^{\frac{\alpha}{k}} \left(f'(m\varphi(a) + t\eta(\varphi(b), \varphi(a), m))\right)^q dt\right]^{\frac{1}{q}} \right\}$$

$$\leq \frac{\|g\|_\infty^{\frac{\alpha}{k}} \eta^{\frac{\alpha}{k}+1}(\varphi(b), \varphi(a), m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times \left\{ \left[\int_0^1 t^{\frac{\alpha}{k}} \left[mh_1(t)(f'(a))^{rq} + h_2(t)(f'(b))^{rq}\right]^{\frac{1}{r}} dt\right]^{\frac{1}{q}} \right.$$

$$+\left[\int_0^1 (1-t)^{\frac{\alpha}{k}}\left[mh_1(t)(f'(a))^{rq}+h_2(t)(f'(b))^{rq}\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}\Bigg\}$$

$$\le \frac{\|g\|_\infty^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\Bigg\{\left[\left(\int_0^1 m^{\frac{1}{r}}(f'(a))^q t^{\frac{\alpha}{k}} h_1^{\frac{1}{r}}(t)dt\right)^r+\left(\int_0^1 (f'(b))^q t^{\frac{\alpha}{k}} h_2^{\frac{1}{r}}(t)dt\right)^r\right]^{\frac{1}{rq}}$$

$$+\left[\left(\int_0^1 m^{\frac{1}{r}}(f'(a))^q (1-t)^{\frac{\alpha}{k}} h_1^{\frac{1}{r}}(t)dt\right)^r+\left(\int_0^1 (f'(b))^q (1-t)^{\frac{\alpha}{k}} h_2^{\frac{1}{r}}(t)dt\right)^r\right]^{\frac{1}{rq}}\Bigg\}$$

$$= \frac{\|g\|_\infty^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\Bigg\{\left[m(f'(a))^{rq}I^r(h_1(t);\alpha,k,r)+(f'(b))^{rq}I^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{rq}}$$

$$+\left[m(f'(a))^{rq}\overline{I}^r(h_1(t);\alpha,k,r)+(f'(b))^{rq}\overline{I}^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is complete.

Remark 6. *For $h_1(t) = 1-t$, $h_2(t) = t$, $r = m = q = 1$, and $\eta(\varphi(b),\varphi(a),m) = \varphi(b) - m\varphi(a)$, where $\varphi(x) = x$, $\forall x \in I$, we get ([10], Theorem 2.2).*

We point out some special cases of Theorem 4.

Corollary 8. *In Theorem 4 for $q = 1$, we have the following Hermite-Hadamard type inequality for generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \le \|g\|_\infty^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)$$

$$\times\Bigg\{\left[m(f'(a))^rI^r(h_1(t);\alpha,k,r)+(f'(b))^rI^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{r}}$$

$$+\left[m(f'(a))^r\overline{I}^r(h_1(t);\alpha,k,r)+(f'(b))^r\overline{I}^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{r}}\Bigg\}. \tag{28}$$

Corollary 9. *In Theorem 4 for $g(s) \equiv 1$, we have the following Hermite-Hadamard type inequality for generalized relative semi-$(r;m,h_1,h_2)$-preinvex mappings via k-fractional integrals:*

$$\left|\frac{f(m\varphi(a))+f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))}{2}-\frac{\Gamma_k(\alpha+k)}{2\eta^{\frac{\alpha}{k}}(\varphi(b),\varphi(a),m)}\right.$$

$$\left.\times\left[I^{\alpha,k}_{(m\varphi(a))+}f(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))+I^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(b),\varphi(a),m))-}f(m\varphi(a))\right]\right|$$

$$\leq\frac{\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{2\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\left\{\left[m(f'(a))^{rq}I^r(h_1(t);\alpha,k,r)+(f'(b))^{rq}I^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m(f'(a))^{rq}\overline{I}^r(h_1(t);\alpha,k,r)+(f'(b))^{rq}\overline{I}^r(h_2(t);\alpha,k,r)\right]^{\frac{1}{rq}}\right\}. \tag{29}$$

Corollary 10. *In Theorem 4 for $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $f'(x)\leq K$, $\forall x\in I$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-$(r;m,h)$-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right|\leq\frac{K\|g\|_\infty^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\left\{\left[mI^r(h(1-t);\alpha,k,r)+I^r(h(t);\alpha,k,r)\right]^{\frac{1}{rq}}+\left[mI^r(h(t);\alpha,k,r)+I^r(h(1-t);\alpha,k,r)\right]^{\frac{1}{rq}}\right\}. \tag{30}$$

Corollary 11. *In Corollary 10 for $h_1(t)=(1-t)^s$, $h_2(t)=t^s$, we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-$(r;m,s)$-Breckner-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right|\leq\frac{K\|g\|_\infty^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\left\{\left[m\beta^r\left(\frac{s}{r}+1,\frac{\alpha}{k}+1\right)+\left(\frac{1}{\frac{s}{r}+\frac{\alpha}{k}+1}\right)^r\right]^{\frac{1}{rq}}+\left[m\left(\frac{1}{\frac{s}{r}+\frac{\alpha}{k}+1}\right)^r+\beta^r\left(\frac{s}{r}+1,\frac{\alpha}{k}+1\right)\right]^{\frac{1}{rq}}\right\}. \tag{31}$$

Corollary 12. *In Corollary 10 for* $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *and* $0 < s < r$, *we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-*$(r; m, s)$*-Godunova-Levin-Dragomir-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{K\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\left\{\left[m\beta^r\left(1-\frac{s}{r},\frac{\alpha}{k}+1\right)+\left(\frac{1}{\frac{\alpha}{k}-\frac{s}{r}+1}\right)^r\right]^{\frac{1}{rq}}+\left[m\left(\frac{1}{\frac{\alpha}{k}-\frac{s}{r}+1}\right)^r+\beta^r\left(1-\frac{s}{r},\frac{\alpha}{k}+1\right)\right]^{\frac{1}{rq}}\right\}. \tag{32}$$

Corollary 13. *In Theorem 4 for* $h_1(t) = h_2(t) = t(1-t)$ *and* $f'(x) \leq K$, $\forall x \in I$, *we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-*(m, tgs)*-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \frac{2K(m+1)^{\frac{1}{rq}}\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}\beta^{\frac{1}{q}}\left(1+\frac{1}{r},\frac{\alpha}{k}+\frac{1}{r}+1\right). \tag{33}$$

Corollary 14. *In Theorem 4 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $f'(x) \leq K$, $\forall x \in I$, *we get the following Hermite-Hadamard-Fejér type inequality for generalized relative semi-*$(r; m)$*-MT-preinvex mappings via k-fractional integrals:*

$$\left|I_{f,g,\eta,\varphi}(\alpha,k,m,a,b)\right| \leq \left(\frac{1}{2}\right)^{\frac{1}{rq}}\frac{K\|g\|_{\infty}^{\frac{\alpha}{k}}\eta^{\frac{\alpha}{k}+1}(\varphi(b),\varphi(a),m)}{\left(\frac{\alpha}{k}+1\right)^{1-\frac{1}{q}}}$$

$$\times\left\{\left[m\beta^r\left(\frac{\alpha}{k}-\frac{1}{2r}+1,1+\frac{1}{2r}\right)+\beta^r\left(\frac{\alpha}{k}+\frac{1}{2r}+1,1-\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m\beta^r\left(\frac{\alpha}{k}+\frac{1}{2r}+1,1-\frac{1}{2r}\right)+\beta^r\left(\frac{\alpha}{k}-\frac{1}{2r}+1,1+\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right\}. \tag{34}$$

Remark 7. *For* $k = 1$, *by our Theorems 3 and 4, we can get some new special Hermite-Hadamard-Fejér type inequalities associated with generalized relative semi-*$(r; m, h_1, h_2)$*-preinvex mappings via fractional integrals of order* $\alpha > 0$.

Remark 8. *Also, applying our Theorems 3 and 4, we can deduce some new inequalities using special means associated with generalized relative semi-*$(r; m, h_1, h_2)$*-preinvex mappings.*

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