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On Flows Spectrum on Closed Trio of Contours with Uniform Load

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Abstract. Considered dynamical system is a flow of clusters with the same length l on contours of unit length connected in polar-remote points into closed chain. When clusters move trough common node, the left-priority rule of conflict resolution works.
In the paper it is shown that in the case of chain consisted third contours the dynamical system has a spectrum of velocity and mode periodicity consisted on not more two components. Distribution of spectrum in dependence on load l is developed. Hypothesis on discrete spectrum in the case of arbitrary number of contours are formulated.

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1. Introduction

1.1. System description

We consider a closed chain of 3 *contours - circles* of unit length (C_1, C_2, C_3). On each contour there is defined a standard coordinate system from 0 to 1 in counterclockwise direction. The coordinates of *nodes - common points of neighboring contours* (C_1, C_2), (C_2, C_3), (C_3, C_1) are equal to $(0, 1/2)$, $(0, 1/2)$, $(0, 1/2)$ correspondingly. On all contours C_i, $i = 1, 2, 3$, clusters of length l move counterclockwise and with a velocity 1 equal to the complete circle per unit time. *A system state at time* t is a vector

$$(\alpha_1(t), \alpha_2(t), \alpha_3(t)),$$

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where $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ are coordinates of the leading points of clusters C_1, C_2, and C_3 correspondingly.

The trailing points of clusters C_1, C_2 and C_3 at the moment t are located in points with coordinates $\alpha_1(t) - l$, $\alpha_2(t) - l$, $\alpha_3(t) - l$ respectively (subtraction by modulo 1).

The system state is called *admissible state*, if no node is covered by more than one cluster. The delay of cluster movement occurs when the cluster approaches the node at the time that cluster on adjacent contour covers this node. If two clusters approach the common node at the same time, then there is *a conflict.* For the dynamical system it needs to determinate a rule of conflict resolution. After a result of conflict resolution one of these clusters is delayed at the node, and another cluster begins to pass through the node.

Let us define the *left-priority* conflict resolution rule that means the following. If the conflict occurs between the clusters of contours C_i and C_{i+1}, (addition modulo 3), then the cluster on contour C_i, $i = 1, 2, 3$, moves though the common node (advantage over priority), and cluster on contour C_{i+1} stops.

Periodic spectral point is called the admissible state $S_0 = (\alpha_1(0), \alpha_2(0), \alpha_3(0))$, for which there exits the minimal time values $T^{**} \geq 1$, $T^* \geq 0$, such that $\forall\ t \geq T^*$

$$\alpha_1(t + T^{**}) \equiv \alpha_1(t),\ \ \alpha_2(t + T^{**}) \equiv \alpha_2(t),\ \ \alpha_3(t + T^{**}) \equiv \alpha_3(t). \tag{1}$$

Collapse (congestion) is a mode of dynamical system where all clusters do not have the capability of movement [2]. The dynamical system is in *the free movement state at the time* t_0, if at any time $t \geq t_0$ all clusters move without delays, [4]. We say that *self-organization* of the dynamical system is a property of the system such that it should result in free movement state over finite time from any admissible initial state, [1].

2. Movement and collapse

2.1. Collapse, $l > \frac{1}{2}$

Proposition 1. *If cluster length*

$$l > \frac{1}{2}, \tag{2}$$

then for any admissible initial state the dynamical system falls into the state collapse no later than through $1/2$ *time units .*

Proof.

From (2) we have that at each time unit any cluster covers at least one node. As the number of contours is equal to nodes number, then for any system state no cluster can cover two nodes simultaneously. And, therefore, the cluster approaching the node can not cross the node. Given that a moving cluster approaches one of the nodes no more than over 1/2 time units, the proposition is proved.

2.2. Movement (life), $l < \frac{1}{2}$

Proposition 2. *If $l < \frac{1}{2}$, then the instantaneous velocity of each cluster and instantaneous average velocity of the system are strictly separated from zero.*

Proof. If the condition is satisfied, then at any time unit at least one of three clusters is moving. Hence the instantaneous velocity of the system is not less than 1/3. On the other hand, each cluster either moves without conflicts, or after a delay equals to not more than l time units at least 1/2 moves without delay. Thus each cluster makes a turn during a limited time.

2.3. System velocity

The basic studied characteristic of the system is the *average cluster velocity,* determined by the limit $\lim\limits_{T\to\infty} \frac{V(T)}{3T}$, where $V(T)$ is the total distance, such that all clusters passed during the time interval $(0, T)$. There arises *problem of the existence of velocity and periodic spectra.* Since the system is deterministic and, as it will be shown by direct verification, for any initial state of the system state, the system states are periodically repeating, starting from some finite time unit, then the limit exists and thus *the average system velocity is determined.*

Suppose that the initial state of the system is such that the average cluster velocity is less than 1. Cluster delays can occur at the node located on the left, at the point with the coordinate $\frac{1}{2}$, and at the node located on the right, at the point with coordinate 0. In the first case, the cluster delay is ending when the coordinate of the leading point of cluster on the left contour takes the value l. In the second case, the cluster delay is ending when the coordinate of the leading point of cluster on right contour takes the value $\frac{1}{2} + l$. Then we can reduce the study of the spectrum of possible values of velocity to the consideration of behavior systems for two one-parameter sets of initial states.

Proposition 3. *If the state*

$$\left(\alpha_1(0), \quad \frac{1}{2}, \quad \alpha_3(0)\right), \quad 0 < \alpha_1(0) < l < \frac{1}{2}$$

is periodic spectral point, then the follow state $\left(l, \quad \frac{1}{2}, \quad \alpha_3(0) + l - \alpha_1(0) = \alpha_{30}\right)$ *is also periodic spectral point.*

If the state

$$(\alpha_1(0), \quad 0, \quad \alpha_3(0)), \quad \frac{1}{2} < \alpha_3(0) < \frac{1}{2} + l$$

is periodic spectral point, then the following state $\left(\alpha_{10} = \alpha_1(0) + \frac{1}{2} + l - \alpha_3(0), \quad 0, \quad \frac{1}{2} + l\right)$ *is also periodic spectral point.*

Proof. We suppose that at time $t = 0$ the system is in the state

$$\left(\alpha_1(0), \frac{1}{2}, \alpha_3(0)\right), \ 0 < \alpha_1(0) < l < \frac{1}{2}.$$

Then during the time interval $t \in \left(0, \frac{1}{2} + l - \alpha_1(0)\right)$ the cluster of contour C_2 does not move, but clusters of contours C_1 C_3 are moving.

Thus, at the moment $\frac{1}{2} + l - \alpha_1(0)$ the system is in the state

$$\left(\alpha_1(t_0) + \frac{1}{2} + l - \gamma_0, 0, \frac{1}{2} + l\right).$$

Similarly, we consider the case when at the moment 0 the system is in the state

$$(\alpha_1(0), 0, \alpha_3(0)), \ \frac{1}{2} < \alpha_3(0) < \frac{1}{2} + l.$$

Proposition 3 has been proved.

Thus, if the cluster velocity is not equal to 1, then the system approached to one of the two states $\left(l, \frac{1}{2}, \alpha_{30}\right)$, $0 \leq \alpha_{30} \leq 1$, or $\left(\alpha_{10}, \frac{1}{2}, \frac{1}{2} + l\right)$, $0 \leq \alpha_{10} \leq 1$, up to a shift, during a finite time

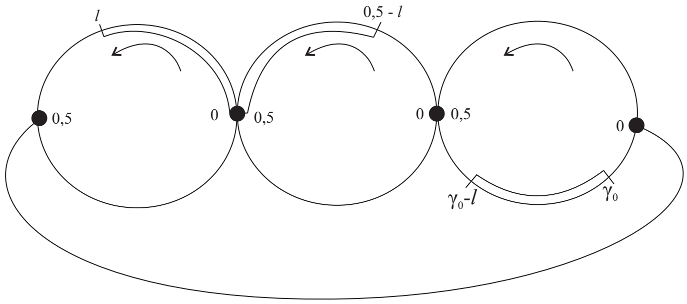

Figure 1: Fig1

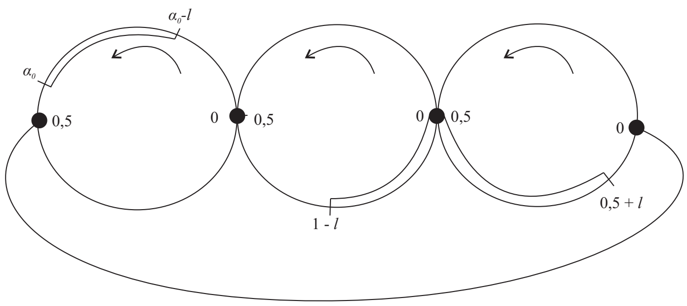

Figure 2: Fig2

2.4. Potential of delay and properties

Let us denote

$$\beta_i^+(t) = (\alpha_{i+1}(t) - \alpha_i(t))_{mod(1)},$$

$$\beta_i^-(t) = (\alpha_i(t) - \alpha_{i+1}(t))_{mod(1)},$$

where the indices are computed by modulo 3. Let $\theta(t)$ be Heaviside function, $\psi(t) = (\theta(t-1/2)\theta(l+1/2-t))$ Suppose that the coordinates of the leading points of clusters on adjacent contours C_i C_{i+1}, $i = 1, 2, 3$, at time t_0 satisfy the relation

$$\frac{1}{2} \leq \beta_i^-(t_0) < \frac{1}{2} + l, \tag{5}$$

Then at the time moment $t = \frac{1}{2} - \alpha_{i+1}(t_0)$ the cluster delay of contour C_{i+1} begins, if the delay does not occur earlier. We define *potential delay at the moment t_0 of cluster of contour C_{i+1} relative of the cluster of contour C_i* the value equals to

$$h_{i+1,i}(t_0) = \frac{1}{2} + l - \beta_i^-(t_0),$$

if condition (3) fulfils, and $h_{i+1,i}(t_0) = 0$, if condition (5) does not fulfill, $i = 1, 2, 3$.

If the coordinates of the leading points of the clusters on neighbor contours C_i and C_{i-1}, $i = 1, 2, 3$, at time unit t_0 satisfy the relation

$$\frac{1}{2} < \beta_{i-1}^+(t_0) < \frac{1}{2} + l, \tag{6}$$

then at the moment $t = \frac{1}{2} - \alpha_{i-1}(t_0)$ the delay of the cluster on contour C_{i+1} will begin, if the delay does not begin earlier.

We shall say that *potential delay at the time t_0 of the cluster on C_{i-1} with respect to cluster of the contour C_i*, is the value equal to

$$h_{i-1,i}(t_0) = \frac{1}{2} + l - \beta_{i-1}^+(t_0),$$

if condition (6) fulfils, and

$$h_{i-1,i}(t_0) = 0,$$

if condition (6) does not fulfill, $i = 1, 2, 3$. Thus,

$$h_{i+1,i}(t_0) = \left(\frac{1}{2} + l - \beta_i^-(t_0)\right)\psi(\beta_i^-(t_0)), \tag{7}$$

$$h_{i-1,i}(t_0) = \left(\frac{1}{2} + l - \beta_{i-1}^+(t_0)\right)\psi(\beta_{i-1}^+(t_0)). \tag{8}$$

We say that *delay potential* is a function of time

$$H(t) = h_{1,2}(t) + h_{2,1}(t) + h_{2,3}(t) + h_{3,2}(t) + h_{3,1}(t) + h_{1,3}(t) =$$

$$= \sum_{i=1}^{i=3} \left((1/2 + l - \beta_i^-(t))\psi(\beta_i^-(t)) + (1/2 + l - \beta_i^+(t_0)\psi(\beta_i^+(t)))\right). \tag{9}$$

Proposition 4. *Delay potential is non-negative piecewise linear function of time, and its derivative has one of the following values* -2, -1, 0 *or* 1.

Proof.

Proposition 4 follows from the fact that the velocity of changes of potential delay for any cluster relative to neighbor cluster at each time unit has one of values -1, 0 or 1.

And in a closed chain of 3 contours, a delay of not more than one cluster simultaneously takes place, such that can have non-zero potential delays relative to one or two clusters.

Proposition 5. *The delay potential is nonincreasing function of time.*

Proof. Suppose at some time unit t the function $H(t)$ increases.

This is possible only in the case such that at least one term on the right-hand side of equation (3) is increasing function. Suppose the term $h_{i_0,i_0+1}(t)$ increases (index addition by modulo n). We note the term can increase only with velocity 1. This term can only increase if the cluster on contour C_{i_0} moves, and the cluster on contour C_{i_0+1} does not move. If cluster on contour C_{i_0+1} is near a node being common with contour C_{i_0}, then term $h_{i_0,i_0+1}(t)$ needs to decrease, but not increase. Hence cluster on contour C_{i_0+1} does not move near node being common for contours C_{i_0+1} and C_{i_0+2}, and cluster on contour C_{i_0+2} passes the node. But the value $h_{i_0+1,i_0+2}(t)$ at time t decreases with velocity 1.

Similarly, it is proved that if at a given value t_0 the term $h_{i_0,i_0-1}(t_0)$ increases, then at t_0 term $h_{i_0-1,i_0-2}(t_0)$ decreases. Thus, each increasing term in the sum on the right-hand side of equality (3) corresponds the decreasing term with the same velocity, and different increasing terms are associated with various decreasing terms.

Thus Proposition 5 has been proved.

Proposition 6. *For any i the following equation holds*

$$h_{i+1,i}(t)h_{i,i+1}(t) = 0 \tag{8}$$

Proof. From definitions (5)-(6) it is following because the identity

$$\beta_i^-(t) + \beta_i^+(t) = 1 \tag{9}$$

holds.

Proposition 6 is proved.

Proposition 7. *The system at the time t_0 is in the free movement state is and only if $H(t_0) = 0$.*

Proof. If $H(t_0) = 0$, then not for any $i = 1, 2, 3$ the condition (5) or (6) fulfils, and therefore, after the time t_0 there are not delays.

If $H(t_0) \neq 0$, then for some $i = 1, 2, 3$ the condition (5) or (6) fulfils, and therefore, after the time t_0 there is at least one delay in cluster movement.

2.5. Concept of closed clusters group

We introduce the following definitions. *Clusters at the time unit t form closed clusters group* if for any i, $i = 1, 2, 3$ it fulfils either $h_{j,j+1}(t) + h_{j+1,j}(t) > 0$, or $\beta_j^-(t) = \frac{1}{2} + l$, or $\beta_j^+(t) = \frac{1}{2} + l$.

Note that if equality $\beta_i^-(t_0) = \frac{1}{2} + l$ is true, then if in interval of time $t \in (t_0, t_0 + \varepsilon)$ cluster on contour C_i does not move, and cluster on contour C_{i+1} moves, then for any arbitrarily small $\varepsilon > 0$ the value $h_{i+1,i}(t_0 + \varepsilon)$ is positive. Analogically, equality $\beta_i^+(t_0) = \frac{1}{2} + l$ means, that if in interval of time $t \in (t_0, t_0 + \varepsilon)$ cluster on contour C_i moves, and cluster on contour C_{i+1} does not move, then for any arbitrarily small $\varepsilon > 0$ the value $h_{i,i+1}(t_0 + \varepsilon)$ is positive.

Clusters $C_i, \dots, C_{i+k-1}$ form at time unit t closed group of left-type, if for any j, $j = 1, \dots, i + k - 2$ either $h_{j+1,j}(t) > 0$ (addition modulo 3), nor $\beta_j^-(t) = \frac{1}{2} + l$. *Clusters form at time unit t closed group of right-type,* if for any j, $j = 1, \dots, i + k - 2$ either $h_{j,j+1}(t) > 0$, nor $\beta_j^+(t) = \frac{1}{2} + l$. We call closed group by *closed of mixed type,* if it is not neither an open left-group, nor an open group of the right-type.

Note that in the case of left-type group the cluster can not delay at node located right from it. And in the case of right-type group the cluster can not delay at node located left from it.

Proposition 8. *Suppose that from some time unit t_0 the delay potential is positive and does not change*

$$H(t) \equiv H(t_0) > 0, \ t \geq t_0.$$

Then clusters form closed group left- or right-type for $t > t_0$.

Proof. As $H(t_0) > 0$, then the system at time t_0 is not in the free movement state.

Suppose, that in the time interval (t_0, t_1) all clusters move, and at the moment t_1 the delay of cluster on contour C_i begins near the node common with contour C_{i+1}. then $h_{i,i+1}(t_0) = h_{i,i+1}(t_1) > 0$ and terms $h_{i,i+1}$ decrease in the time interval (t_1, t_2), at the same time the value $h_{i+1,i}$ is equal to 0 in this time interval. The values $h_{i-1,i+1}$ and $h_{i+1,i-1}$ in interval (t_1, t_2) don't change, because in this interval the clusters on contours C_{i-1} and C_{i+1} move.

In the function $H(t)$ no decreases in the interval (t_1, t_2), then one terms from $h_{i-1,i+1}$ needs to increase, at the same time another term is equal to 0. As cluster on contour C_i does not move , then term $h_{i-1,i}$ needs to increase. If in some time unit t_3 $(t_1 \leq t_3 < t_2)$ the equality $\beta_{i-1}^+(t_3) = \frac{1}{2}$ fulfils, then in time t_3 the value of $h_{i-1,i}$ abruptly decreases from l to 0, and value $h_{i,i-1}$ with jump increases from 0 to l. In time interval (t_2, t_3) there will decrease both term $h_{i,i+1}$ and term $h_{i-1,i}$. So the value $H(t)$ will decrease. If the value β_{i-1}^+ does not achieve the value 1/2 in the interval (t_1, t_2) (it is equivalent that $h_{i,i-1}$ does not achieve the value l), then the following relations hold

$$h_{i,i+1}(t_2) = h_{i+1,i}(t_2) = h_{i,i-1}(t_2) = 0,$$

$$h_{i-1,i}(t_2) > 0.$$

If $h_{i+1,i}(t_0) > 0$, then at time t_0 the clusters form closed group of right-type. If $h_{i-1,i+1}(t_0) > 0$, then $h_{i-1,i+1}(t_2) > 0$, $h_{i+1,i-1}(t_0) = h_{i+1,i-1}(t_0) > 0$. Therefore, at some time unit $t_4 > t_2$ the delay of cluster on contour C_{i-1} begins near node common with contour C_i, or near node common with contour C_{i+1}. In both cases after the time t_2 both values $h_{i-1,i+1}$

and $h_{i+1,i-1}$ will decrease. Thus, the value H will decrease. If

$$h_{i-1,i+1}(t_0) = h_{i-1,i+1}(t_2) = h_{i+1,i-1}(t_0) = h_{i+1,i-1}(t_2) = 0,$$

then in some time unit a delay of cluster on contour C_{i-1} will begin near node common with contour C_i. Thus, immediately after time t_5, the potential delay $h_{i-1,i}$ will decrease, and in order to the delay potential H does not decrease, it is necessary that the value $h_{i+1,i-1}$ increases. As $h_{i+1,i-1}(t_0) = h_{i+1,i-1}(t_5) = 0$, then the condition

$$\beta^+_{i+1}(t_0) = \beta^+_{i+1}(t_5) = \frac{1}{2} + l$$

needs to hold, and so at time t_0 the clusters form the group of right-type.

Analogically it is proved that if at time t_1 the delay of cluster on contour C_i begins near the common node with the contour C_{i-1}, then at time t_0 the clusters form a group of left-type.

The Proposition 8 has been proved.

3. Self-organization as a simple spectrum of a dynamical system

Theorem 1. *For a closed chain of 3 contours the sufficient condition for self-organization is the following condition*

$$l \leq \frac{1}{6}.$$

From Proposition 5 we have that delay potential is a nonincreasing function of time. From Proposition 8, it is true, that either clusters from some moment t_0, or there are time intervals such that clusters are delayed and delay potential decrease, and, in this case, **the delay potential will reach the value 0 for a finite time** . Then consequently, the system results in free movement state.

But in the case $l < 1/6$, clusters can not form a closed group. And in the case $l = 1/6$ clusters can form a closed group, only if the delay potential is 0.

Hence Theorem 1 is true.

4. Case of a multiple spectrum $\left(\frac{1}{6} < l < \frac{1}{2}\right)$

Proposition 9. *Suppose* $\frac{1}{6} < l < \frac{1}{2}$ *and at time* t_0 *clusters form left- or right type. Then*

$$H(t_0) = 3l - \frac{1}{2}. \tag{12}$$

Proof.

Let it be left-type group. For right-type group the proof is analogically.

We have

$$H(t_0) = h_{13}(t_0) + h_{21}(t_0) + h_{32}(t_0) =$$

$$= \frac{3}{2} + 3l - \beta_3^-(t_0) - \beta_2^-(t_0) - \beta_1^-(t_0).$$

As

$$\frac{1}{2} \le \beta_i^-(t_0) \le \frac{1}{2} + l,\ i = 1, 2, 3,$$

then

$$\beta_1^-(t_0) + \beta_2^-(t_0) + \beta_3^-(t_0) =$$

$$= (\alpha_2(t_0) - \alpha_1(t_0))_{mod(1)} + (\alpha_3(t_0) - \alpha_2(t_0))_{mod(1)} +$$

$$+ (\alpha_2(t_0) - \alpha_3(t_0))_{mod(1)} + (\alpha_3(t_0) - \alpha_1(t_0))_{mod(1)} = 2.$$

Proposition 10. *Let $1/6 < l \le 1/2$ and the system does not result in free movement state over finite time. Then from some time unit t_0 clusters form a closed group of left- or right-type. And after time unit t_0, the vector state of the system is cyclically shifted one position to the right over time interval $\frac{1}{2} + l$ (in the case of left-type group), or one position to the left (in the case of right-type group). Also for this interval the total delay of clusters*

$$H(t) = 3l - 1/2 = H(t_0),$$

and the average clusters velocity equals

$$4/(3 + 6l).$$

Proof. According to Proposition condition the system does not result in free movement state, then we consider with taking to account the Proposition 3 the system behavior with initial condition

$$\left(l, \frac{1}{2}, \alpha_{3,0}\right),\ 0 \le \alpha_{3,0} < 1.$$

The case of initial condition

$$\left(\alpha_{1,0}, 0, \frac{1}{2} + l\right),\ 0 \le \alpha_{1,0} < 1$$

is considered analogically.

At initial time unit one of the following conditions fulfils:

$$0 \le \alpha_{,0} < l, \tag{13}$$

$$l \le \alpha_{3,0} \le \frac{1}{2}, \tag{14}$$

$$\frac{1}{2} < \alpha_{3,0} < \frac{1}{2} + l,\ l < \frac{1}{4}, \tag{15}$$

$$\frac{1}{2} + l \le \alpha_{3,0} \le 1 - l,\ l \le \frac{1}{4}, \tag{16}$$

$$1 - l < \alpha_{3,0} < \frac{1}{2} + 2l,\ l < \frac{1}{4}, \tag{17}$$

$$\frac{1}{2} + 2l \le \alpha_{3,0} < 1,\ l < \frac{1}{4}, \tag{18}$$

$$\frac{1}{2} < \alpha_{3,0} \le 1 - l,\ l \ge \frac{1}{4}, \tag{19}$$

$$1 - l < \alpha_{3,0} < \frac{1}{2} + l,\ l > \frac{1}{4}, \tag{20}$$

$$\frac{1}{2} + l < \alpha_{30} < 1,\ l \ge \frac{1}{4}. \tag{21}$$

For any initial state belonging the considered one-parameter family we have

$$h_{1,2}(0) = h_{2,1}(0) = 0.$$

Each of cases (13) – (21) is characterized by which from values $h_{1,3}(0)$, $h_{3,1}(0) = 0$, $h_{2,3}(0)$, $h_{3,2}(0)$ is nonzero.

We describe the system behavior at the fulfilling of inequalities (13)-(21).

We will show below that if condition (14) is satisfied, at the initial time the system is in free movement state . In this case, the delay potential $H(t)$ is equal to 0 at any time. If conditions (13), (15), (19), (20) fulfill, the system results in free movement state over finite time. The delay potential $H(t)$ is a piecewise-linear nonincreasing function equaled to 0 over some finite time.

If conditions (16), (17), (18), (21) hold, then beginning from finite time unit the system states periodically repeat, and average cluster velocity equals

$$v = \frac{4}{3 + 6l}.$$

At fulfilling of conditions (16), (17), (21), cluster delays are arising cyclically in the following order: delay of cluster C_1 with the duration $2l - \alpha_{30} + \frac{1}{2}$, delay of cluster C_3 with the duration $\alpha_{3,0} + l - 1$, delay of cluster C_2 with the duration $2l - \alpha_{3,0} + \frac{1}{2}$, delay of cluster C_1 with the duration $\alpha_{3,0} + l - 1$, delay of cluster C_3 with the duration $2l - \alpha_{3,0} + \frac{1}{2}$, delay of cluster C_2 with the duration $\alpha_{3,0} + l - 1$.

Then after the end of the period, the system results in a state $\left(\frac{1}{2}, \quad \alpha'_{3,0}, \quad l\right)$, such that its vector is obtained from initial state vector by cyclic shift of one position to the right and substitution $\alpha_{3,0}$ to $\alpha'_{3,0}$, $\alpha_{3,0} + \alpha'_{3,0} = \frac{3}{2} + l$. During the following part of the period the clusters are delayed in the same order, while the delay times are equal to $\alpha'_{3,0} + l - 1$ $2l - \alpha_{3,0} + \frac{1}{2}$.

At fulfilling of conditions (14), clusters are delayed in the period the following order: cluster delay C_3, cluster delay C_1, cluster delay C_2, while the duration of each of these delays is equal to $3l-1$.

Delay potential at fulfilling of conditions (16), (17), (21) has constant value equal to $3l-1$. At fulfilling of conditions (14) delay potential is piece-linear function, such that its values do not change from some time and equal to $3l-1$, if $\frac{1}{2}+2l \le \alpha_{3,0} < 1$, and equal to $3l-1$ from initial time, if $\alpha_{3,0} = \frac{1}{2}+2l < 1$.

As we will show below the behavior of the system with the initial state $\left(l, \frac{1}{2}, \alpha_{30}\right)$, $0 \le \alpha_{30} < 1$ and various values of α_{30}, it holds the following. vskip3pt If condition (14) is satisfied, the system is in free movement state from the initial time unit. And if conditions (15) , (19), (20) fulfill, then the system results in free movement state over a finite time.

At fulfilling of conditions (16), (17), (21)

$$H(0) = 3l - \frac{1}{2}$$

and over time interval with the duration $\frac{1}{2}+l$ the state vector cyclicly moves to one position to the right, and total cluster delay over this interval equals to $H(0)$.

At fulfilling of conditions (18) clusters form a closed group from time unit $t = 1/2 - 2l$ (at condition (18) $l < 1/4$). Then at time unit $t = 1/2 - 2l$ the delay potential $H(t)$ has value $3l - \frac{1}{2}$, and after this time the value delay potential of does not change. From time $t = 1/2 - 2l$ the state vector over time interval with duration $\frac{1}{2}+l$ cyclicly moves to one position to the right, and total cluster delay over this interval equals to $H(0)$.

Thus, at initial state $\left(l, \frac{1}{2}, \alpha_{30}\right)$, with any α_{30} the state behavior satisfies the condition of Proposition 10.

We have an average cluster velocity

$$v = 1 - \frac{H(t_0)}{\frac{1}{2}+l} = 1 - \frac{3l - \frac{1}{2}}{3\left(\frac{1}{2}+l\right)} = \frac{4}{3+6l}.$$

Let us prove the statement by direct verification of each conditions (13) - (21).

a) Suppose the conditions (13) fulfils. Then in time interval $t \in \left(0, \frac{1}{2}\right)$ all clusters move. At time $t \in \left[0, \frac{1}{2}\right]$

$$h_{1,2}(t) = h_{2,1}(t) = h_{1,3}(t) = h_{3,1}(t) = h_{3,2}(t) = 0,$$

$$h_{2,3}(t) = l - \alpha_{3,0}.$$

Thus,

$$H\left(\frac{1}{2} + l - \alpha_{3,0}\right) = 0.$$

At moment $t = \frac{1}{2}$ the system results in the state

$$\alpha_1\left(\frac{1}{2}\right) = \frac{1}{2} + l,\ \alpha_2\left(\frac{1}{2}\right) = 0,\ \alpha_3\left(\frac{1}{2}\right) = \frac{1}{2} + \alpha_{3,0}.$$

Over time interval $\left(\frac{1}{2}, \frac{1}{2}+l-\alpha_{3,0}\right)$ cluster on contour C_2 does not move. We have $t \in \left(\frac{1}{2}, \frac{1}{2}+l-\alpha_{3,0}\right)$,

$$h_{1,2}(t) = h_{2,1}(t) = h_{1,3}(t) = h_{3,1}(t) = h_{3,2}(t) = 0,$$

$$h_{2,3}(t) = l - \alpha_{3,0} - \left(t - \frac{1}{2}\right),$$

$$h_{1,2}\left(\frac{1}{2}+l-\alpha_{3,0}\right) = h_{2,1}\left(\frac{1}{2}+l-\alpha_{3,0}\right) = h_{1,3}\left(\frac{1}{2}+l-\alpha_{3,0}\right) =$$

$$= h_{3,1}\left(\frac{1}{2}+l-\alpha_{3,0}\right) = h_{2,3}\left(\frac{1}{2}+l-\alpha_{3,0}\right) =$$

$$= h_{3,2}\left(\frac{1}{2}+l-\alpha_{3,0}\right) = 0,$$

$$H(t) = \begin{cases} l-\alpha_{3,0}, \ 0 < t < \frac{1}{2}, \\ l-\alpha_{3,0} - \left(t-\frac{1}{2}\right), \ \frac{1}{2} < t < \frac{1}{2}+l-\alpha_{3,0}, \end{cases}$$

$$H\left(\frac{1}{2}+l-\alpha_{3,0}\right) = 0.$$

Therefore, delay potential is constant and equals to $l-\alpha_{3,0}$ from initial time unit to time $t=\frac{1}{2}$ then in time interval $t \in \left(\frac{1}{2}, \frac{1}{2}+l-\alpha_{3,0}\right)$ it linear decreases to 0 with velocity 1. After time unit $\frac{1}{2}+l-\alpha_{3,0}$ the delay potential equals to 0.

At the time unit $t = \frac{1}{2}+l-\alpha_{3,0}$ the system results in the state

$$\alpha_1\left(\frac{1}{2}+l-\alpha_{3,0}\right) = \frac{1}{2}+2l-\alpha_{3,0},$$

$$\alpha_2\left(\frac{1}{2}+l-\alpha_{3,0}\right) = 0,$$

$$\alpha_3\left(\frac{1}{2}+l-\alpha_{3,0}\right) = \frac{1}{2}+l,$$

that is free movement state.

b) Suppose the conditions (14) fulfils. Then $t \in [0, +\infty)$.

$$h_{1,2}(t) = h_{2,1}(t) = h_{1,3}(t) = h_{3,1}(t) = h_{2,3}(t) = h_{3,2}(t) = 0,$$

$$H(t) \equiv 0.$$

Thus, in this case the delay potential equals 0 as at initial time and at any time unit. The system is in free movement state from initial time unit.

c) Suppose that there fulfils either condition (15) or (19). Then in time interval $t \in (0, 1-\alpha_{3,0})$ all clusters move. At time $t \in [0, 1-\alpha_{3,0}]$

$$h_{1,2}(t) = h_{2,1}(t) = h_{1,3}(t) = h_{2,3}(t) = h_{3,2}(t) = 0,$$

$$h_{3,1}(t) = \alpha_{3,0} - \frac{1}{2}.$$

At time $t = 1 - \alpha_{3,0}$ the system results in the state

$$\alpha_1 (1 - \alpha_{3,0}) = 1 + l - \alpha_{3,0},\ \alpha_2 (1 - \alpha_{3,0}) = \frac{3}{2} - \alpha_{3,0},\ \alpha_3 (1 - \alpha_{3,0}) = 0.$$

In time interval $t \in \left(1 - \alpha_{3,0}, \frac{1}{2}\right)$ cluster on contour C_3 does not move.

We have $t \in \left(1 - \alpha_{3,0}, \frac{1}{2}\right)$

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = h_{3,2}(t) = 0,$$

$$h_{1,3}(t) = \alpha_{30} - \frac{1}{2} - (t - 1 + \alpha_{3,0}),$$

$$H(t) = \alpha_{30} - \frac{1}{2} - (t - 1 + \alpha_{3,0}).$$

At time unit $t = \frac{1}{2}$

$$h_{1,2}\left(\frac{1}{2}\right) = h_{2,1}\left(\frac{1}{2}\right) = h_{1,3}\left(\frac{1}{2}\right) =$$

$$= h_{3,1}\left(\frac{1}{2}\right) = h_{2,3}\left(\frac{1}{2}\right) = h_{3,2}\left(\frac{1}{2}\right) = 0,$$

Thus, at initial time unit the delay potential equals to $\alpha_{3,0} - \frac{1}{2}$ *and preserves the value until the time unit* $1 - \alpha_{3,0}$. *Then in time interval* $t \in \left(1 - \alpha_{3,0}, \frac{1}{2}\right)$ *delay potential linear decreases with velocity 1 and equals to 0 from time* $t = \frac{1}{2}$.

At time $t = \frac{1}{2}$ the system results in the state

$$\alpha_1\left(\frac{1}{2}\right) = \frac{1}{2} + l,\ \alpha_2\left(\frac{1}{2}\right) = 0,\ \alpha_3\left(\frac{1}{2}\right) = 0,$$

that is free movement state.

d) Suppose condition (20) fulfils. Then in time interval $t \in (0, 1 - \alpha_{3,0})$ all clusters move. At $t \in [0, 1 - \alpha_{3,0}]$

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = 0,$$

$$h_{3,1}(t) = \alpha_{3,0} - \frac{1}{2},$$

$$h_{3,2}(t) = \alpha_{3,0} + l - 1,$$

$$H(t) = 2\alpha_{3,0} + l - \frac{3}{2}.$$

At time unit $t = 1 - \alpha_{3,0}$ the system results in the state

$$\alpha_1 (1 - \alpha_{3,0}) = 1 + l - \alpha_{3,0},\ \alpha_2 (1 - \alpha_{3,0}) = \frac{3}{2} - \alpha_{3,0},\ \alpha_3 (1 - \alpha_{3,0}) = 0.$$

In time interval $t \in \left(1-\alpha_{3,0}, \frac{1}{2}\right)$ cluster on contour C_3 does not move. We have $t \in \left[1-\alpha_{3,0}, \frac{1}{2}\right]$

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = 0,$$

$$h_{3,1}(t) = \alpha_{3,0} - \frac{1}{2} - (t - 1 + \alpha_{3,0}).$$

For potential delay $h_{3,2}(t)$ we have

$$h_{3,2}(t) = l + \alpha_{3,0} - 1 - (t - 1 + \alpha_{3,0}) \in [1-\alpha_{3,0}, l],$$

$$h_{3,2}(t) = 0 \in \left(l, \frac{1}{2}\right],$$

$$H(t) = l + \alpha_{3,0} - 1 - (t - 1 + \alpha_{3,0}) \in [1-\alpha_{3,0}, l],$$

$$h_{3,2}(t) = 0 \in \left(l, \frac{1}{2}\right].$$

Therefore

$$H(t) = 2\alpha_{3,0} + l - \frac{3}{2} - 2(t - 1 + \alpha_{3,0}),\ 1-\alpha_{3,0} \le t \le l,$$

$$H(l) = \frac{1}{2} - l,$$

$$H(t) = \frac{1}{2} - l - (t - l),\ l \le t \le \frac{1}{2}.$$

At time $t = \frac{1}{2}$ we have

$$h_{1,2}\left(\frac{1}{2}\right) = h_{2,1}\left(\frac{1}{2}\right) = h_{1,3}\left(\frac{1}{2}\right) =$$

$$= h_{3,1}\left(\frac{1}{2}\right) = h_{2,3}\left(\frac{1}{2}\right) = h_{3,2}\left(\frac{1}{2}\right) = 0,$$

$$H\left(\frac{1}{2}\right) \equiv 0.$$

Thus, delay potential equals to $2\alpha_{3,0} + l - \frac{3}{2}$ *at initial time and preserves this value until time unit* $t = 1 - \gamma$. *After this time the delay potential decreases with velocity 2 to the value* $\frac{1}{2} - l$ *at time unit* $t = l$, *and in time interval* $t \in \left(l, \frac{1}{2}\right)$ *it decreases to value 0, that is preserved at any next time.* At time unit $t = \frac{1}{2}$ the system is in the state

$$\alpha_1\left(1-\alpha_{3,0}\right) = \frac{1}{2} + l,\ \alpha_2\left(1-\alpha_{3,0}\right) = 0,\ \alpha_3\left(1-\alpha_{3,0}\right) = 0,$$

that is free movement state.

e) Suppose there fulfils condition either (16), or (17), or (21). Then in time interval $t \in \left(0, \frac{1}{2} - l\right)$ all clusters move. And at time $t = \frac{1}{2} - l$ the system results in the state

$$\alpha_1\left(\frac{1}{2} - l\right) = \frac{1}{2},\ \alpha_2\left(\frac{1}{2} - l\right) = 1 - l,\ \alpha_3\left(\frac{1}{2} - l\right) = \alpha_{3,0} - \frac{1}{2} - l.$$

At time $t \in \left[0, \frac{1}{2} - l\right]$

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = 0,$$

$$h_{1,3}(t) = 2l - \alpha_{3,0} + \frac{1}{2},$$

$$h_{3,2}(t) = \alpha_{3,0} + l - 1,$$

$$H(t) \equiv 3l - \frac{1}{2}.$$

In time interval $\left(\frac{1}{2} - l, 1 + l - \alpha_{3,0}\right)$ cluster on contour C_1 does not move. At time $t = 1 + l - \alpha_{3,0}$ the system results in the state

$$\alpha_1 \left(1 + l - \alpha_{3,0}\right) = \frac{1}{2},\ \alpha_2 \left(1 + l - \alpha_{3,0}\right) = \frac{3}{2} + l - \alpha_{3,0},\ \alpha_3 \left(1 + l - \alpha_{3,0}\right) = l.$$

In time interval $\left(\frac{1}{2} - l, 1 + l - \alpha_{3,0}\right)$ cluster on contour C_1 does not move. In time interval $t \in \left[\frac{1}{2} - l, 1 + l - \alpha_{3,0}\right]$

$$h_{1,2}(t) = h_{3,1}(t) = h_{2,3}(t) = 0,$$

$$h_{1,3}(t) = 2l - \alpha_{30} + \frac{1}{2} - \left(t - \frac{1}{2} + l\right),$$

$$h_{2,1}(t) = t - \frac{1}{2} + l,$$

$$h_{3,2}(t) = \alpha_{3,0} + l - 1,$$

$$H(t) \equiv 3l - \frac{1}{2}.$$

Suppose

$$\alpha'_{3,0} = \frac{3}{2} + l - \alpha_{3,0}.$$

Then

$$\alpha_{3,0} = \frac{3}{2} + l - \alpha'_{3,0}.$$

We obtain

$$\frac{1}{2} + l < \alpha'_{3,0} < \frac{1}{2} + 2l,$$

i. e. for α'_{30} it is safisfied the condition, analogical the condition for α_{30}. We have

$$\alpha_{3,0} + \alpha'_{3,0} = \frac{3}{2} + l. \tag{22}$$

Vector of system state at time $t = 1 + l - \alpha_{3,0}$ is obtained from vector of initial system state by shift on one position to left and substitution $\alpha_{3,0}$ to $\alpha'_{3,0}$ Delay potential equals to

$$H(t) = 3l - \frac{1}{2}$$

at initial time unit and reserves constant on any time. *At any time unit the value $H(t)$ does not change*

$$H(t) \equiv 3l - \frac{1}{2}.$$

Thus at time

$$t = 6(1+l) - 3(\alpha_{3,0} + \alpha'_{3,0}) = \frac{3}{2} + 3l$$

the system returns to initial state. Over this time duration each cluster delays twice: first time it delays for $(3l - \alpha_{3,0})$ and second time it delays for $(3l - \alpha'_{3,0})$. According to (22), we obtain that total delay of cluster over period equals to $3l - \frac{1}{2}$ time units.

f) Suppose condition (18) fulfils. In time interval $t \in \left(0, \frac{1}{2} - l\right)$ all clusters move. In time interval $t \in \left(0, \frac{3}{2} - \alpha_{3,0}\right)$ all clusters move. We have

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{1,3}(t) = h_{2,3}(t) = 0,$$

$$h_{3,2}(t) = \alpha_{3,0} + l - 1,$$

$$H(t) \equiv \alpha_{3,0} + l - 1,$$

$$0 \le t \le 1 - \alpha_{3,0}.$$

At time $t = \frac{3}{2} - \alpha_{3,0}$ the system results in the state

$$\alpha_1\left(\frac{3}{2} - \alpha_{3,0}\right) = \frac{3}{2} - \alpha_{3,0} + l,\ \alpha_2\left(\frac{3}{2} - \alpha_{3,0}\right) = 1 - \alpha_{3,0},\ \alpha_3\left(\frac{3}{2} - \alpha_{3,0}\right) = \frac{1}{2}.$$

In time interval $t \in \left(\frac{3}{2} - \gamma_0, \frac{1}{2} + l\right)$ cluster on contour C_3 does not move, while other clusters move. The function $h_{3,2}$ decreases in this interval, and beginning from time $t = 1 - 2l$, function $h_{1,3}$ increases (note that at $\alpha_{3,0} = 1 + 2l$). The equality $\frac{3}{2} - \alpha_{3,0} = 1 + 2l$ holds, thus, in this case function $h_{1,3}$ increases when time $t = \frac{3}{2} - \alpha_{3,0}$ comes. At $t \in \left[\frac{3}{2} - \alpha_{30}, \frac{1}{2} + l\right]$ we have

$$h_{1,2}(t) = h_{2,1}(t) = h_{2,3}(t) = h_{3,1}(t) = 0,$$

$$h_{3,2}(t) = \alpha_{30} + l - 1 - \left(t - \frac{3}{2} - \alpha_{3,0})\right),$$

$$h_{1,3}(t) = 0,\ \frac{3}{2} - \alpha_{3,0} \le t \le 1 - 2l,$$

$$h_{1,3}(t) = 0,\ 1 - 2l \le t \le \frac{1}{2} + 2l,$$

$$H(t) = \alpha_{3,0} + l - 1 - \left(t - \frac{3}{2} - \alpha_{3,0}\right),$$

$$\frac{3}{2} - \alpha_{3,0} \le t \le \frac{1}{2} + l.$$

At time $t = \frac{1}{2} + l$ the system results in the state

$$\alpha_1\left(\frac{1}{2}+l\right) = \frac{1}{2}+2l,\ \alpha_2\left(\frac{1}{2}+l\right) = l,\ \alpha_3\left(\frac{1}{2}+l\right) = \frac{1}{2},$$

that differs from initial system state by shift on one position to the right and substitution $\alpha_{3,0}$ on $1+2l$, while

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{1,3}(t) = h_{2,3}(t) = 0,$$

$$h_{3,2}(t) = 3l - \frac{1}{2},$$

$$H(t) = 3l - \frac{1}{2}.$$

Thus, *in time interval* $t \in \left(0, \frac{3}{2} - \alpha_{3,0}\right)$ *function* $H(t)$ *has value* $\alpha_{3,0} + l - 1$, *then in the time interval* $t \in \left(\frac{3}{2} - \alpha_{3,0}, 1 - 2l\right)$ *function* $H(t)$ *linear decreases to value* $3l - \frac{1}{2}$, *and beginning from time* $\frac{1}{2} - 2l$ *value of* $H(t)$ *preserves constant.* Next system states return with period

$$T = \frac{1}{2} + l.$$

Over this time period one of clusters has delay with duration $3l - \frac{1}{2}$ time units.

All possible cases are considered. Proposition 10 has been proved.

5. Remarks, generalizations, hypotheses

5.1. Behavior of contours trio at $l = \frac{1}{2}$

In [3] a discrete analog of considered system is studied. There are 2 cells and one particle on each contour of closed chain consisting of n contours. Particle moves at discrete instants of time. If $n = 3$ and rule of conflict resolution is conflict left-hand, this system is equivalent to the following case of continuous considered system.

The cluster length on each contour is $\frac{1}{2}$. In the initial state, each cluster is located at a point with coordinate 0 or $\frac{1}{2}$.

Example 1. Suppose that at initial time the system was in the state

$$\alpha_1(0) = l = \frac{1}{2},\ \alpha_2(0) = l = \frac{1}{2},\ \alpha_3(0) = \alpha_{3,0}.$$

If $\alpha_{3,0} = \frac{1}{2}$, then $H(0) = 0$, and the system is in free movement state.

If $\alpha_{30} = 0$, i. e. initial state is

$$\alpha_1(0) = l = \frac{1}{2},\ \alpha_2(0) = l = \frac{1}{2},\ \alpha_3(0) = 0,$$

then

$$h_{1,2}(0) = h_{2,1}(0) = h_{3,1}(0) = h_{2,3}(0) = h_{3,2}(0) = 0,$$

$$h_{1,3}(0) = \frac{1}{2},$$

$$H(0) = \frac{1}{2}$$

and at initial time there is conflict of clusters C_1 and C_3. In according of left-priority rule of conflict resolution in time interval $t \in \left(0, \frac{1}{2}\right)$ cluster on contour C_1 does not move, while clusters on contours C_2 and C_3 move. So at time $t = \frac{1}{2}$ the system results in the state

$$\alpha\left(\frac{1}{2}\right) = \frac{1}{2},\ \beta\left(\frac{1}{2}\right) = 0,\ \beta\left(\frac{1}{2}\right) = \frac{1}{2},$$

that is obtained from initial state vector by cyclic shift on one position to the right.

At $t \in [0, \frac{1}{2}]$ we have

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = 0,$$

$$h_{1,3}(t) = \frac{1}{2} - t,$$

$$h_{3,2}(t) = t,$$

$$H(t) \equiv \frac{1}{2}$$

and, thus,

$$h_{1,2}\left(\frac{1}{2}\right) = h_{2,1}\left(\frac{1}{2}\right) = h(1,3)\left(\frac{1}{2}\right) = h_{3,1}\left(\frac{1}{2}\right) = h_{2,3}\left(\frac{1}{2}\right) = 0,$$

$$h_{3,2}\left(\frac{1}{2}\right) = \frac{1}{2},$$

$$H\left(\frac{1}{2}\right) = \frac{1}{2},$$

i. e. *the potential preserves constant value from initial time to any time unit.*

Initial system state repeats over time interval with duration $T = \frac{3}{2}$. Over period the delay of each clusters comes with duration equals to $\frac{1}{2}$.

Average velocity of each clusters equals to

$$v = \frac{2}{3},$$

that corresponds to formula

$$v = \frac{4}{3 + 6l}$$

at $l = \frac{1}{2}$ and results of [1].

5.2. Behavior of contours trio at $l = \frac{1}{6}$

In Sect. 2 it was proved, that at $l \le \frac{1}{6}$ the system results to self-organization over finite time .

Example 2. Let cluster length be $l = \frac{1}{6}$ and at time $t = 0$ system be in the state

$$\alpha(0) = \frac{1}{6},\ \beta(0) = \frac{1}{2},\ \gamma(0) = \frac{3}{4}.$$

Then at $t \in \left(0, \frac{1}{3}\right)$

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = h_{3,2}(t) = 0,$$

$$h_{1,3}(t) = \frac{1}{12} - \left(t - \frac{1}{3}\right),$$

$$H(t) = \frac{1}{12}.$$

In time interval $\left(0, \frac{1}{3}\right)$ all clusters move and in $t = \frac{1}{3}$ the system results in state

$$\alpha\left(\frac{1}{3}\right) = \frac{1}{2},\ \beta\left(\frac{1}{3}\right) = \frac{5}{6},\ \gamma\left(\frac{1}{3}\right) = \frac{1}{12}.$$

In $\left(\frac{1}{3}, \frac{5}{12}\right)$ cluster on contour C_1 does not move, and other clusters move. While $t \in \left(\frac{1}{3}, \frac{5}{12}\right)$

$$h_{1,2}(t) = h_{2,1}(t) = h_{3,1}(t) = h_{2,3}(t) = h_{3,2}(t) = 0,$$

$$h_{1,3}(t) = \frac{1}{12} - \left(t - \frac{1}{3}\right),$$

$$h_{2,1}(t) = t - \frac{1}{3},$$

$$H(t) = \frac{1}{12}.$$

At time $t = \frac{5}{12}$ the system is in state

$$\alpha\left(\frac{5}{12}\right) = \frac{1}{2},\ \beta\left(\frac{5}{12}\right) = \frac{11}{12},\ \gamma\left(\frac{5}{12}\right) = \frac{1}{6},$$

$$h_{1,2}\left(\frac{5}{12}\right) = h_{2,1}\left(\frac{5}{12}\right) = h_{1,3}\left(\frac{5}{12}\right) = h_{3,1}\left(\frac{5}{12}\right) = h_{2,3}\left(\frac{5}{12}\right) =$$

$$= h_{3,2}\left(\frac{5}{12}\right) = 0,$$

$$h_{2,1}\left(\frac{5}{12}\right) = \frac{1}{12},$$

$$H(t) = \frac{1}{12}.$$

In time interval $\left(\frac{5}{12}, \frac{1}{2}\right)$ the system is at time $t = \frac{1}{2}$ in state

$$\alpha\left(\frac{5}{12}\right) = \frac{7}{12},\ \beta\left(\frac{5}{12}\right) = 0,\ \gamma\left(\frac{5}{12}\right) = \frac{1}{4}.$$

In time interval $\left(\frac{1}{2}, \frac{7}{12}\right)$ cluster on contour C_1 does not move, while at $t \in \left(\frac{1}{2}, \frac{7}{12}\right)$

$$h_{1,2}(t) = h_{1,3}(t) = h_{3,1}(t) = h_{2,3}(t) = h_{3,2}(t) = 0,$$

$$h_{2,1}(t) = \frac{1}{12} - \left(t - \frac{1}{2}\right).$$

We have

$$h_{1,2}\left(\frac{7}{12}\right) = h_{2,1}\left(\frac{7}{12}\right) = h_{1,3}\left(\frac{7}{12}\right) = h_{3,1}\left(\frac{7}{12}\right) = h_{2,3}\left(\frac{7}{12}\right) =$$

$$= h_{3,2}\left(\frac{7}{12}\right) = 0,$$

$$H\left(\frac{7}{12}\right) = 0.$$

Thus, *delay potential value equals* $\frac{1}{12}$ *from initial time to time* $t = \frac{1}{2}$*, and delay potential linear decreases with velocity 1 in time interval* $t \in \left(\frac{1}{2}, \frac{7}{12}\right)$ *to 0, and it reserved value 0 in all next time.*

At time $t = \frac{7}{12}$ the system is in the state

$$\alpha_1\left(\frac{7}{12}\right) = \frac{7}{12},\ \alpha_2\left(\frac{7}{12}\right) = 0,\ \alpha_3\left(\frac{7}{12}\right) = \frac{1}{4},$$

that is free movement state.

Thus, according to the above, following is true.

Depending on the value of cluster length l we have

1) If

$$l \le \frac{1}{6},$$

then cluster velocity equals to 1.

2) If

$$\frac{1}{6} < l < \frac{1}{2},$$

then, depending on the initial state, the average clusters velocity is equal to 1 or $\frac{4}{3+6l}$

3) If

$$l > \frac{1}{2},$$

then cluster velocity equals to 0.

5.3. Hypotheses about velocity spectrum for closed chain consisting of arbitrary number of contours

We formulate hypotheses about the behavior of system that analogical to considered system, but contour number is arbitrary and equals to n.

Hypothesis 1.

1) The system results in free movement state over finite time, if number n is odd, and condition

$$l < \frac{1}{n}$$

fulfils, od n is even and

$$l < \frac{1}{2n}.$$

2) For given $n \geq 3$ and l, that satisfy the condition

$$\frac{1}{n} < l \leq \frac{1}{2},$$

if n is even, or condition

$$\frac{1}{2n} < l \leq \frac{1}{2},$$

if n is odd, in the spectrum of possible values of average velocity for various initial states of the system, except for value 1, there contains no more than $\left[\frac{n}{3}\right]$ values , such that less than 1 (square brackets denote the whole part of a number).

These values are calculated as follows.

Let n be even number; k_0 be maximal natural number k, such that

$$k < nl. \tag{23}$$

Then all possible value of average velocity

$$v = \frac{\frac{n}{2} + k}{\frac{n}{2} + nl},$$

where k is natural number satisfying the inequalities (23) and

$$k \geq k_0 - \left[\frac{n}{3}\right] + 1.$$

Let n — ; k_0 be maximal natural number k, satisfying condition

$$k < nl + \frac{1}{2}. \tag{24}$$

Then all possible value of average velocity

$$v = \frac{\frac{n}{2} + k - \frac{1}{2}}{\frac{n}{2} + nl},$$

where k is natural number satisfying the inequalities (24) and

$$k \geq k_0 - \left[\frac{n}{3}\right] + 1.$$

5.4. The behavior of closed chain with 4 contours

In the case of closed chain with 4 contours, we have results of computer simulation modeling and formulate the following.

1) If

$$l \leq \frac{1}{4},$$

then the system results in a state of self-organization over finite time.

2) If

$$\frac{1}{4} < l \leq \frac{1}{2},$$

then, depending on the initial state, the average cluster velocity is equal to 1 or $\frac{3}{2+4l}$,

3) If

$$l > \frac{1}{2}.$$

then the system results in a state of collapse.

Thus, hypothesis 1 is confirmed.

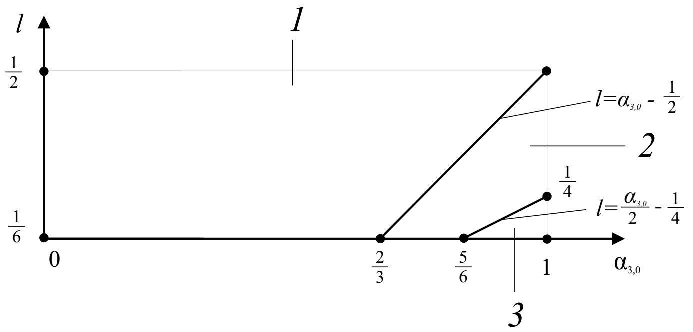

Figure 3: Fig3

6. Conclusion

A deterministic dynamical system is introduced and considered. It is a closed chain of contours, on which there are movement of clusters with length l in accordance with the specified rules.

It is found that, if $l \leq 1/6$, then self-organization takes place. i.e., the system results in free movement state over finite time from any initial state. For $l > 1/2$, the system results in in collapse state over a finite time. And for any value of l, satisfying the condition $1/6 < l \leq 1/2$, the spectrum of possible values of average velocity contains two possible values: o 1 and $4/(3 + 6l)$. While what value will average velocity take, depends on the initial system state. We have developed method to to analyze the system behavior. The concept of delay potential is introduced. The properties of delay potential function are studied. In particular, it is proved that the delay potential is a nonincreasing function of time and has a value 0, when the system results in free movement state.

Possible generalizations of the results to a closed chain with arbitrary number of contours are considered. Hypothesis about the average clusters velocity and condition of free movement state of the system is discussed.

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