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On the existence of roots of some p-adic exponential-polynomials

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Abstract. In this paper, we apply Newton polygon method in order to derive sufficient conditions for the existence of zeros of some p-adic exponential-polynomials.

1. Introduction

Let K be an algebraically closed field of characteristic zero, and let exp be a (partial) exponential map $\exp : E \to K^{\times}$, $E \subset K$ being the domain of the exponential map. An exponential-polynomial is any expression of the form

$$a(X) = P_1(X)\exp(w_1X) + ... + P_d(X)\exp(w_dX), \tag{1}$$

where the P_i's $(i = 1, \dots, d)$ are polynomials in $K[X]$, and $w_i \in K$ for $i = 1, 2, \dots, d$. The theory of exponential polynomials is an important topic in transcendental number theory. A remarkable work has been made by P. D'A quino, A. Macintyre and G. Terzo [1], where they proved that Shapiro's Conjecture (over an algebraically closed exponential field of characteristic zero having an infinite cyclic group of periods and the exponential is surjective onto the multiplicative group) is true with an extra assumption, Schanuel's Conjecture. In 2017, they proved the following [2]:
Assume Schanuel's Conjecture. Let $Z(f)$ be the zero set of the exponential polynomial

$$f(X) = \alpha_1\exp(w_1X) + ... + \alpha_d\exp(w_dX),$$

where α_i, w_i are constants in K. If $Z(f)$ is infinite, then each infinite subset $X \subseteq Z(f)$ has an infinite transcendence degree over $\mathbb{Q}$.
In this paper, we work in the non-archimedean fields, namely the $p-$adic fields. Here, the situation is different to some extend. In fact, Poorten and Rumely [PR] proved that each exponential polynomial of the form (1.1) has at most finitely many roots in the domain of convergence. Their proof relies on a geometric approach, namely The Newton polygon of power series, where they proved that The Newton polygon of the exponential

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polynomial (1.1) ends with a straight line. That guarantees the existence of a bound on the number of zeros. Furthermore, the p-adic exponential function and the p-adic trigonometric functions are not periodic. Many results, however, have been made in the $p-$adic exponential polynomials. For example, Poorten, [5] , used Strassmann Theorem to prove that the exponential polynomial

$$b(z) = P_1[z]\exp(w_1 z) + ... + P_d[z]\exp(w_d z), \tag{2}$$

with $P_i[z] \in \mathbb{C}_p[z]$ and $\mathrm{ord}(w_i) > \frac{1}{p-1}+\epsilon, i = 1, 2, .., d$, has at most $(d-1+\sum_{i=1}^{d} \deg P_i[z])(1+\frac{1}{\epsilon(p-1)})$ roots in the unit disk. Poorten and Rumely, [6], proved that the exponential polynomial (1.2) over $\mathbb{Q}_p$ (for large enough p) has at most $(d-2+\sum_{i=1}^{d} \deg P_i[z])p$ roots in its domain of convergence using the Newton polygon method and concepts of recurrence sequences and generalized sums.

In this work, we consider some $p-$adic exponential polynomials, where we put a least bound on the number of zeros (counting multiplicity) of the $p-$adic exponential polynomials. We mainly use here the Newton Polygon of power series method which assures that the roots of the power series yield from the finite segments of the polygon. In other words, if the Newton polygon of the power series f has a finite segment with projection length on the x-axis equals to m, then there exist m roots (counting multiplicity) of f of the same order. Similarly, we use The Newton polygon method to put sufficient conditions on some polynomials $P[X,Y] \in \mathbb{Q}[X,Y]$ to have roots of the form $(x, \exp(x))$.

2. Notation and preliminaries

Let p be a prime number, $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the $p-$adic absolute value $|.|$ and $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. The absolute value $|\cdot|$ on $\mathbb{C}_p$ is the extension of the $p-$adic absolute value $|.|$.

Starting from $|\cdot|$, one can define a map $\mathrm{ord} : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$, as follows: $\mathrm{ord}(0) = \infty$, and $\mathrm{ord}(x) = -\log(|x|)$. This map satisfies the properties:

$$\begin{aligned} \mathrm{ord}(x \pm y) &\geq \min\{\mathrm{ord}(x), \mathrm{ord}(y)\}, \\ \mathrm{ord}(xy^{\pm 1}) &= \mathrm{ord}(x) \pm \mathrm{ord}(y), \\ \text{if } \mathrm{ord}(x) &\neq \mathrm{ord}(y), \text{ then } \mathrm{ord}(x \pm y) = \min\{\mathrm{ord}(x), \mathrm{ord}(y)\}. \end{aligned}$$

The set $\mathcal{O} := \{x \in \mathbb{C}_p : \mathrm{ord}(x) \geq 0\}$ forms a local ring called the ring of integers in $\mathbb{C}_p$.
Let $n \in \mathbb{N}$ with $n \geq 1$. It is well known that

$$\mathrm{ord}(n!) = \frac{n - S_n}{p - 1},$$

where S_n is the sum of digits of n when it is written in the base p. In particular, if $n = p^m, m \geq 1$, then $S_n = 1$. It is clear that $S_n \geq 1, \forall n \geq 1$. This implies that

$$\mathrm{ord}(\frac{1}{n!}) \geq -\frac{n-1}{p-1}.$$

We also recall some basic concepts and results concerning The Newton polygon method. For more details, see [4] and [3].

2.1. The Newton polygon for polynomials

Let $f(X) = 1 + a_1X + \cdots + a_nX^n \in 1 + X\mathbb{C}_p[X]$ be a polynomial with degree n and the constant term is 1. We plot the following points in the Euclidean space $\mathbb{R}^2$:

$$(0,0),(1,\mathrm{ord}(a_1)),(2,\mathrm{ord}(a_2)),\dots,(n,\mathrm{ord}(a_n)).$$

If $a_i = 0$ for some i, we omit this point (considering it as a point at infinity). The Newton polygon of the polynomial f is defined as the convex hull of the points $(0,0),(1,\mathrm{ord}(a_1)),(2,\mathrm{ord}(a_2)),\dots,(n,\mathrm{ord}(a_n))$. That is the highest convex polygonal line joining $(0,0)$ with $(n,\mathrm{ord}(a_n))$ and passing through or below all the points $(i,\mathrm{ord}(a_i)), i = 1,2,..,n-1$. Practically, the Newton polygon of polynomials is obtained by the following steps:
1) Start with the vertical half-line which is the negative part of the y-axis.
2) Rotate the line counter-clockwise until it hits one of the points we have plotted.
3) Break the line at that point, and continue rotating the remaining part until another point is hit.
4) Continue until all the points have either been hit or lie strictly above a portion of the polygon.
A vertex of the Newton polygon is a point $(i,\mathrm{ord}(a_i))$ where the slopes change. If a segment joins the point (i,m) to the point (i',m'), then the slope is the quantity $\frac{m-m'}{i-i'}$. By the length of the slope we mean the quantity $i-i'$.
If the polygon has a segment ends by a point $(i,\mathrm{ord}(a_i))$ and continues by another segment of different slope, we say that the Newton polygon has a "break" at the point $(i,\mathrm{ord}(a_i))$.

Theorem 1. *Let $f(X) = 1 + a_1X + \cdots + a_nX^n \in 1 + X\mathbb{C}_p[X]$. If λ is a slope of the Newton polygon associated to the polynomial f with the length m, then there exist the numbers $\alpha_1,\alpha_2,..,\alpha_m \in \mathbb{C}_p$ (counting multiplicity) such that $f(\alpha_i) = 0$ and $\mathrm{ord}(\alpha_i) = -\lambda$, $\forall i = 1,2,..,m$.*

2.2. The Newton polygon for power series

The definition is formally identical to that given for polynomials: Consider the power series

$$f(X) = 1 + a_1X + a_2X^2 + \cdots + a_nX^n + \dots$$

We plot the points $(i,\mathrm{ord}(a_i)), i = 1,2,...$, ignoring as before any points where $a_i = 0$. The Newton polygon of $f(X)$ is again obtained by the rotating line procedure. In this case, the things become more complicated than the case of polynomials. For example, the Newton polygon of the power series $f(X) = 1 + pX + pX^2 + .. + pX^n + ...$ is just the horizontal line OX which does not hit any of the points $(i,\mathrm{ord}(a_i)), i = 1,2,....$ For this case and other cases, we must modify the rules to obtain the Newton polygon of power series as the following steps:
Start with the half-line which is the negative part of the y-axis. Rotate that line counter-clockwise until one of the following happens:

i) The line simultaneously hits infinitely many of the points we have plotted. In this case, stop and the polygon is complete. For example the Newton polygon of the power series $f(X) = 1 + \sum_{i=1}^{\infty} p^i X^i$ is just the line $Y = X$.
ii) The line reaches a position where it contains only one of our points that serves as a center of rotation, but can be rotated no further without leaving behind some points. In this case, stop and the polygon is complete. We will counter this case in Section 4.
iii) The line hits a finite number of points. In this case, break the line at the last point it was hit, and repeat the whole procedure again.
Using the above procedure, it can be seen that the Newton polygon of power series either ends by a ray (see the Appendix) or has an infinite number of finite segments (for example, the Newton polygon of the power series $1 + \sum_{i=1}^{\infty} p^{i^2} X^i$).
Furthermore, it is well known that if the Newton polygon of a power series f ends by a ray, then f has at most finitely many zeros in its disk of convergence. The following lemma is a connection between the domain of convergence of a power series and the slopes of its polygon.

Lemma 1. *Let m be the* sup *of all slopes appearing in the Newton polygon of a power series $f(X) = 1 + \sum_{i=1}^{\infty} a_i X^i$. Then, the domain of f is the set*

$$\{x \in \mathbb{C}_p : \mathrm{ord}(x) > -m\}.$$

In the case m is infinite. Then f converges on all of $\mathbb{C}_p$. In particular, if the Newton polygon of f ends by a ray of slope m, then the domain is $\{x \in \mathbb{C}_p : \mathrm{ord}(x) > -m\}$.

Finally, we need the following:

Corollary 1. *([4], p.106) If a segment of the Newton polygon of $f(X) \in 1 + \mathbb{C}_p[[X]]$ has finite length N and slope λ, then there are precisely N values of x counting multiplicity for which $f(x) = 0$ and $\mathrm{ord}(x) = -\lambda$.*

We summarize what we need as the following:
Fact 1. The points $(i, \mathrm{ord}(\frac{1}{i!}))$; $i \geqslant 1$ are on or above the line $Y = \frac{-1}{p-1}(X-1)$. It algebraically means that $\mathrm{ord}(\frac{1}{i!}) \geqslant \frac{-1}{p-1}(i-1), \forall i \geqslant 1$.
Fact 2. A finite segment of the length m of the Newton polygon of the power series f determines at least m roots (counting multiplicity) of f of the same order.

3. The Main Results

Keep the notation as above. We prove the following:

Theorem 2. *Consider the polynomials over $\mathcal{O}$:*

$$P_1(z) = \sum_{j=0}^{n_1} a_j^{(1)} z^j, P_2(z) = \sum_{j=0}^{n_2} a_j^{(2)} z^j, \ldots, P_d(z) = \sum_{j=0}^{n_d} a_j^{(d)} z^j,$$

where $n_d > \max_{1\le i\le d-1}\{\deg P_i, 1\}$. *Let* $w_1, \dots, w_d \in \mathbb{C}_p$ *with* $\mathrm{ord}(w_i) > \frac{1}{p-1}, i = 1, 2, .., d$. *Then, the exponential polynomial*

$$b(z) = P_1(z)\exp(w_1 z) + P_2(z)\exp(w_2 z) + \dots. + P_d(z)\exp(w_d z),$$

with $\mathrm{ord}(a_0^{(1)} + \dots + a_0^{(d)}) = \mathrm{ord}(a_{n_d}^{(d)}) = 0$, *has at least* n_d *roots (counting multiplicity) in the unit disk.*

Proof. As $\mathrm{ord}(w_j) > \frac{1}{p-1}$ for $j = 1, 2, .., d$, the domain of convergence is the unit disk. Let $c = a_0^{(1)} + ... + a_0^{(d)}$, then, by expanding $a(z)$ as a power series in z, one finds:

$$\frac{a(z)}{c} = 1 + c^{-1}\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{d} a_0^{(j)}\frac{w_j^i}{(i)!} + ... + a_{n_j}^{(j)}\frac{w_j^{i-n_j}}{(i-n_j)!}\Big)z^i =: 1 + \sum_{i=1}^{\infty} m_i z^i.$$

We have

$$\mathrm{ord}(w_j) > \frac{1}{p-1} = \frac{i}{i(p-1)} > \frac{i - S_i}{i(p-1)}, \forall i \ge 1, \forall j = 1, 2, .., d.$$

Hence,

$$i\,\mathrm{ord}(w_j) > \frac{i - S_i}{p-1} = \mathrm{ord}(i!) \Rightarrow i\,\mathrm{ord}(w_j) - \mathrm{ord}(i!) > 0.$$

Therefore,

$$\mathrm{ord}(\frac{w_j^i}{i!}) > 0. \tag{3}$$

Using (3), the assumptions of the theorem that $\mathrm{ord}(c) = 0$ and the coefficients of $P_j, j = 1, 2, ..., d$ are in $\mathcal{O}$, we find that

$$\mathrm{ord}(m_i) \ge 0, i = 1, 2, ...$$

That means the points $(i, \mathrm{ord}(m_i)), i = 1, 2, ..$ are on or above the x-axis. Consider the coefficient m_{n_d} in the series $\frac{a(z)}{c}$. The assumption that $\mathrm{ord}(a_{n_d}) = \mathrm{ord}(c) = 0$ and (3) guarantee that

$$\mathrm{ord}(m_{n_d}) = 0.$$

Assume that $\mathrm{ord}(w_1) = \min\{\mathrm{ord}(w_j), j = 1, 2, .., d\}$ (the other cases can be done similarly). Therefore, we have for all $i > n_d$

$$\min\Big\{\mathrm{ord}(\frac{1}{(i-j)!}) : 0 \le j \le n_d\Big\} = \mathrm{ord}(\frac{1}{i!}).$$

Hence,

$$\begin{aligned}\mathrm{ord}(m_i) &\ge \min\Big\{\mathrm{ord}(\mathrm{ord}(\frac{w_j^{i-k}}{(i-k)!})) : j = 1, \dots, d, 0 \le k \le n_j\Big\}\\ &\ge (i - n_d)\mathrm{ord}(w_1) + \mathrm{ord}(\frac{1}{i!})\end{aligned}$$

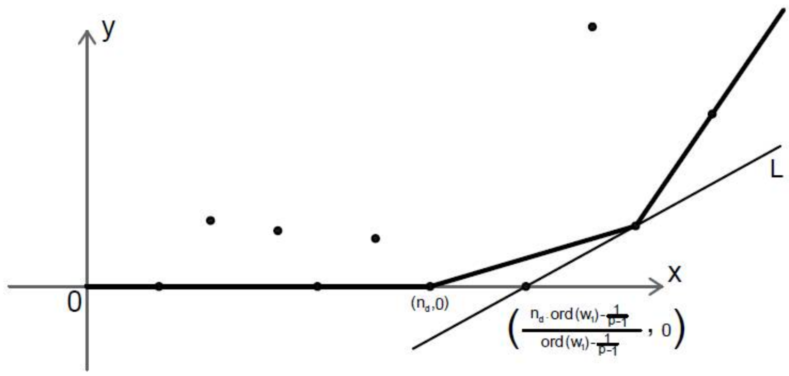

Figure 1: The Newton polygon of $\frac{a(z)}{c}$

$$\geq \quad (i-n_d)\mathrm{ord}(w_1) - \frac{i-1}{p-1}.$$

Therefore, for all $i > n_d$ the points $(i, \mathrm{ord}(m_i))$ are on or above the line

$$L : Y + n_d.\mathrm{ord}(w_1) - \frac{1}{p-1} = (\mathrm{ord}(w_1) - \frac{1}{p-1})X.$$

This line has a positive slope $\lambda = \mathrm{ord}(w_1) - \frac{1}{p-1} > 0$ and intersects with the x-axis in the point $(\frac{n_d.\mathrm{ord}(w_1)-\frac{1}{p-1}}{\mathrm{ord}(w_1)-\frac{1}{p-1}}, 0)$ which lies on the right of the point $(n_d, 0)$ since $n_d > 1$.
The points $(i, \mathrm{ord}(m_i)), i = 1, 2, 3, \ldots$ are thus distributed as follows:
1) The points $(1, \mathrm{ord}(m_1)), \ldots, (n_d - 1, \mathrm{ord}(m_{n_d-1}))$ are on or above the x-axis.
2) The point $(n_d, \mathrm{ord}(m_{n_d}))$ is on the x-axis.
3) The points $(i, \mathrm{ord}(m_i)), i > n_d$ are on or above the line L and above x-axis. It follows that there exists a finite number of points $(i, \mathrm{ord}(m_i))$ lying on a horizontal line above the x-axis.
Now, we apply the previous steps to obtain The Newton polygon of $\frac{a(z)}{c}$ as follows: Rotate the vertical half-line of the negative part of the y-axis until it hits the point $(n_d, 0)$. Break the line at this point (the existence of the break is because there is at most finitely many points $(i, \mathrm{ord}(m_i))$ lying on a horizontal line above the x-axis). Rotate it around this point until it hits another point or continues until it reaches a position parallel to a line with a positive slope. In all cases, the Newton polygon of $\frac{a(z)}{c}$ starts with a segment of the length n_d and has a break at the point $(n_d, 0)$ (see figure 1). Therefore, using Fact 2, we find that $\frac{a(z)}{c}$ (and hence $a(z)$) has at least n_d roots.

Remark 1. *Theorem 2 covers only certain cases of exponential polynomials. The assumption* $\mathrm{ord}(w_j) = 0, \forall j = 1, 2, .., d$ *is crucial. In fact, there exists a very big class of exponential polynomials that have no roots. For example, the exponential polynomial over* $\mathcal{O}$*:*

$$a(z) = a\exp(w_1 z) + (a_1 z + \cdots + a_n z^n)\exp(w_2 z) + (b_1 z + \cdots + b_m z^m)\exp(w_3 z),$$

where $\mathrm{ord}(w_j) = 0, \forall j = 1, 2, 3, \ldots, n < m$*, has no roots it its domain even if* $\mathrm{ord}(a) = \mathrm{ord}(b_m) = 0$.

The above follows since, if $z_0 \in \{z \in \mathbb{C}_p : \mathrm{ord}(z) > \frac{1}{p-1}\}$ is a root of $a(z)$, then

$$a\exp(w_1 z_0) = -\Big((a_1 z_0 + \cdots + a_n z_0^n)\exp(w_2 z_0) + (b_1 z_0 + \cdots + b_m z_0^m)\exp(w_3 z_0)\Big).$$

Therefore,

$$\begin{aligned} |a\exp(w_1 z_0)| &= \Big|(a_1 z_0 + \cdots + a_n z_0^n)\exp(w_2 z_0) + (b_1 z_0 + \cdots + b_m z_0^m)\exp(w_3 z_0)\Big| \\ &\leq \max\{|(a_1 z_0 + \cdots + a_n z_0^n)\exp(w_2 z_0)|, |(b_1 z_0 + \cdots + b_m z_0^m)\exp(w_3 z_0)|\} \\ &< p^{\frac{-1}{p-1}} < 1, \end{aligned}$$

since $|\exp(w_j z_0)| = 1, j = 2, 3$ and $|z_0| < p^{\frac{-1}{p-1}} < 1$. On the other hand, we have $|a\exp(w_1 z_0)| = |\exp(w_1 z_0)| = 1$. This contradiction shows that $a(z)$ has no roots in its domain.

Corollary 2. *Consider the polynomial*

$$P[X, Y] = a + bY^m + a_{(i_1,j_1)}X^{i_1}Y^{j_1} + \cdots + a_{(i_d,j_d)}X^{i_d}Y^{j_d} \in \mathbb{Z}[X, Y],$$

with $p|\gcd(m, j_1, .., j_d), 0 < i_1 < \cdots < i_d$*, and* $(a + b, p) = (a_{(i_d,j_d)}, p) = 1$*. Then, P has at least* i_d *roots of the form* $(x, \exp(x))$*.*

Proof. Consider the exponential polynomial

$$b(z) = (a \cdot z^0) + (b \cdot z^0)\exp(mz) + a_{(i_1,j_1)}z^{i_1}\exp(j_1 z) + \cdots + a_{(i_d,j_d)}z^{i_d}\exp(j_d z).$$

Then, we have

$$\mathrm{ord}(m) \geq 1 > \frac{1}{p-1}, \mathrm{ord}(j_k) \geq 1 > \frac{1}{p-1}, k = 1, 2, \ldots, d.$$

Also, the polynomial $P_d(z) = a_{(i_d,j_d)}z^{i_d}$ has the largest degree with $\mathrm{ord}(a_{(i_d,j_d)}) = \mathrm{ord}(a + b) = 0$. This implies, by Theorem 2, that $b(z)$ has at least i_d roots. This proves the Corollary.

Corollary 3. *Let* $P[X, Y] \in \mathbb{Q}[X, Y]$ *be polynomial defined by the conditions of the previous Corollary.*
Then there exists a tuple $(x, \exp(x)), x \in E$ *(domain of the exponential function) such that* $P(x, \exp(x)) = 0$*. In other words, the elements* $x, \exp(x)$ *are* $\mathbb{Q}-$*algebraically dependent. Hence,* $td_{\mathbb{Q}}\mathbb{Q}(x, \exp(x)) \leq 1$*, where td stands for the transcendence degree.*

4. Further Applications of the Newton polygon method

One can also use the Newton polygon of power series in order to obtain sufficient conditions on a polynomial $P[X,Y] \in \mathbb{Q}[X,Y]$ to have roots of the form $(x, \exp(x))$, as done in what follows.

Theorem 3. *For any polynomial $P[X,Y]$ having the form*

$$P[X,Y] = dY^n + cX^m + c_1X^{m_1}Y^{n_1} + \cdots + c_rX^{m_r}Y^{n_r} \in \mathbb{Q}[X,Y],$$

with $m_i, m, n, n_i \in \mathbb{Z}_{\geq 1}$, $(n_i, p) = (n, p) = 1$, $\mathrm{ord}(d) = \mathrm{ord}(c_i) = 0$, for $i = 1, 2, \ldots r$, *and such that* $\mathrm{ord}(c) < \frac{-1}{p-1}m$, *has a root of the form* $(x, \exp(x))$, $x \in \mathbb{C}_p$ *with* $\mathrm{ord}(x) > \frac{1}{p-1}$.

Proof. Let $f(X) := P[X, \exp(X)]$ be the corresponding power series associated with the original polynomial $P[X,Y]$. Then, $f(X)$ can be written as

$$f(X) = d(1 + b_1X + .. + b_{m-1}X^{m-1} + (b_m + c.d^{-1})X^m + b_{m+1}X^{m+1} + \dots),$$

where $b_i = \frac{n^i}{i!} + e_1d^{-1}\frac{n_1^{i-m_1}}{(i-m_1)!} + \cdots + e_rd^{-1}\frac{n_r^{i-m_r}}{(i-m_r)!}$; $e_k = 0$ or c_k, for $k = 0, .., r$.
According to the assumption of theorem (the coefficients d, c_i, and the degrees n_j have the same order which is zero) and fact (1), we find that the numbers b_i satisfy the inequality

$$\mathrm{ord}(b_i) \geq \frac{-1}{p-1}(i-1).$$

Since $\mathrm{ord}(c) < \frac{-1}{p-1}m < \frac{-1}{p-1}(m-1)$, it follows that

$$\mathrm{ord}(b_m + cd^{-1}) = \min\{\mathrm{ord}(b_m), \mathrm{ord}(cd^{-1})\} = \mathrm{ord}(c) < \frac{-1}{p-1}m.$$

Also, if i is sufficiently large index of the form p^j, then $\mathrm{ord}(\frac{n^i}{i!}) = \mathrm{ord}(\frac{1}{i!}) = -\frac{i-1}{p-1}$.
Clearly, $m_k + S_{i-m_k} > 1, \forall k = 1, 2, .., r$ (since $S_n \geq 1, \forall n \geq 1$). This is equivalent to $-\frac{i-1}{p-1} < -\frac{i-m_k-S_{i-m_k}}{p-1}$. In other words,

$$\mathrm{ord}(\frac{1}{i!}) < \mathrm{ord}(\frac{1}{(i-m_k)!}), k = 1, 2, \ldots, r.$$

Hence,

$$\mathrm{ord}(b_i) \quad = \quad \min\left\{\mathrm{ord}(\frac{n^i}{i!}), \mathrm{ord}(\frac{n_1^{i-m_1}}{(i-m_1)!} + ... + \frac{n_r^{i-m_r}}{(i-m_r)!})\right\} = \mathrm{ord}(\frac{1}{i!}).$$

Therefore, the points of the power series $\frac{f(X)}{d}$ are distributed as follows (for more details, see the Appendix):
The points $(i, \mathrm{ord}(b_i))$ are on or above the line $Y = \frac{-1}{p-1}(X-1)$, the subsequence $(p^i, \mathrm{ord}(b_{p^i}))$, for large enough i lies on the previous line and the point $(m, \mathrm{ord}(b_m + d^{-1}c))$ is below the line $Y = -\frac{1}{p-1}X$.

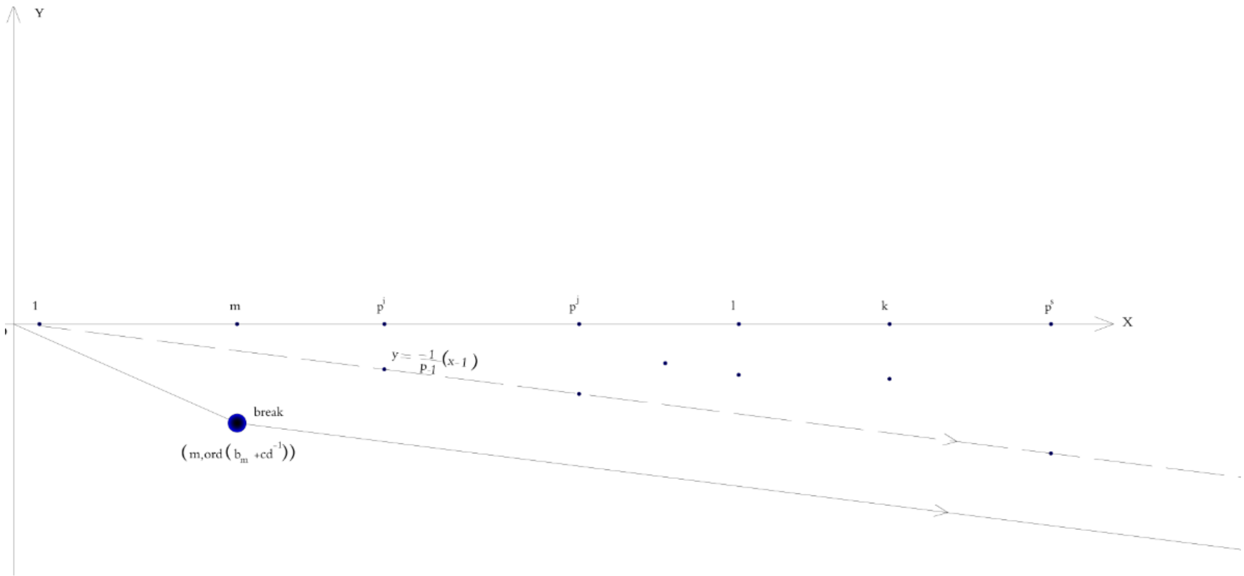

Figure 2: The Newton polygon of $\frac{f(X)}{d}$

Now, we apply the previous steps to obtain The Newton polygon of the power series $\frac{f(X)}{d}$ as follows:
Rotate the vertical half-line of the negative part of the y-axis until it hits the point $(m, \mathrm{ord}(b_m + d^{-1}c))$. Then rotate it around this point until it reaches a position parallel to the line $Y = -\frac{1}{p-1}(X-1)$. Stop here and the polygon is complete. Any further rotation would leave behind some points $(i, \mathrm{ord}(b_i))$ (see figure 2 and the Appendix). Therefore, the Newton polygon of $\frac{f(X)}{d}$ has a break at the point $(m, \mathrm{ord}(b_m + d^{-1}c))$. This implies, that the Newton polygon of $\frac{f(X)}{d}$ has a finite segment of the length m. Using Fact 2 , $\frac{f(X)}{d}$ (and hence f) has at least m roots. So, the original polynomial $P[X,Y]$ has m roots in $\mathbb{C}_p \times \mathbb{C}_p^*$ of the form $(x, \exp(x))$.

Remark 2. *Newton polygon method does not only guarantee the existence of polynomials that admit roots of the form $(x, \exp(x))$, but it also gives us information about the order of x.*

Example 1. *Consider the polynomial*

$$P[X,Y] = p^{-1}X^2 + Y^2; p \geqslant 5.$$

Let

$$f(X) := P[X, \exp(X)] = p^{-1}X^2 + (\exp(X))^2 = p^{-1}X^2 + \exp(2X).$$

Then we have,

$$f(X) = 1 + \frac{2X}{1!} + (\frac{2^2}{2!} + p^{-1})X^2 + \frac{2^3}{3!}X^3 + \cdots + \frac{2^i}{i!}X^i + \ldots.$$

Therefore,

$$\mathrm{ord}(\frac{2}{1!}) = 0, \mathrm{ord}(\frac{2^i}{i!}) = \mathrm{ord}(\frac{1}{i!}), \forall i \geqslant 3.$$

This is because, $\mathrm{ord}(2) = 0$ for $p \geqslant 5$. Furthermore, we find

$$\mathrm{ord}(\frac{2^2}{2!} + p^{-1}) = -1.$$

The assumption $p \geqslant 5$ guarantees that the point $(2, -1)$ is below the line $Y = \frac{-1}{p-1}X$. This means that the Newton polygon of $f(X)$ starts with a segment of the slope $(\frac{-1}{2})$ and ends with the point (2,-1)while the second segment is a half line which starts with the point (2,-1) and has the slope $\frac{-1}{p-1}$. Therefore, the Newton polygon of $f(X)$ has a break at the point (2,-1). Using fact (2), we find that the power series $f(X)$ has at least two roots of the order $\frac{1}{2}$. So, the original polynomial $P[X,Y]$ has at least two roots of the form $(x,\exp(x))$.

5. Appendix

The following known Lemma, (see, e.g., [4], p. 143), determines the distribution of vertices $(i, \mathrm{ord}(\frac{1}{i!}))$ that appear in the Newton polygon of the exponential map.

Lemma 2. *The Newton polygon of the exponential function is a straight line from* $(0, 0)$ *with the slope* $\frac{-1}{p-1}$.

Proof. We know that the exponential function $\exp(X)$ is defined as

$$\exp(X) = 1 + \frac{X}{1} + \frac{X^2}{2!} + \cdots + \frac{X^i}{i!} + \ldots$$

We first show that, for all $i \geqslant 0$, the points $(p^i,\mathrm{ord}(a_{p^i}))$ belong to the line $Y = \frac{-1}{p-1}(X-1)$ and the other points are on or above this line. That is, for all $j \geqslant 1$, the points $(j,\mathrm{ord}(a_j))$ are on or above the line $Y = \frac{-1}{p-1}(X-1)$.
To do that, we prove that the slope of the line which passes through any two points $(p^i,\mathrm{ord}(a_{p^i}))$, $(p^j,\mathrm{ord}(a_{p^j}))$;$(i > j)$ has a slope independent of i and j and indeed has the value $\frac{-1}{p-1}$. Let m be the slope of the line through any points $(p^i,\mathrm{ord}(a_{p^i}))$, $(p^j,\mathrm{ord}(a_{p^j}))$. Then

$$\begin{aligned} m &= \frac{\mathrm{ord}(a_{p^i}) - \mathrm{ord}(a_{p^j})}{p^i - p^j} = \frac{-\mathrm{ord}(p^i!) + \mathrm{ord}(p^j!)}{p^i - p^j} = \frac{\mathrm{ord}(p^j!) - \mathrm{ord}(p^i!)}{p^i - p^j} \\ &= \frac{\frac{p^j - S_{p^j}}{p-1} - \frac{p^i - S_{p^i}}{p-1}}{p^i - p^j}. \end{aligned}$$

Therefore,

$$m = \frac{\frac{p^j-1}{p-1} - \frac{p^i-1}{p-1}}{p^i - p^j} = \frac{\frac{p^j-p^i}{p-1}}{p^i - p^j} = \frac{-1}{p-1}.$$

So, all the points $(p^i, \mathrm{ord}(a_{p^i})); i \geqslant 0$ belong to the line with the slope $\frac{-1}{p-1}$ and passes through the point $(p^0$,ord$(a_{p^0}))$(which is the point (1,0)). This line has the equation $Y = \frac{-1}{p-1}(X-1)$. Also, for all $i \geqslant 1$, we have

$$\mathrm{ord}(a_i) = \mathrm{ord}(\frac{1}{i!}) \geq -\frac{1}{p-1}(i-1).$$

This implies that all the points $(i$,ord$(a_i))$; $(i \geqslant 1)$ are on or above the line $Y = \frac{-1}{p-1}(X-1)$. From that argument, we deduce that the points $(i$,ord$(a_i))$; $(i \geq 1)$ are distributed as follows:
1) The subsequence $(p^j, \mathrm{ord}(a_{p^j})), j \geq 0$ lies on the line $Y = \frac{-1}{p-1}(X-1)$ which has the slope $-\frac{1}{p-1}$.
2) The other points are on or above this line.
Now apply the previous steps to obtain The Newton polygon of $\exp(X)$ as follows: Rotate the vertical half-line of the negative part of the y-axis until it reaches to a position parallel to the line $Y = -\frac{1}{p-1}(X-1)$. We stop here without any further rotation. This is because, for any $\epsilon > -\frac{1}{p-1}$, the line $Y = \epsilon X$ would leave behind it some points of the form $(p^i, \mathrm{ord}(\frac{1}{p^i!}))$. Since $\epsilon > \frac{-1}{p-1}$, it follows that there exists some positive real number $\delta > 0$ such that $\epsilon = \frac{-1}{p-1} + \delta$. Therefore,

$$\mathrm{ord}(a_i) < \epsilon i \Leftrightarrow -\frac{i - S_i}{p-1} < \epsilon i \Leftrightarrow -\frac{i - S_i}{p-1} < (\frac{-1}{p-1} + \delta)i \Leftrightarrow$$

$$i - S_i > i - (p-1)\delta i \Leftrightarrow (p-1)\delta i > S_i \Leftrightarrow i > \frac{S_i}{(p-1)\delta}.$$

We can choose i to be sufficiently large and has the form p^j. In this case, we find that $S_i = 1$, so the relation $i > \frac{1}{\delta(p-1)}$ holds true for the index $i := p^j$. This implies that there is no further rotation of the line $Y = \frac{-1}{p-1}X$. Hence, the Newton polygon of the exponential function is the straight line $Y = \frac{-1}{p-1}X$ from $(0,0)$.

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