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Invited Paper

Some Theta-Function Identities Related to Jacobi's Triple-Product Identity

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Abstract. The main object of this paper is to present some q-identities involving some of the theta functions of Jacobi and Ramanujan. These q-identities reveal certain relationships among three of the theta-type functions which arise from the celebrated Jacobi's triple-product identity in a remarkably simple way. The results presented in this paper are motivated by some recent works by Chaudhary *et al.* (see [4] and [5]) and others (see, for example, [1] and [13]).

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1. Introduction, Definitions and Preliminaries

As long ago as 1829, Carl Gustav Jacob Jacobi (1804-1851) introduced a set of four theta functions $\vartheta_j(z,q)$ $(j = 1,2,3,4)$, which we recall here in the following forms (see [9] and [15, pp. 463 *et seq.*]; see also [12, p. 86]):

$$\vartheta_1(z,q) = -\mathrm{i} \sum_{n=-\infty}^{\infty} (-1)^n \, \mathrm{e}^{(2n+1)\mathrm{i}z}$$

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$$= 2\sum_{n=0}^{\infty}(-1)^n\, q^{\left(\frac{n+\frac{1}{2}}{2}\right)^2}\ \sin[(2n+1)z]$$

$$= 2q^{\frac{1}{4}}\ \sum_{n=0}^{\infty}(-1)^n\, q^{n(n+1)}\ \sin[(2n+1)z], \tag{1}$$

$$\vartheta_2(z,q) = \sum_{n=-\infty}^{\infty} q^{\left(\frac{n+\frac{1}{2}}{2}\right)^2}\ \mathrm{e}^{(2n+1)\mathrm{i}z}$$

$$= 2\sum_{n=0}^{\infty} q^{\left(\frac{n+\frac{1}{2}}{2}\right)^2}\ \cos[(2n+1)z]$$

$$= 2q^{\frac{1}{4}}\ \sum_{n=0}^{\infty} q^{n(n+1)}\ \cos[(2n+1)z], \tag{2}$$

$$\vartheta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2}\ \mathrm{e}^{2n\mathrm{i}z} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\ \cos(2nz) \tag{3}$$

and

$$\vartheta_4(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n^2}\ \mathrm{e}^{2n\mathrm{i}z} = 1 + 2\sum_{n=1}^{\infty}(-1)^n\, q^{n^2}\ \cos(2nz), \tag{4}$$

where $z \in \mathbb{C}$ and $|q| < 1$, $\mathbb{C}$ being the set of complex numbers. We also recall here that, in Chapter 16 of his celebrated Notebooks, Srinivasa Ramanujan (1887-1920) defined the general theta function $\mathfrak{f}(a,b)$ as follows (see, for example, [10] and [11]; see also [1] and [13]):

$$\mathfrak{f}(a,b) := 1 + \sum_{n=1}^{\infty}(ab)^{\frac{n(n-1)}{2}}\ (a^n + b^n)$$

$$= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}}\ b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b,a) \qquad (|ab| < 1), \tag{5}$$

so that, for any integer n, it is easily seen that

$$\mathfrak{f}(a,b) = a^{\frac{n(n+1)}{2}}\ b^{\frac{n(n-1)}{2}}\ \mathfrak{f}\left(a(ab)^n, \frac{b}{(ab)^n}\right) = \mathfrak{f}(b,a). \tag{6}$$

Ramanujan also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by

$$\mathfrak{f}(a,b) = (-a;ab)_\infty\ (-b;ab)_\infty\ (ab;ab)_\infty, \tag{7}$$

or, equivalently, by (see [9])

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \prod_{n=1}^{\infty} \left(1-q^{2n}\right)\left(1+zq^{2n-1}\right)\left(1+\frac{1}{z}\,q^{2n-1}\right)$$
$$= \left(q^2;q^2\right)_\infty \left(-zq;q^2\right)_\infty \left(-\frac{q}{z};q^2\right)_\infty \qquad (|q|<1;\ z\neq 0), \tag{8}$$

which was, in fact, first proved by Carl Friedrich Gauss (1777-1855).

As usual, in the above equations as well as throughout our present investigation, we denote the set of complex numbers by $\mathbb{C}$ and the set of positive integers by $\mathbb{N}$ with, of course, $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. Moreover, for $q, \lambda, \mu \in \mathbb{C}$ $(|q|<1)$, the basic (or q-) shifted factorial $(\lambda;q)_\mu$ is defined by (see, for example, [12, Chapter 3, Section 3.2.1] and [14, pp. 346 *et seq.*])

$$(\lambda;q)_\mu := \prod_{j=0}^{\infty}\left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right) \qquad (|q|<1;\ \lambda,\mu\in\mathbb{C}), \tag{9}$$

so that

$$(\lambda;q)_n := \begin{cases} 1 & (n=0) \\ \prod\limits_{j=0}^{n-1}\left(1-\lambda q^j\right) & (n\in\mathbb{N}) \end{cases} \tag{10}$$

and

$$(\lambda;q)_\infty := \prod_{j=0}^{\infty}\left(1-\lambda\, q^j\right) \qquad (|q|<1;\ \lambda\in\mathbb{C}). \tag{11}$$

The theory of Jacobi's theta functions $\vartheta_j(z,q)$ $(j=1,2,3,4)$, which are defined above by the equations (1) to (4), has a long history and many applications in a wide variety of research fields such as number theory (*especially* in quadratic forms and elliptic functions) and quantum physics. Besides, the subject of q-analysis, which is popularly known as the *quantum analysis*, has its roots in such important areas as (for example) Mathematical Physics, Analytic Number Theory, and the Theory of Partitions. Motivated essentially by the potential for applications of q-series and q-products, we investigate here the following three most interesting functions which are related closely to such entities as Jacobi's theta functions in the equations (1) to (4), Ramanujan's general theta function in (5) and Jacobi's triple-product identity in (7) or (8) (see also [1] and [13]):

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{n(3n-1)}{2}} = (q;q)_\infty$$
$$= \frac{1}{\sqrt{3}}\, q^{-\frac{1}{24}}\, \vartheta_2\left(\frac{\pi}{6}, x^{\frac{1}{6}}\right), \tag{12}$$

$$\begin{aligned}\varphi(q) := \mathfrak{f}(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} &= \frac{(-q;q^2)_\infty \, (q^2;q^2)_\infty}{(q;q^2)_\infty \, (-q^2;q^2)_\infty} \\ &= \vartheta_3(0,q) \end{aligned} \tag{13}$$

and

$$\begin{aligned}\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\ &= \frac{1}{2}\, q^{-\frac{1}{8}} \left[\vartheta_2\big(0,\sqrt{q}\big) - 1\right]. \end{aligned} \tag{14}$$

The main object of the present article is to prove two (presumably new) q-identities which provide interesting relationships among the above-defined three ϑ-type functions $f(-q)$, $\varphi(q)$ and $\psi(q)$, each of which arises from Jacobi's triple-product identity (8) in a remarkably simple way. For more details and further results, the interested reader may be referred to the works presented in [1], [2], [3], [6], [7], [8] and [13].

2. The Main Results

In this section, we begin by expressing the functions $f(-q)$, $\varphi(q)$ and $\psi(q)$ in rising powers of q as follows:

$$\begin{aligned} f(-q) &= 1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}}\right) \\ &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots, \end{aligned} \tag{15}$$

$$\varphi(q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots \tag{16}$$

and

$$\psi(q) = 1 + \sum_{n=1}^{\infty} q^{\frac{n(n+1)}{2}} = 1 + q + q^3 + q^6 + q^{10} + q^{15} + \cdots. \tag{17}$$

We now state our main results as the following Theorem.

Theorem. *Each of the following relationships holds true*:

$$2qf(-q^3)\psi\left(q^9\right) = f\left(-q^6\right)\left[\varphi\left(-q^9\right) - \varphi(-q)\right] \tag{18}$$

and

$$2f(-q)f\left(-q^2\right) = \varphi\left(q^4\right)\left[\psi(q) + q\psi\left(q^9\right)\right] + \varphi\left(q^{36}\right)\left[\psi(q) - 3q\psi\left(q^9\right)\right]$$

$$- 2q\psi(q)\left[\psi\left(q^8\right)+q^8\psi\left(q^{72}\right)\right]-2q^2\psi\left(q^9\right)\left[\psi\left(q^8\right)-3q^8\psi\left(q^{72}\right)\right], \tag{19}$$

where the functions $f(q)$, $\varphi(q)$ *and* $\psi(q)$ *are given by* (12), (13) *and* (14), *respectively.*

Proof. First of all, we shall prove our first q-identity (18). Let $\mathcal{L}_1(q)$ and $\mathcal{R}_1(q)$ denote the left-hand and the right-hand sides of the q-identity (18), respectively. Then, in order to compute the value for $\mathcal{L}_1(q)$, by using (15) (for $q \mapsto q^3$) and (17) (for $q \mapsto q^9$), we have

$$\begin{aligned}\mathcal{L}_1(q) = 2q&\left(1-q^3-q^6+q^{15}+q^{21}-\cdots\right)\\&\cdot\left(1+q^9+q^{27}+q^{54}+q^{90}+q^{135}+q^{189}+q^{252}+\cdots\right),\end{aligned}$$

which, after multiplication and further simplification, yields

$$\begin{aligned}\mathcal{L}_1(q) = 2q &- 2q^4 - 2q^7 + 2q^{10} - 2q^{13} + 2q^{22} + 2q^{25} + 2q^{28}\\&- 2q^{34} - 2q^{37} + 2q^{43} - 4q^{46} + 2q^{49} - 2q^{58} - 2q^{61} - 2q^{64} + 2q^{67}\\&+ 2q^{70} - 2q^{73} + 4q^{76} + 2q^{79} + 2q^{88} - 2q^{97} - 2q^{100} + \cdots .\end{aligned} \tag{20}$$

In a similar way, we can compute the value for $\mathcal{R}_1(q)$ by applying (15) (for $q \mapsto q^6$) and (16) (for $q \mapsto -q$ and $q \mapsto -q^9$) as follows:

$$\begin{aligned}\mathcal{R}_1(q) = &\left(1-q^6-q^{12}+q^{30}+q^{42}-q^{72}-q^{90}+\cdots\right)\\&\cdot\Big[\left(1-2q^9+2q^{36}-2q^{81}+2q^{144}-2q^{225}+2q^{324}-2q^{441}+2q^{576}-\cdots\right)\\&\quad-\left(1-2q+2q^4-2q^9+2q^{16}-2q^{25}+2q^{36}-2q^{49}+2q^{64}-2q^{81}+\cdots\right)\Big],\end{aligned}$$

which, after simplification and by using algebraic manipulation, becomes

$$\begin{aligned}\mathcal{R}_1(q) = &\,2q - 2q^4 - 2q^7 + 2q^{10} - 2q^{13} + 2q^{22}\\&+ 2q^{25} + 2q^{28} - 2q^{34} - 2q^{37} + 2q^{43} - 4q^{46} + 2q^{49} - 2q^{58} - 2q^{61}\\&- 2q^{64} + 2q^{67} + 2q^{70} - 2q^{73} + 4q^{76} + 2q^{79} + 2q^{88} - 2q^{97} - 2q^{100} + \cdots .\end{aligned} \tag{21}$$

By comparing the equations (20) and (21), we readily arrive at the q-identity (18).

We next prove the second q-identity (19). Let $\mathcal{L}_2(q)$ and $\mathcal{R}_2(q)$ denote the left-hand and the right-hand sides of (19), respectively. Then, in order to compute the value for $\mathcal{L}_2(q)$, we make use of (15) (for $q \mapsto q$ and $q \mapsto q^2$) as follows:

$$\begin{aligned}\mathcal{L}_2(q) = 2&\left(1-q-q^2+q^5+q^7-q^{12}-q^{15}+\cdots\right)\\&\cdot\left(1-q^2-q^4+q^{10}+q^{14}-q^{24}-q^{30}+q^{44}+q^{52}-\cdots\right),\end{aligned}$$

which, after simplification and by using algebraic manipulation, yields

$$\begin{aligned}\mathcal{L}_2(q) = 2\big(&1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + q^{10} - 2q^{11} - 2q^{12} + 2q^{14} - q^{15}\\&+ 2q^{17} + 2q^{19} + q^{21} - 2q^{24} - q^{28} - 2q^{29} - 2q^{30} + 2q^{32} - 2q^{35} + 3q^{36} + 2q^{39}\\&+ 2q^{42} + 2q^{44} - q^{45} - 2q^{46} - 2q^{50} + 2q^{51} - 2q^{53} - 2q^{54} - q^{55} - 2q^{56}\end{aligned}$$

$$+ 2q^{57} + q^{59} - q^{60} + 2q^{65} + q^{66} + 2q^{71} + 2q^{72} + 2q^{74} - \cdots\big). \tag{22}$$

We now compute the value for

$$\varphi\left(q^4\right)\left[\psi(q) + q\psi\left(q^9\right)\right],$$

which occurs in (19). By applying (16) (for $q \mapsto q^4$) and (17) (for $q \mapsto q$ and $q \mapsto q^9$), we have

$$\begin{aligned}\varphi\left(q^4\right)\left[\psi(q) + q\psi\left(q^9\right)\right] &= (1 + 2q^4 + 2q^{16} + 2q^{36} + 2q^{64} + \cdots) \\ &\cdot\left[\left(1 + q + q^3 + q^6 + q^{10} + q^{15} + \cdots\right) + q\left(1 + q^9 + q^{27} + q^{54} + q^{90} + q^{135} + \cdots\right)\right],\end{aligned}$$

which, after simplification and by using algebraic manipulation, assumes the following form:

$$\begin{aligned}\varphi\left(q^4\right)\left[\psi(q) + q\psi\left(q^9\right)\right] &= 1 + 2q + q^3 + 2q^4 + 4q^5 + q^6 + 2q^7 + 4q^{10} + 4q^{14} \\ &+ q^{15} + 2q^{16} + 4q^{17} + 4q^{19} + q^{21} + 2q^{22} + 2q^{25} + 4q^{26} \\ &+ 2q^{28} + 2q^{31} + 4q^{32} + 3q^{36} + 6q^{37} + 2q^{39} + 2q^{40} + 2q^{42} + 4q^{44} + q^{45} \\ &+ 4q^{46} + 2q^{49} + 2q^{51} + 2q^{52} + 2q^{55} + \cdots.\end{aligned} \tag{23}$$

We next compute the value for

$$\varphi\left(q^{36}\right)\left[\psi(q) - 3q\psi\left(q^9\right)\right],$$

which occurs in (19). By applying (16) (for $q \mapsto q^{36}$) and (17) (for $q \mapsto q$ and $q \mapsto q^9$), we find that

$$\begin{aligned}\varphi\left(q^{36}\right)\left[\psi(q) - 3q\psi\left(q^9\right)\right] &= \left(1 + 2q^{36} + 2q^{144} + 2q^{324} + \cdots\right) \\ &\cdot\left[\left(1 + q + q^3 + q^6 + q^{10} + q^{15} + \cdots\right)\right. \\ &\left.- 3q\left(1 + q^9 + q^{27} + q^{54} + q^{90} + q^{135} + \cdots\right)\right],\end{aligned}$$

which, after simplification and by using algebraic manipulation, yields

$$\begin{aligned}\varphi\left(q^{36}\right)\left[\psi(q) - 3q\psi\left(q^9\right)\right] &= 1 - 2q + q^3 + q^6 - 2q^{10} + q^{15} + q^{21} - 2q^{28} \\ &+ 3q^{36} - 4q^{37} + 2q^{39} + 2q^{42} + q^{45} - 4q^{46} + 2q^{51} \\ &- 2q^{55} + 2q^{57} - 4q^{64} + q^{66} + 2q^{72} + \cdots.\end{aligned} \tag{24}$$

In order to compute the value for

$$-2q\psi(q)\left[\psi\left(q^8\right) + q^8\psi\left(q^{72}\right)\right],$$

which occurs in (19), by making use of (17) (for $q \mapsto q$, $q \mapsto q^8$ and $q \mapsto q^{72}$), we obtain

$$-2q\psi(q)\left[\psi\left(q^8\right) + q^8\psi\left(q^{72}\right)\right] = -2q\left(1 + q + q^3 + q^6 + q^{10} + q^{15} + \cdots\right)$$

$$\cdot\left[\left(1+q^8+q^{24}+q^{48}+q^{80}+q^{120}+\cdots\right)+q^8\left(1+q^{72}+q^{216}+q^{432}+\cdots\right)\right],$$

which, after simplification and by using algebraic manipulation, becomes

$$\begin{aligned}
&-2q\psi(q)\left[\psi\left(q^8\right)+q^8\psi\left(q^{72}\right)\right]=-2q-2q^2-2q^4-2q^7-4q^9-4q^{10}\\
&\quad-2q^{11}-4q^{12}-4q^{15}-2q^{16}-4q^{19}-2q^{22}-4q^{24}-2q^{25}-2q^{26}-2q^{28}\\
&\quad-2q^{29}-4q^{30}-2q^{31}-2q^{35}-6q^{37}-2q^{40}-4q^{45}-4q^{46}-2q^{49}-2q^{50}\\
&\quad-2q^{52}-2q^{53}-4q^{54}-2q^{55}-2q^{56}-2q^{59}-2q^{61}-6q^{64}-\cdots.
\end{aligned}\tag{25}$$

Finally, we compute the value for

$$-2q^2\psi\left(q^9\right)\left[\psi\left(q^8\right)-3q^8\psi\left(q^{72}\right)\right],$$

which occurs in (19). By making use of (17) (for $q\mapsto q^8$, $q\mapsto q^9$ and $q\mapsto q^{72}$), it is easily observed that

$$\begin{aligned}
&-2q^2\psi\left(q^9\right)\left[\psi\left(q^8\right)-3q^8\psi\left(q^{72}\right)\right]=-2q^2\big(1+q^9+q^{27}+q^{54}+q^{90}+q^{135}\\
&\quad+q^{189}+\cdots\big)\big[\big(1+q^8+q^{24}+q^{48}+q^{80}+q^{120}+\cdots\big)\\
&\quad-3q^8(1+q^{72}+q^{216}+q^{432}+\cdots)\big],
\end{aligned}$$

which, after simplification and by using algebraic manipulation, yields

$$\begin{aligned}
&-2q^2\psi\left(q^9\right)\left[\psi\left(q^8\right)-3q^8\psi\left(q^{72}\right)\right]=-2q^2+4q^{10}-2q^{11}+4q^{19}-2q^{26}-2q^{29}-2q^{35}\\
&\quad+4q^{37}-2q^{50}-2q^{53}-2q^{56}-2q^{59}+4q^{64}-2q^{77}-2q^{80}+\cdots.
\end{aligned}\tag{26}$$

Thus, by applying the equations (23) to (26), we find that

$$\begin{aligned}
\mathcal{R}_2(q)&=\varphi\left(q^4\right)\left[\psi(q)+q\psi\left(q^9\right)\right]+\varphi\left(q^{36}\right)\left[\psi(q)-3q\psi\left(q^9\right)\right]\\
&\quad-2q\psi(q)\left[\psi\left(q^8\right)+q^8\psi\left(q^{72}\right)\right]-2q^2\psi\left(q^9\right)\left[\psi\left(q^8\right)-3q^8\psi\left(q^{72}\right)\right]\\
&=2\big[1-q-2q^2+q^3+2q^5+q^6-2q^9+q^{10}-2q^{11}-2q^{12}+2q^{14}-q^{15}\\
&\quad+2q^{17}+2q^{19}+q^{21}-2q^{24}-q^{28}-2q^{29}-2q^{30}+2q^{32}-2q^{35}+3q^{36}\\
&\quad+2q^{39}+2q^{42}+2q^{44}-q^{45}-2q^{46}-2q^{50}+2q^{51}-2q^{53}-2q^{54}-q^{55}\\
&\quad-2q^{56}+2q^{57}+q^{59}-q^{60}+2q^{65}+q^{66}+2q^{71}+2q^{72}+2q^{74}-\cdots\big].
\end{aligned}\tag{27}$$

The q-identity (19) now follows upon comparing the equations (22) and (27).

We thus have completed our proof of the Theorem.

3. Concluding Remarks and Observations

Our present article is motivated essentially by the potential for applications of q-series and q-products. We have investigated here the three most interesting functions $f(q)$, $\varphi(q)$

and $\psi(q)$, which are related closely to such celebrated entities as Jacobi's theta functions in the equations (1) to (4), Ramanujan's general theta function in (5) and Jacobi's triple-product identity in (7) or (8). Our main results are stated and proved as the above Theorem and provide a sequel to some recent works by Chaudhary *et al.* (see [4] and [5]), by Adiga *et al.* [1], and by Srivastava and Chaudhary [13].

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