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The Proofs of product inequalities in a generalized vector space

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Abstract. In this paper, we introduce the proofs of product inequalities:
$\|u\|\|v\| \leq \|u\| + \|v\|$, for all $u, v \in [0,2]$, and $\|u\| + \|v\| \leq \|u\|\|v\|$, for all $u, v \in [2, \infty)$. By applying the first product inequality to the L^p spaces, we observed that if $f : \Omega \to [0,1]$, and $g : \Omega \to \mathbf{R}$, then $\|f\|_p \|g\|_p \leq \|f\|_p + \|g\|_p$. Also, if $f, g : \Omega \to \mathbf{R}$, then $\|f\|_p + \|g\|_p \leq \|f\|_p \|g\|_p$.

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1. Introduction

Inequalities are inevitable tools in the mathematical analysis as they provide bases for sound arguments. Due to the enormous applications of inequalities, most researchers are shifting to this line of research by introducing new inequalities. From the historical standpoint, the triangle inequality was first discovered, see [1]. Since then many researchers across the globe had obtained different ways of proving triangle inequality. For example, see a research paper by authors in [2]. After the discovery of triangle inequality the so-called arithmetic-geometric mean AGM inequality:

$$\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \forall \ n \in \mathbf{N} \ and \ x_i \in \mathbf{R}^+,$$

with equality occurs when $x_1 = x_2 = \ldots = x_n$, was observed. Specifically, for any two positive real numbers a and b the AGM inequality becomes

$$ab \leq \left(\frac{a+b}{2}\right)^2,$$

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see [3]. Several ways of proving the AGM inequality have been observed, see research papers by authors in [4] and [5].

However, the author in [6] proved that the sum of vector points and one is less than the product of sum of one and the vector points:

$$1+\sum_{i=1}^{n} x_i < \Pi_{i=1}^{n}(1+x_i).$$

The AGM inequality has been applied to establish the relationships between areas of plane figures and their perimeters of what is called isoperimetric inequality. For example, see [7] . Although the inequality of two real numbers has been in history for long time with many researchers looking at inequalities involving products of real numbers. Newton showed that the square of a real number between the first and third consecutive real numbers is greater than their product

$$P_{r-1}P_{r+1} < P_r^2, \quad \forall \ 1 \leq r < n,$$

see [8]. In [9], the author obtained another way of proving the Weierstrass inequality by applying AGM inequality and also, extended the Weierstrass inequalities by the use of majorization. The author in [10], generalized the Weierstrass inequality in the Euclidean space. Inequalities involving functions have received much attention in the 21^{st} century. Jordan as cited in [11] introduced fractional inequality:

$$\frac{2}{\pi} \leq \frac{\sin(x)}{x} < 1, \qquad 0 < x \leq \frac{\pi}{2}.$$

Hilbert constructed double series inequality:

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi\Big(\sum_{m=1}^{\infty} a_m^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\infty} b_m^2\Big)^{\frac{1}{2}},$$

see [12].

Unlike AGM inequality, the product of two positive real numbers and their sum without any factor multiplier, in Euclidean space, delineate unique inequalities within certain intervals. In this paper, we provide two inequalities in the interval $[0,\infty)$; the first product inequality $\|u\|\|v\| \leq \|u\|+\|v\|$ holds for all values of u and v in the interval $[0,2]$, and the second product inequality $\|u\|+\|v\| \leq \|u\|\|v\|$ holds for all u and v in interval $[2,\infty)$. Each of these inequalities is proved by induction. The section 1 contains the introduction. The section 2 contains the proofs of the first product inequality in generalized linear space and the second product inequality in the linear space. In section 3, we illustrated the first product inequality to a real line, norms and trigonometric functions. We extend and also, apply the second product inequality to the L^p spaces in section 4 of this paper. In section 5, we discuss our main findings and summarize these results in section 6.

2. Main Result

In this section, the two product inequalities are introduced: the first product inequality

$$\|u\|\|v\| \leq \|u\| + \|v\|, \quad \forall\, u, v \in [0, 2], \tag{1}$$

and the second product inequality,

$$\|u\| + \|v\| \leq \|u\|\|v\|, \quad \forall\, u, v \in [0, 2]. \tag{2}$$

Before we proceed to provide the proofs of the product inequalities some of relevant derintions regarding the introduction of product inequalities are given in this paper.

Definition 1 (Inner product). *Let V be a linear vector space defined over the real number field $\mathbf{R}$. A scalar-valued function $p : V \times V \to \mathbf{R}$ that associates with each pair u, v of vectors in V a scalar, denoted (u, v), is called an inner product on V if and only if*

(i) $(u, u) > 0$ *whenever* $u \neq 0$, *and* $(u, u) = 0$ *if and only if* $u = 0$

(ii) $(u, v) = (v, u)$, $\forall\, u,\ v \in V$

(iii) $(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v)$, $\forall\, \alpha,\ \beta \in \mathbf{R}$, *and* $u_1, u_2, v \in V$, *see* [13].

Definition 2 (Norm). *Let V be a linear space over $\mathbf{R}$. A norm on V is a real-valued function*

$$\|\cdot\| : V \to [0, \infty)$$

such that for any $u, v \in V$ and $\alpha \in \mathbf{R}$ the following conditions are met:

$$\begin{aligned} \|u\| &\geq 0,\ \text{and}\ \|u\| = 0,\ iff\ u = 0 \\ \|\alpha u\| &= |\alpha|\|u\|,\ \forall\, u \in V\ \text{and}\ \alpha \in \mathbf{R} \\ \|u \pm v\| &\leq \|u\| + \|v\|,\ \forall\, u, v \in V \end{aligned}$$

The norm of a vector u can be generated by the inner product $(,)$

$$\|u\| = \sqrt{(u, u)}.$$

See [14].

Definition 3 (Cauchy-Schwarz inequality). *Let E be an inner product space. Then*

$$|\langle u, v\rangle|^2 \leq \|x\|.\|y\|, \quad \forall\, x, y \in E.$$

The equality holds if and only if x and y are linearly dependent, see [15].

Definition 4 (boundedness)**.** *Let $T : X \to Y$ be a linear map. Then T is said to be a bounded linear map if there exists $K \geq 0$ such that*

$$\|Tx\| \leq K\|x\|, \quad \forall\, x \in X,$$

and K is the boundedness constant for T. The boundedness of the linear map implies continuity of T. See [16]

Definition 5 (Integral Operator and Monotonicity of the Integral Operator)**.** *Let $f(x)$ and $g(x)$ be simple functions defined on a set of finite measure E. Then for any α and β,*

$$\int_E (\alpha f(x) + \beta g(x))dx = \alpha \int_E f(x)dx + \beta \int_E g(x)dx, \;\; \forall \;\; f, g \in E.$$

and α, β are scalars. Moreover, if $f(x) \leq g(x)$, then

$$\int_E f(x)dx \leq \int_E g(x)dx,$$

see [17].

2.1. The Proof of the First Product Inequality

We provide the proofs of product inequalities, through binomial inequalities, by induction. Firstly, we consider a positive integer $n = 2$ as follows.

$$\begin{aligned}
(u+v)^2 &\geq 0 \\
(u,u) + 2(u,v) + (v,v) &\geq 0 \\
-2(u,v) &\leq \{(u,u) + (v,v)\} \\
-2(u,v) &\leq (u,u) + (v,v) + 2(u,v) \\
-2(u,v)\frac{1}{2}(u,v) &\leq (u+v)^2 \\
-(u,v)^2 &= (u+v)^2 \\
\| -(u,v)^2\| &= \|(u+v)^2\| \\
\|(u,v)\|^2 &\leq \|(u+v)\|^2 \\
\Rightarrow \|u\|\|v\| &\leq \|u\| + \|v\|.
\end{aligned}$$

For $n = 4$, we observe the following inequalities:

$$\begin{aligned}
(u+v)^4 &\geq 0 \\
-6(u,u)(v,v) &\leq \{(u,u)^2 + 4(u,u)(u,v) + 4(u,v)(v,v) + (v,v)^2\} \\
-6(u,u)(v,v) &\leq \{(u,u)^2 + 4(u,u)(u,v) + 6(u,u)(v,v) + 4(u,v)(v,v) + (v,v)^2\} \\
-6(u,u)(v,v).\frac{1}{6}(u,u)(v,v) &\leq (u+v)^4 \\
-\{(u,u)(v,v)\}^2 &= (u+v)^4
\end{aligned}$$

$$\begin{aligned}
\|-\{(u,u)(v,v)\}^2\| &= \|(u+v)^4\| \\
\{\|u\|\|v\|\}^4 &= \|(u+v)\|^4 \\
\Rightarrow \|u\|\|v\| &\leq \|u\|+\|v\|.
\end{aligned}$$

In order to generalize the first product inequality, we observed that for any positive integer n, we obtain:

$$\begin{aligned}
& (u+v)^n \geq 0 \\
\Rightarrow\ & (u,u)^{\frac{n}{2}} +^n C_1(u,v)(u,u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v) +^n C_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v) \\
+\ & {}^nC_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v) +^n C_6(u,u)^{\frac{n-6}{2}}(v,v)^3 \\
+\ & {}^nC_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots +^n C_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} + \ldots + (u,u)^{\frac{n}{2}} \geq 0 \\
\Rightarrow\ & -^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} \leq \{(u,u)^{\frac{n}{2}} +^n C_1(u,v)(u,u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v) \\
+\ & {}^nC_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v) +^n C_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v) \\
+\ & {}^nC_6(u,u)^{\frac{n-6}{2}}(v,v)^3 +^n C_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots + (u,u)^{\frac{n}{2}}\} \\
\Rightarrow\ & -^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} \leq \Big\{(u,u)^{\frac{n}{2}} +^n C_1(u,v)(u,u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v) \\
+\ & {}^nC_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v) +^n C_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v) \\
+\ & {}^nC_6(u,u)^{\frac{n-6}{2}}(v,v)^3 +^n C_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots +^n C_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} + \ldots + (u,u)^{\frac{n}{2}}\Big\} \\
\Rightarrow\ & -^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}}\frac{1}{{}^nC_{\frac{n}{2}}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} \leq (u+v)^n \\
\Rightarrow\ & \|-(u,u)^{\frac{n}{2}}(v,v)^{\frac{n}{2}}\| = \|(u+v)^n\| \\
\Rightarrow\ & \{\|u\|\|v\|\}^n = \|(u+v)\|^n \\
\Rightarrow\ & \|u\|\|v\| \leq \|u\|+\|v\|, \quad \forall\ u,v \in [0,2]. \qquad (3)
\end{aligned}$$

2.2. The Proof of the Second Product Inequality

In this subsection, we show that the sum of norms of two vectors less than or equal to the norms of product of their vectors. Setting $n=2$, we obtain:

$$\begin{aligned}
(u+v)^2 &\geq 0 \\
(u,u)+2(u,v)+(v,v) &\geq 0 \\
-\{(u,u)+(v,v)\} &\leq 2(u,v) \\
-\{(u,u)+(v,v)+2(u,v)\} &\leq 2(u,v) \\
-\{(u,u)+(v,v)+2(u,v)\} &\leq 2(u,v).\frac{1}{2}(u,v) \\
-(u+v)^2 &= (u,v)^2 \\
\|-(u+v)^2\| &= \|(u,v)^2\| \\
\|(u+v)\|^2 &= \|(u,v)\|^2
\end{aligned}$$

$$\Rightarrow \|u\| + \|v\| \quad \leq \quad \|u\|\|v\|.$$

Also, we set $n = 4$, which gives as the following result:

$$\begin{aligned}
(u+v)^4 &\geq 0\\
-\{(u,u)^2 + 4(u,u)(u,v) + 4(u,v)(v,v) + (v,v)^2\} &\leq 6(u,u)(v,v)\\
-\{(u,u)^2 + 4(u,u)(u,v) + 6(u,u)(v,v) + 4(u,v)(v,v) + (v,v)^2\} &\leq 6(u,u)(v,v)\\
-(u+v)^4 &\leq 6(u,u)(v,v).\frac{1}{6}(u,u)(v,v)\\
-(u+v)^4 &= \{(u,u)(v,v)\}^2\\
\|-(u+v)^4\| &= \|\{(u,u)(v,v)\}^2\|\\
\|(u+v)\|^4 &= \{\|u\|\|v\|\}^4\\
\Rightarrow \|u\| + \|v\| &\leq \|u\|\|v\|.
\end{aligned}$$

Similarly, we generalize the second product inequality by observing the following:

$$\begin{aligned}
& (u+v)^n \geq 0\\
\Rightarrow\quad & (u,u)^{\frac{n}{2}} +^n C_1(u,v)(u.u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v) +^n C_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v)\\
+\quad & {}^nC_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v) +^n C_6(u,u)^{\frac{n-6}{2}}(v,v)^3\\
+\quad & {}^nC_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots +^n C_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} + \ldots + (u,u)^{\frac{n}{2}} \geq 0\\
\Rightarrow\quad & -\{(u,u)^{\frac{n}{2}} +^n C_1(u,v)(u,u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v)\\
+\quad & {}^nC_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v) +^n C_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v)\\
+\quad & {}^nC_6(u,u)^{\frac{n-6}{2}}(v,v)^3 +^n C_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots + (u,u)^{\frac{n}{2}}\}\\
\leq\quad & {}^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}}\\
\Rightarrow\quad & -\Big\{(u,u)^{\frac{n}{2}} +^n C_1(u,v)(u,u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v)\\
+\quad & {}^nC_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v) +^n C_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v)\\
+\quad & {}^nC_6(u,u)^{\frac{n-6}{2}}(v,v)^3 +^n C_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots +^n C_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} + \ldots + (u,u)^{\frac{n}{2}}\Big\}\\
\leq\quad & {}^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}}\\
\Rightarrow\quad & -(u+v)^n \leq^n C_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}}\frac{1}{{}^nC_{\frac{n}{2}}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}}\\
\Rightarrow\quad & \|-(u+v)^n\| = \|(u,u)^{\frac{n}{2}}(v,v)^{\frac{n}{2}}\|\\
\Rightarrow\quad & \|(u+v)\|^n = \{\|u\|\|v\|\}^n\\
\Rightarrow\quad & \|u\| + \|v\| \leq \|u\|\|v\|, \quad \forall\ u, v \in [2,\infty).
\end{aligned} \tag{4}$$

3. Illustration of the First Product Inequality

In this section of the paper, we illustrate some of the aspects of mathematics where the first product inequality is feasible. Notwithstanding, the areas of applications of the first product inequality are highlighted.

3.1. Illustration of the First Product Inequality to the Real Line

Firstly, we illustrate the first product inequality in (1) to any two real numbers $a, b \in [0, 2]$.

Example 1. *Let* $a = \frac{5}{6}$ *and* $b = \frac{13}{18}$*, then*

$$\begin{aligned} \left\|\frac{5}{6}\right\|\left\|\frac{13}{18}\right\| &\leq \left\|\frac{5}{6}\right\| + \left\|\frac{13}{18}\right\| \\ \frac{65}{108} &< \frac{28}{18}. \end{aligned}$$

Example 2. *Let* $a = \frac{-1}{2}$ *and* $b = \frac{-3}{5}$*, then*

$$\begin{aligned} \left\|\frac{-1}{2}\right\|\left\|\frac{-3}{5}\right\| &\leq \left\|\frac{-1}{2}\right\| + \left\|\frac{-3}{5}\right\| \\ \frac{3}{10} &< \frac{11}{10}. \end{aligned}$$

Example 3. *Let* $a = \frac{19}{10}$ *and* $b = \frac{9}{5}$*, then*

$$\begin{aligned} \left\|\frac{19}{10}\right\|\left\|\frac{9}{5}\right\| &\leq \left\|\frac{19}{10}\right\| + \left\|\frac{9}{10}\right\| \\ \frac{171}{50} &< \frac{37}{10}. \end{aligned}$$

3.2. Illustration of the First Product Inequality to the Euclidean Space

Again, the first product inequality is illustrated to the vector points of real numbers as follows. In this paper, three basic norms: $1-$ norm, $2-$ norm and ∞-norm are considered.

Example 4. *Let* $a = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}$ *and* $b = \begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}$*, then*

$$\begin{aligned} \left\|\begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}\right\|_1 \left\|\begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}\right\|_1 &\leq \left\|\begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}\right\|_1 + \left\|\begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}\right\|_1 \\ \Rightarrow \frac{119}{90} &< \frac{69}{30}. \end{aligned}$$

Example 5. *Let* $a = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}$ *and* $b = \begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}$*, then*

$$\begin{aligned} \left\|\begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}\right\|_2 \left\|\begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}\right\|_2 &\leq \left\|\begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}\right\|_2 + \left\|\begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}\right\|_2 \\ \Rightarrow \frac{13}{18} &< \frac{28}{18}. \end{aligned}$$

Example 6. *Let* $a = \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix}$ *and* $b = \begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix}$, *then*

$$\begin{aligned} \left\| \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix} \right\|_\infty \left\| \begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix} \right\|_\infty &\leq \left\| \begin{pmatrix} \frac{1}{2} \\ \frac{2}{3} \end{pmatrix} \right\|_\infty + \left\| \begin{pmatrix} \frac{4}{5} \\ \frac{-1}{3} \end{pmatrix} \right\|_\infty \\ \Rightarrow \frac{8}{15} &< \frac{22}{15}. \end{aligned}$$

3.3. Illustration of the First Product Inequality to Trigonometric Functions

Undoubedly, we construct the inequalities involving the product of sine of an angle and cosine of an angle. We can see that:

$$\|\sin(\theta)\| \leq 1,$$

and

$$\|\cos(\theta)\| \leq 1.$$

By induction, we start at $n = 0$, by setting

$$\Big(\sin(\theta)\cos(\theta)\Big)^0$$

Taking the magnitude of $\Big(\sin(\theta)\cos(\theta)\Big)^0$ and applying the Cauchy-Schwarz inequality to it, we obtain:

$$\Big(|\langle\sin(\theta), \cos(\theta)\rangle|\Big)^0 \leq \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^0. \tag{5}$$

Applying the first product inequality to the expression on the right hand side of equation (5) yields

$$\begin{aligned} \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^0 &\leq \Big(\|\sin(\theta)\| + \|\cos(\theta)\|\Big)^0 \\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^0 &= (1+1)^0 \\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^0 &= 2^0 \end{aligned}$$

For $n = 1$, the following result is obtained:

$$\begin{aligned} &\Big(\sin(\theta)\cos(\theta)\Big)^1 \\ \Rightarrow \Big(|\langle\sin(\theta), \cos(\theta)\rangle|\Big)^1 &\leq \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^1. \end{aligned} \tag{6}$$

Applying the first product inequality to the expression on the right hand side of equation (6) yields

$$\begin{aligned}\Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^1 &\le \Big(\|\sin(\theta)\|+\|\cos(\theta)\|\Big)^1\\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^1 &= (1+1)^1\\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^1 &< 2^1\end{aligned}$$

Similarly, we observed that when $n = 2$, the following expression is obtained:

$$\begin{aligned}&\Big(\sin(\theta)\cos(\theta)\Big)^2\\ \Rightarrow &\Big(|\langle\sin(\theta),\cos(\theta)\rangle|\Big)^2 \le \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^2\end{aligned} \tag{7}$$

Applying the first product inequality to the expression on the right hand side of equation (7) yields

$$\begin{aligned}\Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^2 &\le \Big(\|\sin(\theta)\|+\|\cos(\theta)\|\Big)^2\\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^2 &= (1+1)^2\\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^2 &< 2^2\end{aligned}$$

Generalizing the product inequality of $\sin(\theta)$ and $\cos(\theta)$, we observed that for any integer $n \in [0,\infty)$, we obtain:

$$\begin{aligned}&\Big(\sin(\theta)\cos(\theta)\Big)^n\\ \Rightarrow &\Big(|\langle\sin(\theta),\cos(\theta)\rangle|\Big)^n \le \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^n.\end{aligned} \tag{8}$$

Applying the first product inequality to the expression on the right hand side of equation (6), we obtain:

$$\begin{aligned}\Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^n &\le \Big(\|\sin(\theta)\|+\|\cos(\theta)\|\Big)^n\\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^n &= (1+1)^n\\ \Big(\|\sin(\theta)\|\|\cos(\theta)\|\Big)^n &\le 2^n.\end{aligned} \tag{9}$$

Again, the first product inequality is applied to functions which give the following results.

Theorem 1. *Suppose that f be the space of all measurable functions $f : [0,2] \to \mathbf{R}$, and $g : \Omega \to [0,1]$, then following inequalities hold:*

$$\|e^x\|\|\sin(\theta)\| \le \|e^x\| + \|\sin(\theta)\|$$

$$\begin{aligned}
\|e^x\|\|\cos(\theta)\| &\leq \|e^x\| + \|\cos(\theta)\| \\
\|\tan(\theta)\|\|\sin(\theta)\| &\leq \|\tan(\theta)\| + \|\sin(\theta)\| \\
\|\tan(\theta)\|\|\cos(\theta)\| &\leq \|\tan(\theta)\| + \|\cos(\theta)\| \\
\|sec(\theta)\|\|\sin(\theta)\| &\leq \|sec(\theta)\| + \|\sin(\theta)\| \\
\|cosec(\theta)\|\|\cos(\theta)\| &\leq \|cosec(\theta)\| + \|\cos(\theta)\| \\
\|cot(\theta)\|\|\cos(\theta)\| &\leq \|cot(\theta)\| + \|\cos(\theta)\|
\end{aligned}$$

We observed that the norms of the trigonometric functions: $\sin(\theta)$ and $\cos(\theta)$ are seen as delayed growth norms.

Definition 6. *If u and v are any two vectors in the Euclidean space, then*

$$\begin{aligned}
|u+v| &\leq \|u\| + \|v\| \leq 2\max\{|u|, |v|\} \\
\Rightarrow |u+v|^p &\leq 2^p\{|u|^p + |v|^p\},
\end{aligned}$$

see [17].

With the above inequalities, the first product inequality is extended to $p-$normed spaces as follows:

Theorem 2. *If p is any positive integer, then*

$$\|u\|^p\|v\|^p \leq 2^p\{\|u\|^p + \|v\|^p\}. \tag{10}$$

Proof: We see from the above inequality that:

$$|u+v| \leq \|u\| + \|v\|$$

Raising the expression on the both sides of inequality to the power p, we get:

$$\begin{aligned}
|u+v|^p &\leq \Big|\|u\| + \|v\|\Big|^p \\
\Rightarrow |u+v|^p &\leq \|u\|^p + \|v\|^p \\
\Rightarrow |u+v|^p &\leq 2^p\{\|u\|^p + \|v\|^p\} \\
\Rightarrow \|u\|^p + \|v\|^p &\leq 2^p\{\|u\|^p + \|v\|^p\}.
\end{aligned}$$

Substituting inequality in (3) into the inequality in (10) yields

$$\begin{aligned}
|\|u\|\|v\||^p &\leq 2^p\{|u|^p + |v|^p\} \\
\|u\|^p\|v\|^p &\leq 2^p\{\|u\|^p + \|v\|^p\}, \quad \forall\, u, v \in [0,2]
\end{aligned}$$

Theorem 3. *If u and v are any two vectors in the Euclidean space, then*

$$\|u\|^p\|v\|^p \leq \|u\|^p + \|v\|^p, \quad \forall\ \|u\|,\ \|v\| \in [0,2].$$

Proof: The result in theorem (3) follows from theorem (2).

Theorem 4. *Suppose that at least one of the measurable functions over the domain* Ω *is such that*

$$f : \Omega \to [0, 1]$$

and the other measurable function

$$g : \Omega \to \mathbf{R},$$

then

$$\|fg\|_p \le \|f\|_p + \|g\|_p.$$

Proof:

$$\Big| \int_\Omega f(x)g(x)dx\Big|^p \le \int_\Omega \Big|f(x)g(x)\Big|^p dx$$

Applying the first product inequality, we obtain

$$\begin{aligned}
\Big| \int_\Omega f(x)g(x)dx\Big|^p &\le \int_\Omega \Big\{\Big|f(x)|^p + |g(x)\Big|^p\Big\}dx \\
\Rightarrow \Big(\Big| \int_\Omega f(x)g(x)dx\Big|^p\Big)^{\frac{1}{p}} &= \Big(\int_\Omega \Big|f(x)|^p dx + \int_\Omega |g(x)\Big|^p dx\Big)^{\frac{1}{p}} \\
\Rightarrow \Big(\Big| \int_\Omega f(x)g(x)dx\Big|^p\Big)^{\frac{1}{p}} &\le \Big(\int_\Omega \Big|f(x)|^p dx\Big)^{\frac{1}{p}} + \Big(\int_\Omega |g(x)\Big|^p dx\Big)^{\frac{1}{p}} \\
\Rightarrow \|fg\|_p &\le \|f\|_p + \|g\|_p. \qquad (11)
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we observe that:

$$\|fg\|_p \le \|f\|_p\|g\|_p. \qquad (12)$$

Substituting inequality (11) into inequality (12) yields

$$\|f\|_p\|g\|_p \le \|f\|_p + \|g\|_p.$$

We show how theorem (4) is applied to estimate two smooth functions defined on domains.

Theorem 5. *Suppose that* $f(x), g(x)$ *are* $C^\infty(\Omega)$ *such that* $f : \Omega \to [0, 1]$ *and* $g : \Omega \to [0, 1]$*, then the following inequality holds:*

$$\Big| \int_a^b \cos(nx)\sin(nx)dx\Big| \le 2(b-a).$$

Proof:

$$\begin{aligned}
\Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &\leq \int_a^b \Big|\cos(nx)\sin(nx)\Big|dx \\
\Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &\leq \int_a^b \Big|\cos(nx)\Big|\Big|\sin(nx)\Big|dx
\end{aligned}$$

Applying the first product inequality, we obtain:

$$\begin{aligned}
\Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &\leq \int_a^b \{\Big|\cos(nx)\Big| + \Big|\sin(nx)\Big|\}dx \\
\Rightarrow \Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &= \int_a^b \Big|\cos(nx)\Big|dx + \int_a^b \Big|\sin(nx)\Big|dx \\
\Rightarrow \Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &\leq \int_a^b 1dx + \int_a^b 1dx \\
\Rightarrow \Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &= \Big[x\Big]_a^b + \Big[x\Big]_a^b \\
\Rightarrow \Big|\int_a^b \cos(nx)\sin(nx)dx\Big| &\leq 2(b-a).
\end{aligned}$$

4. Extension and applications of the Second Product Inequality to the L^p Spaces

In this section, we apply the second product inequality to L^p spaces.

Theorem 6. *If u and v are any two vectors in the Euclidean space, then*

$$\|u\|^p + \|v\|^p \leq \|u\|^p\|v\|^p, \quad \forall\, u, v \in [2, \infty)$$

Proof: We can see that:

$$\begin{aligned}
\|u\| + \|v\| &\leq \|u\|\|v\| \\
\Rightarrow \Big(\|u\| + \|v\|\Big)^p &\leq \Big(\|u\|\|v\|\Big)^p. \qquad (13)
\end{aligned}$$

Also, we observed that:

$$\|u\|^p + \|v\|^p \leq \Big(\|u\| + \|v\|\Big)^p. \qquad (14)$$

By the transitivity of the inequalities (13) and (14), we obtain

$$\begin{aligned}
\|u\|^p + \|v\|^p &\leq \Big(\|u\|\|v\|\Big)^p \\
\|u\|^p + \|v\|^p &\leq \|u\|^p\|v\|^p.
\end{aligned}$$

Theorem 7. *Suppose that the measurable functions over the domain* Ω *are such that*

$$f : \Omega \to \mathbf{R}$$

and the other measurable function

$$g : \Omega \to \mathbf{R},$$

then

$$\|f\|_p + \|g\|_p \leq \|f\|_p \|g\|_p.$$

Proof:

$$\Big| \int_\Omega f(x)g(x)dx \Big|^p \leq \int_\Omega \Big| f(x)g(x) \Big|^p dx$$

Applying the second product inequality, we obtain

$$\begin{aligned} \int_\Omega \Big(|f(x)| + |g(x)| \Big) dx &\leq \int_\Omega |f(x)g(x)| dx \\ \Rightarrow \int_\Omega \Big(|f(x)| + |g(x)| \Big)^p dx &= \int_\Omega \Big(|f(x)g(x)| \Big)^p dx. \end{aligned} \tag{15}$$

We see that:

$$\int_\Omega |f(x)|^p + |g(x)|^p dx \leq \int_\Omega \Big(|f(x)| + |g(x)| \Big)^p dx. \tag{16}$$

Applying the transitive law to inequalities (15) and (16), we obtain:

$$\begin{aligned} \int_\Omega |f(x)|^p + |g(x)|^p dx &\leq \int_\Omega \Big(|f(x)g(x)| \Big)^p dx \\ \Rightarrow \int_\Omega |f(x)|^p + |g(x)|^p dx &= \int_\Omega |f(x)|^p |g(x)|^p dx \\ \Rightarrow \Big(\int_\Omega |f(x)|^p + |g(x)|^p dx \Big)^{\frac{1}{p}} &= \Big(\int_\Omega |f(x)|^p |g(x)|^p dx \Big)^{\frac{1}{p}} \\ \Rightarrow \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}} + \Big(\int |g(x)|^p dx \Big)^{\frac{1}{p}} &\leq \Big(\int_\Omega |f(x)|^p dx \Big)^{\frac{1}{p}} \Big(\int |g(x)|^p dx \Big)^{\frac{1}{p}} \\ \Rightarrow \|f\|_p + \|g\|_p &\leq \|f\|_p \|g\|_p. \end{aligned}$$

5. Discussion

We deduce from (9) that, applying the first product inequality to the trigonometric functions $\sin(\theta)$ and $\cos(\theta)$, another new inequality $1 \leq 2^n, \ \forall n = 0, 1, 2, \dots$ is uncovered. The first and second product inequalities have given birth to new inequalities in the generalized space. These new inequalities give additional information about embeddings of one space into another space.

6. Conclusion

In summary, we observed that the first product inequality holds for any two vectors in a genearalized linear space. On the other hand, the second product inequality holds for any two vectors only in the Euclidean space. In addition, by applying the first product inequality to the L^p spaces, we observed that if, $f : \Omega \to [0,1]$, and $g : \Omega \to \mathbf{R}$, then $\|f\|_p\|g\|_p \leq \|f\|_p + \|g\|_p$. But if, $f, g : \Omega \to \mathbf{R}$, then $\|f\|_p + \|g\|_p \leq \|f\|_p\|g\|_p$. Last but not the least, we have shown in this paper that the first product inequality holds for $C^\infty[0,1]$.

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