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Subalgebras and ideals in BCK/BCI-algebras based on Uni-hesitant fuzzy set theory

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Abstract. In the present paper, the notions of uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals in BCK-algebras and BCI-algebras are introduced, and several related properties are investigated. Characterizations of uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals are considered, and a new uni-hesitant fuzzy algebra (resp. uni-hesitant fuzzy (closed) ideal) from old one is established. Relations between uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals are discussed, and conditions for a uni-hesitant fuzzy ideal to be hesitant closed are provided.

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1. Introduction

The hesitant fuzzy set which is introduced by Torra [14] is a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [14] and [15]). The hesitant fuzzy set therefore provides a more accurate representation of people's hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy set theory has been applied to several practical problems, primarily in the area of decision making (see [13], [15] [16], [17], [18], [19], [20]). Furthermore, Jun et al. applied the notion of hesitant fuzzy sets to BCK/BCI-algebras, MTL-algebras, EQ-algebras and semigroups (see [2], [3], [4] and [5]). Recently, Muhiuddin et al. applied the notion of hesitant fuzzy sets to residuated lattices, BCK/BCI-algebras and lattice implication algebras (see [8], [9],

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[10], [11] and [12]).

In this paper, we introduce the notions of uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals in BCK/BCI-algebras, and investigate several related properties. We consider characterizations of uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals. Given a unihesitant fuzzy algebra (resp. uni-hesitant fuzzy (closed) ideal), we make a new uni-hesitant fuzzy algebra (resp. uni-hesitant fuzzy (closed) ideal). We investigate relations between uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals. We provide conditions for a uni-hesitant fuzzy ideal to be hesitant closed.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI*-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK*-algebra*. Any BCK/BCI-algebra X satisfies the following axioms:

(a1) $(\forall x \in X)\ (x * 0 = x)$,

(a2) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,

(a3) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$,

(a4) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. In a BCI-algebra X, the following hold:

(b1) $(\forall x, y \in X)\,(x * (x * (x * y)) = x * y)$,

(b2) $(\forall x, y \in X)\,(0 * (x * y) = (0 * x) * (0 * y))$.

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in A, \tag{1}$$
$$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A \;\Rightarrow\; x \in A)\,. \tag{2}$$

An ideal A of a BCI-algebra X is said to be *closed* if $0 * x \in A$ for all $x \in A$. We refer the reader to the books [1, 6] for further information regarding BCK/BCI-algebras.

Torra [14] defined hesitant fuzzy sets in terms of a function that returns a set of membership values for each element in the domain. We display the basic notions on hesitant fuzzy sets. For more details, we refer to references.

Let X be a reference set. Then we define a hesitant fuzzy set on X in terms of a function $\mathcal{G}$ that when applied to X returns a subset of $[0, 1]$, and the image of $x \in X$ under $\mathcal{G}$ is denoted by $x\mathcal{G}$.

For a hesitant fuzzy set $\mathcal{G}$ on X and a subset λ of $[0, 1]$, the set

$$L(\mathcal{G}; \lambda) := \{x \in X \mid x\mathcal{G} \subseteq \lambda\},$$

is called the *uni-hesitant level set* of $\mathcal{G}$.

Let $\mathcal{G}$ and $\mathcal{H}$ be two hesitant fuzzy sets on X. The *hesitant union* $\mathcal{G} \sqcup \mathcal{H}$ and *hesitant intersection* $\mathcal{G} \sqcap \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ are defined to be hesitant fuzzy sets on X as follows:

$$\mathcal{G} \sqcup \mathcal{H} : X \to P([0, 1]),\ x \mapsto x\mathcal{G} \cup x\mathcal{H} \tag{3}$$

and

$$\mathcal{G} \sqcap \mathcal{H} : X \to P([0, 1]),\ x \mapsto x\mathcal{G} \cap x\mathcal{H}, \tag{4}$$

respectively.

3. Uni-hesitant fuzzy algebras in BCK/BCI-algebras

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

Definition 1. *A hesitant fuzzy set $\mathcal{G}$ on X is called a uni-hesitant fuzzy algebra on X if $(x * y)\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G}$ for all $x, y \in X$.*

Example 1. *(1) Let $X = \{0, a, b\}$ be a BCK-algebra with the Cayley table which is appeared in Table 1.*
Let $\mathcal{G}$ be a hesitant fuzzy set on X defined as follows:

$$\mathcal{G} : X \to P([0, 1]), \quad x \mapsto \begin{cases} (0, 3, 0.7) & \text{if } x = 0, \\ (0.2, 0.8] & \text{if } x = a, \\ (0, 1) & \text{if } x = b. \end{cases}$$

Table 1: Cayley table for the $*$-operation

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

It is routine to verify that $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X.

(2) Let $X = \{0, a, b, c\}$ be a BCK-algebra with the Cayley table which is appeared in Table 2.

Table 2: Cayley table for the $*$-operation

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined as follows:

$$\mathcal{H} : X \to P([0,1]), \quad x \mapsto \begin{cases} (0,3,0.4) \cup \{0.5, 0.6\} & \text{if } x = 0, \\ (0.2, 0.4] \cup \{0.5, 0.6\} & \text{if } x = a, \\ (0.2, 0.7] & \text{if } x = b, \\ (0.2, 0.5) \cup [0.5, 0.6] & \text{if } x = c. \end{cases}$$

It is routine to verify that $\mathcal{H}$ is a uni-hesitant fuzzy algebra on X.

Theorem 1. *A hesitant fuzzy set $\mathcal{G}$ on X a uni-hesitant fuzzy algebra on X if and only if the nonempty uni-hesitant level set $L(\mathcal{G}; \lambda)$ of $\mathcal{G}$ is a subalgebra of X for all $\lambda \in P([0,1])$.*

Proof. Assume that $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X. Let $\lambda \in P([0,1])$ and $x, y \in L(\mathcal{G}; \lambda)$. Then $x\mathcal{G} \subseteq \lambda$ and $y\mathcal{G} \subseteq \lambda$. It follows that $(x * y)\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G} \subseteq \lambda$. Hence $x * y \in L(\mathcal{G}; \lambda)$. Therefore $L(\mathcal{G}; \lambda)$ is a subalgebra of A.

Conversely, suppose that the nonempty uni-hesitant level set of $\mathcal{G}$ is a subalgebra of X for all $\lambda \in P([0,1])$. Let $x, y \in X$ be such that $x\mathcal{G} = \lambda_x$ and $y\mathcal{G} = \lambda_y$. Taking $\lambda = \lambda_x \cup \lambda_y$ implies that $x, y \in L(\mathcal{G}; \lambda)$, and so $x * y \in L(\mathcal{G}; \lambda)$. Hence

$$(x * y)\mathcal{G} \subseteq \lambda = \lambda_x \cup \lambda_y = x\mathcal{G} \cup y\mathcal{G}.$$

Therefore $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X.

Proposition 1. *Every uni-hesitant fuzzy algebra $\mathcal{G}$ on X satisfies $0\mathcal{G} \subseteq x\mathcal{G}$ for all $x \in X$.*

Proof. Since $x * x = 0$ for all $x \in X$, it is straightforward.

Proposition 2. *Let X be a BCI-algebra. If $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X, then $(x * (0 * y))\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G}$ for all $x, y \in X$.*

Proof. Using Proposition 1, we have

$$(x * (0 * y))\mathcal{G} \subseteq x\mathcal{G} \cup (0 * y)\mathcal{G} \subseteq x\mathcal{G} \cup 0\mathcal{G} \cup y\mathcal{G} = x\mathcal{G} \cup y\mathcal{G}$$

for all $x, y \in X$.

Proposition 3. *For any uni-hesitant fuzzy algebra $\mathcal{G}$ on X, we have*

$$(\forall x, y \in X)\left((x * y)\mathcal{G} \subseteq y\mathcal{G} \;\Leftrightarrow\; x\mathcal{G} = 0\mathcal{G}\right).$$

Proof. Assume that $(x * y)\mathcal{G} \subseteq y\mathcal{G}$ for all $x, y \in X$. Taking $y = 0$ induces $x\mathcal{G} = (x * 0)\mathcal{G} \subseteq 0\mathcal{G}$. It follows from Proposition 1 that $x\mathcal{G} = 0\mathcal{G}$ for all $x \in X$.

Conversely, suppose that $x\mathcal{G} = 0\mathcal{G}$ for all $x \in X$. Then

$$(x * y)\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G} = 0\mathcal{G} \cup y\mathcal{G} = y\mathcal{G}$$

for all $x, y \in X$.

Theorem 2. *Given a uni-hesitant fuzzy algebra $\mathcal{G}$ on X, the hesitant fuzzy set $\mathcal{G}^*$ on X defined by*

$$\mathcal{G}^* : X \to P([0,1]),\; x \mapsto \begin{cases} x\mathcal{G} & \text{if } x \in L(\mathcal{G};\lambda), \\ (0,1) & \text{otherwise} \end{cases}$$

is a uni-hesitant fuzzy algebra on X.

Proof. If $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X, then $L(\mathcal{G};\lambda)$ is a subalgebra of A for all $\lambda \in P([0,1])$ with $L(\mathcal{G};\lambda) \neq \emptyset$ by Theorem 1. Let $x, y \in X$. If $x, y \in L(\mathcal{G};\lambda)$, then $x * y \in L(\mathcal{G};\lambda)$ and so

$$(x * y)\mathcal{G}^* = (x * y)\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G} = x\mathcal{G}^* \cup y\mathcal{G}^*.$$

If $x \notin L(\mathcal{G};\lambda)$ or $y \notin L(\mathcal{G};\lambda)$, then $x\mathcal{G}^* = (0,1)$ or $y\mathcal{G}^* = (0,1)$. Thus

$$(x * y)\mathcal{G}^* \subseteq (0,1) = x\mathcal{G}^* \cup y\mathcal{G}^*.$$

Therefore $\mathcal{G}^*$ is a uni-hesitant fuzzy algebra on X.

4. Uni-hesitant fuzzy ideals

Definition 2. *A hesitant fuzzy set $\mathcal{G}$ on X is called a uni-hesitant fuzzy ideal on X if it satisfies $0\mathcal{G} \subseteq x\mathcal{G}$ for all $x \in X$ and*

$$(\forall x, y \in X)\left(x\mathcal{G} \subseteq (x * y)\mathcal{G} \cup y\mathcal{G}\right). \tag{5}$$

Example 2. *Let $X = \{0, 1, 2, a, b\}$ be a BCI-algebra with the Cayley table which is appeared in Table 3.*

Table 3: Cayley table for the $*$-operation

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $\mathcal{H}$ be a hesitant fuzzy set on X defined as follows:

$$\mathcal{H} : X \to P([0,1]), \quad x \mapsto \begin{cases} \{0.2, 0.4, 0.6, 0.8\} & \text{if } x = 0, \\ [0.2, 0.3) \cup \{0.4, 0.6, 0.8\} & \text{if } x = 1, \\ (0.1, 0.2] \cup \{0.4\} \cup [0.6, 0.8] & \text{if } x = 2, \\ (0.1, 0.4] \cup [0.6, 0.8] & \text{if } x = a, \\ [0.1, 0.4] \cup [0.6, 0.9) & \text{if } x = b. \end{cases}$$

It is routine to verify that $\mathcal{H}$ is a uni-hesitant fuzzy ideal on X.

Example 3. *Consider a BCI-algebra $X = \{2^n \mid n \in \mathbb{Z}\}$ with a binary operation "$\div$" (usual division). Let $\mathcal{G}$ be a hesitant fuzzy set on X given as follows:*

$$\mathcal{G} : X \to P([0,1]), \ x \mapsto \begin{cases} \lambda_1 & \text{if } n \geq 0, \\ \lambda_2 & \text{if } n < 0, \end{cases}$$

where λ_1 and λ_2 are subsets of $[0, 1]$ with $\lambda_1 \subsetneq \lambda_2$. Then $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X,

Lemma 1. *Every uni-hesitant fuzzy ideal $\mathcal{G}$ on X satisfies the following condition.*

$$(\forall x, y \in X)\left(x * y = 0 \ \Rightarrow \ x\mathcal{G} \subseteq y\mathcal{G}\right). \tag{6}$$

Proof. It is straightforward by (5) and Proposition 1.

Proposition 4. *Every uni-hesitant fuzzy ideal $\mathcal{G}$ on X satisfies:*

(1) $(\forall x, y, z \in X)\,((x*y)\mathcal{G} \subseteq (x*z)\mathcal{G} \cup (z*y)\mathcal{G})$.

(2) $(\forall x, y \in X)\,((x*y)\mathcal{G} = 0\mathcal{G} \;\Rightarrow x\mathcal{G} \subseteq y\mathcal{G})$.

Proof. Since $((x*y)*(x*z))*(z*y) = 0$, it follows from Lemma 1 that

$$((x*y)*(x*z))\mathcal{G} \subseteq (z*y)\mathcal{G}.$$

Hence

$$(x*y)\mathcal{G} \subseteq ((x*y)*(x*z))\mathcal{G} \cup (x*z)\mathcal{G} \subseteq (x*z)\mathcal{G} \cup (z*y)\mathcal{G}$$

for all $x, y, z \in X$.

(2) If $(x*y)\mathcal{G} = 0\mathcal{G}$, then

$$x\mathcal{G} \subseteq (x*y)\mathcal{G} \cup y\mathcal{G} = 0\mathcal{G} \cup y\mathcal{G} = y\mathcal{G}$$

for all $x, y \in X$.

Proposition 5. *For any uni-hesitant fuzzy ideal $\mathcal{G}$ on X, the following conditions are equivalent:*

(1) $(\forall x, y \in X)\,((x*y)\mathcal{G} \subseteq ((x*y)*y)\mathcal{G})$.

(2) $(\forall x, y, z \in X)\,(((x*z)*(y*z))\mathcal{G} \subseteq ((x*y)*z)\mathcal{G})$.

Proof. Assume that (1) is valid and let $x, y, z \in X$. Since

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z,$$

that is, $(((x*(y*z))*z)*z)*((x*y)*z) = 0$, it follows from (a3), (1) and (6) that

$$\begin{aligned}((x*z)*(y*z))\mathcal{G} &= ((x*(y*z))*z)\mathcal{G} \\ &\subseteq (((x*(y*z))*z)*z)\mathcal{G} \subseteq ((x*y)*z)\mathcal{G}.\end{aligned}$$

Conversely, suppose that (2) holds. If we take $y = z$ in (2), then

$$((x*z)*z)\mathcal{G} \supseteq ((x*z)*(z*z))\mathcal{G} = ((x*z)*0)\mathcal{G} = (x*z)\mathcal{G}$$

by (III) and (a1). This proves (1).

Theorem 3. *Every uni-hesitant fuzzy ideal on a BCK-algebra is a uni-hesitant fuzzy algebra.*

Proof. Let $\mathcal{G}$ be a uni-hesitant fuzzy ideal on a BCK-algebra X. Note that $(x*y)*x = 0$ for all $x, y \in X$. Using Lemma 1 and (5), we have

$$(x*y)\mathcal{G} \subseteq x\mathcal{G} \subseteq (x*y)\mathcal{G} \cup y\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G}$$

for all $x, y \in X$. Hence $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X.

If X is a BCI-algebra, then Theorem 3 is not true as seen in the following example.

Example 4. *Let $(Y, *, 0)$ be a BCI-algebra and let $(\mathbb{Z}, -, 0)$ be the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Then $X := Y \times \mathbb{Z}$ is a BCI-algebra with a binary operation $\otimes$ defined as follows:*

$$(\forall (x, m), (y, n) \in X) \left((x, m) \otimes (y, n) = (x * y, m - n)\right).$$

For a subset $A = Y \times (\mathbb{N} \cup \{0\})$ of X, let $\mathcal{G}$ be a hesitant fuzzy set on X given by

$$\mathcal{G} : X \to P([0,1]),\ (x, m) \mapsto \begin{cases} \lambda & \text{if } x \in A, \\ [0,1] & \text{otherwise} \end{cases}$$

where $\lambda \in P([0,1])$ with $\lambda \neq [0,1]$. Then $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X. Note that $(0,2) \in A$ and $(0,3) \in A$, but $(0,2) \otimes (0,3) = (0,-1) \notin A$. Thus

$$((0,2) \otimes (0,3))\mathcal{G} = [0,1] \nsubseteq \lambda = (0,2)\mathcal{G} \cup (0,3)\mathcal{G}.$$

Therefore $\mathcal{G}$ is not a uni-hesitant fuzzy algebra on X.

Proposition 6. *Every uni-hesitant fuzzy ideal $\mathcal{G}$ on X satisfies the following condition:*

$$(\forall x, y, z \in X) \left((x * y) * z = 0 \ \Rightarrow\ x\mathcal{G} \subseteq y\mathcal{G} \cup z\mathcal{G}\right). \tag{7}$$

Proof. Let $x, y, z \in X$ be such that $(x * y) * z = 0$. Then

$$(x * y)\mathcal{G} \subseteq ((x * y) * z)\mathcal{G} \cup z\mathcal{G} = 0\mathcal{G} \cup z\mathcal{G} = z\mathcal{G}$$

by (5) and Proposition 1. It follows that

$$x\mathcal{G} \subseteq (x * y)\mathcal{G} \cup y\mathcal{G} \subseteq y\mathcal{G} \cup z\mathcal{G}$$

for all $x, y, z \in X$. This completes the proof.

We provide conditions for a hesitant fuzzy set to be a hesitant fuzzy ideal.

Proposition 7. *If a hesitant fuzzy set $\mathcal{G}$ on X satisfies Proposition 1 and (7), then $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X.*

Proof. It is straightforward by (II) and (7).

The following could be easily proved by induction.

Corollary 1. *Let $\mathcal{G}$ be a hesitant fuzzy set on X satisfying Proposition 1. Then $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X if and only if*

$$(\forall x, a_1, a_2, \cdots, a_n \in X) \left(x * \prod_{i=1}^{n} a_i = 0 \ \Rightarrow\ x\mathcal{G} \subseteq \bigcup_{i=1,2,\cdots,n} a_i\mathcal{G} \right),$$

where $x * \prod_{i=1}^{n} a_i = (\cdots(x * a_1) * \cdots) * a_n$.

Proposition 8. *Let X be a BCK-algebra such that*

$$(x * a) * b = 0, \tag{8}$$

$$a * \prod_{i=1}^{n} a_i = 0, \tag{9}$$

$$b * \prod_{j=1}^{m} b_j = 0, \tag{10}$$

for all $x, a, b, a_1, a_2, \cdots, a_n,\ b_1, b_2, \cdots, b_m \in X$. If $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X, then

$$x\mathcal{G} \subseteq \bigcup_{\substack{i=1,2,\cdots,n \\ j=1,2,\cdots,m}} (a_i\mathcal{G} \cup b_j\mathcal{G})$$

for all $x, a_1, a_2, \cdots, a_n,\ b_1, b_2, \cdots, b_m \in X$.

Proof. Using (8) and (a3), we have $(x * b) * a = 0$. It follows from (a2), (a3) and (9) that

$$\left(x * \prod_{i=1}^{n} a_i\right) * b \leq a * \prod_{i=1}^{n} a_i = 0$$

so that $\left(x * \prod_{i=1}^{n} a_i\right) * b = 0$, i.e., $x * \prod_{i=1}^{n} a_i \leq b$. Using (a2), (a3) and (10), we have

$$\left(x * \prod_{i=1}^{n} a_i\right) * \prod_{j=1}^{m} b_j \leq b * \prod_{j=1}^{m} b_j = 0,$$

and so $\left(x * \prod_{i=1}^{n} a_i\right) * \prod_{j=1}^{m} b_j = 0$. Thus, by Corollary 1, we have

$$x\mathcal{G} \subseteq \bigcup_{\substack{i=1,2,\cdots,n \\ j=1,2,\cdots,m}} (a_i\mathcal{G} \cup b_j\mathcal{G})$$

for all $x, a_1, a_2, \cdots, a_n,\ b_1, b_2, \cdots, b_m \in X$.

Theorem 4. *For a hesitant fuzzy set $\mathcal{G}$ on X, the following are equivalent.*

(i) *$\mathcal{G}$ is a uni-hesitant fuzzy ideal on X.*

(ii) *The nonempty uni-hesitant level set $L(\mathcal{G}; \lambda)$ of $\mathcal{G}$ is an ideal of X for all $\lambda \in P([0, 1])$.*

Proof. Assume that $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X. Let $\lambda \in P([0, 1])$ be such that $L(\mathcal{G}; \lambda) \neq \emptyset$. Then $x\mathcal{G} \subseteq \lambda$ for some $x \in X$. It follows from Proposition 1 that

$0\mathcal{G} \subseteq x\mathcal{G} \subseteq \lambda$. Hence $0 \in L(\mathcal{G};\lambda)$. Let $x, y \in X$ be such that $x * y \in L(\mathcal{G};\lambda)$ and $y \in L(\mathcal{G};\lambda)$. Then $(x*y)\mathcal{G} \subseteq \lambda$ and $y\mathcal{G} \subseteq \lambda$. It follows from (5) that

$$x\mathcal{G} \subseteq (x*y)\mathcal{G} \cup y\mathcal{G} \subseteq \lambda.$$

Thus $x \in L(\mathcal{G};\lambda)$. Therefore $L(\mathcal{G};\lambda)$ is an ideal of X.

Conversely suppose that the nonempty uni-hesitant level set of $\mathcal{G}$ is an ideal of X for all $\lambda \in P([0,1])$. Then $0 \in L(\mathcal{G};\lambda)$. If there exists $a \in X$ such that $0\mathcal{G} \nsubseteq a\mathcal{G}$, then $0\mathcal{G} \nsubseteq \lambda$ for $\lambda = a\mathcal{G} \setminus 0\mathcal{G}$. Hence $0 \notin L(\mathcal{G};\lambda)$, a contradiction. Therefore $0\mathcal{G} \subseteq x\mathcal{G}$ for all $x \in X$. Let $x, y \in X$ be such that $(x*y)\mathcal{G} = \lambda_1$ and $y\mathcal{G} = \lambda_2$. Let us take $\lambda = \lambda_1 \cup \lambda_2$. Then $x*y \in L(\mathcal{G};\lambda)$ and $y \in L(\mathcal{G};\lambda)$. Since $L(\mathcal{G};\lambda)$ is an ideal of X, it follows from (2) that $x \in L(\mathcal{G};\lambda)$. Hence $x\mathcal{G} \subseteq \lambda = \lambda_1 \cup \lambda_2 = (x*y)\mathcal{G} \cup y\mathcal{G}$. Consequently, $\mathcal{G}$ is a uni-hesitant fuzzy ideal on X.

5. Uni-hesitant fuzzy closed ideals in BCI-algebras

Definition 3. *A uni-hesitant fuzzy ideal $\mathcal{G}$ on a BCI-algebra X is said to be hesitant closed if the following inclusion is valid.*

$$(\forall x \in X)\left((0*x)\mathcal{G} \subseteq x\mathcal{G}\right). \tag{11}$$

Obviously, every uni-hesitant fuzzy closed ideal is a uni-hesitant fuzzy algebra.

Example 5. *The uni-hesitant fuzzy ideal $\mathcal{H}$ in Example 2 is hesitant closed.*

Note that every uni-hesitant fuzzy ideal on a BCK-algebra X is hesitant closed. The following example shows that there exists a uni-hesitant fuzzy ideal on a BCI-algebra X which is not hesitant closed.

Example 6. *In Example 3, the uni-hesitant fuzzy ideal $\mathcal{G}$ on X is not hesitant closed since*

$$\left(1 \div 2^2\right)\mathcal{G} = 2^{-2}\mathcal{G} = \lambda_2 \nsubseteq \lambda_1 = 1\mathcal{G} \cup 2^2\mathcal{G}.$$

We provide conditions for a uni-hesitant fuzzy ideal to be hesitant closed.

Theorem 5. *Let $\mathcal{G}$ be a uni-hesitant fuzzy ideal on a BCI-algebra X. Then $\mathcal{G}$ is hesitant closed if and only if $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X.*

Proof. Assume that $\mathcal{G}$ is hesitant closed. Then $(0*x)\mathcal{G} \subseteq x\mathcal{G}$ for all $x \in X$. It follows from (5) that

$$(x*y)\mathcal{G} \subseteq ((x*y)*x)\mathcal{G} \cup x\mathcal{G} = (0*y)\mathcal{G} \cup x\mathcal{G} \subseteq x\mathcal{G} \cup y\mathcal{G}$$

for all $x, y \in X$. Hence $\mathcal{G}$ is a uni-hesitant fuzzy algebra on X.

Conversely, suppose that $\mathcal{G}$ is a uni-hesitant fuzzy algebra on a BCI-algebra X. Then

$$(0*x)\mathcal{G} \subseteq 0\mathcal{G} \cup x\mathcal{G} = x\mathcal{G}$$

for all $x \in X$. Therefore $\mathcal{G}$ is hesitant closed.

By the similar way to Theorem 4, we have a characterization of a uni-hesitant fuzzy closed ideal.

Theorem 6. *For a hesitant fuzzy set $\mathcal{G}$ on a BCI-algebra X, the following are equivalent.*

(i) *$\mathcal{G}$ is a uni-hesitant fuzzy closed ideal on X.*

(ii) *The nonempty uni-hesitant level set $L(\mathcal{G};\lambda)$ of $\mathcal{G}$ is a closed ideal of X for all $\lambda \in P([0,1])$.*

Theorem 7. *The hesitant union of two uni-hesitant fuzzy closed ideals on a BCI-algebra X is a uni-hesitant fuzzy closed ideal on X.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$ be uni-hesitant fuzzy closed ideals on a BCI-algebra X. Then $L(\mathcal{G};\lambda)$ and $L(\mathcal{H};\lambda)$ are closed ideals of X for all $\lambda \in P([0,1])$ whenever they are nonempty. Thus $0 \in L(\mathcal{G};\lambda) \cap L(\mathcal{H};\lambda)$, and so $0(\mathcal{G} \sqcup \mathcal{H}) = 0\mathcal{G} \cup 0\mathcal{H} \subseteq \lambda$. Thus $0 \in L(\mathcal{G} \sqcup \mathcal{H};\lambda)$. Let $x, y \in X$ be such that $x * y \in L(\mathcal{G} \sqcup \mathcal{H};\lambda)$ and $y \in L(\mathcal{G} \sqcup \mathcal{H};\lambda)$. Then

$$(x*y)\mathcal{G} \cup (x*y)\mathcal{H} = (x*y)(\mathcal{G} \sqcup \mathcal{H}) \subseteq \lambda \text{ and } y\mathcal{G} \cup y\mathcal{H} = y(\mathcal{G} \sqcup \mathcal{H}) \subseteq \lambda.$$

Hence $(x*y)\mathcal{G} \subseteq \lambda$, $(x*y)\mathcal{H} \subseteq \lambda$, $y\mathcal{G} \subseteq \lambda$ and $y\mathcal{H} \subseteq \lambda$. It follows from (5) that $x\mathcal{G} \subseteq (x*y)\mathcal{G} \cup y\mathcal{G}$ and $x\mathcal{H} \subseteq (x*y)\mathcal{H} \cup y\mathcal{H}$. Hence

$$\begin{aligned} x(\mathcal{G} \sqcup \mathcal{H}) &= x\mathcal{G} \cup x\mathcal{H} \\ &\subseteq ((x*y)\mathcal{G} \cup y\mathcal{G}) \cup ((x*y)\mathcal{H} \cup y\mathcal{H}) \\ &= ((x*y)\mathcal{G} \cup (x*y)\mathcal{H}) \cup (y\mathcal{G} \cup y\mathcal{H}) \\ &= (x*y)(\mathcal{G} \sqcup \mathcal{H}) \cup y(\mathcal{G} \sqcup \mathcal{H}) \end{aligned}$$

and $(0*y)(\mathcal{G} \sqcup \mathcal{H}) = (0*y)\mathcal{G} \cup (0*y)\mathcal{H} \subseteq y\mathcal{G} \cup y\mathcal{H} = y(\mathcal{G} \sqcup \mathcal{H})$. Thus $x \in L(\mathcal{G} \sqcup \mathcal{H};\lambda)$ and $0*y \in L(\mathcal{G} \sqcup \mathcal{H};\lambda)$. Therefore $L(\mathcal{G} \sqcup \mathcal{H};\lambda)$ is a closed ideal of X. It follows from Theorem 6 that $\mathcal{G} \sqcup \mathcal{H}$ is a uni-hesitant fuzzy closed ideal on X.

Theorem 8. *If $\mathcal{G}$ is a uni-hesitant fuzzy closed ideal on a BCI-algebra X, then the set*

$$A := \{x \in X \mid x\mathcal{G} = 0\mathcal{G}\}$$

is a closed ideal of X.

Proof. Clearly $0 \in A$. Let $x, y \in X$ be such that $x*y \in A$ and $y \in A$. Then $(x*y)\mathcal{G} = 0\mathcal{G} = y\mathcal{G}$. It follows from (5) and (11) that $x\mathcal{G} \subseteq (x*y)\mathcal{G} \cup y\mathcal{G} = 0\mathcal{G}$ and $(0*y)\mathcal{G} \subseteq y\mathcal{G} = 0\mathcal{G}$. Since $0\mathcal{G} \subseteq x\mathcal{G}$ for all $x \in X$, we have $x\mathcal{G} = 0\mathcal{G}$ and $(0*y)\mathcal{G} = 0\mathcal{G}$, that is, $x \in A$ and $0*y \in A$. Therefore A is a closed ideal of X.

Let X be a BCI-algebra and $B(X) := \{x \in X \mid 0 \leq x\}$. For any $x \in X$ and $n \in \mathbb{N}$, we define x^n by

$$x^1 = x, \; x^{n+1} = x*(0*x^n).$$

If there is an $n \in \mathbb{N}$ such that $x^n \in B(X)$, then we say that x is of *finite periodic* (see [7]), and we denote its period $|x|$ by

$$|x| = \min\{n \in \mathbb{N} \mid x^n \in B(X)\}.$$

Otherwise, x is of infinite period and denoted by $|x| = \infty$.

Theorem 9. *If X is a BCI-algebra in which every element is of finite period, then every uni-hesitant fuzzy ideal on X is hesitant closed.*

Proof. Let $\mathcal{G}$ be a uni-hesitant fuzzy ideal on X. For any $x \in X$, assume that $|x| = n$. Then $x^n \in B(X)$. Note that

$$\begin{aligned}(0 * x^{n-1}) * x &= (0 * (0 * (0 * x^{n-1}))) * x \\ &= (0 * x) * (0 * (0 * x^{n-1})) \\ &= 0 * (x * (0 * x^{n-1})) \\ &= 0 * x^n = 0,\end{aligned}$$

and so $((0 * x^{n-1}) * x)\,\mathcal{G} = 0\mathcal{G} \subseteq x\mathcal{G}$ by Proposition 1. It follows from (5) that

$$(0 * x^{n-1})\,\mathcal{G} \subseteq ((0 * x^{n-1}) * x)\,\mathcal{G} \cup x\mathcal{G} \subseteq x\mathcal{G}. \tag{12}$$

Also, note that

$$\begin{aligned}(0 * x^{n-2}) * x &= (0 * (0 * (0 * x^{n-2}))) * x \\ &= (0 * x) * (0 * (0 * x^{n-2})) \\ &= 0 * (x * (0 * x^{n-2})) \\ &= 0 * x^{n-1},\end{aligned}$$

which implies from (12) that

$$((0 * x^{n-2}) * x)\,\mathcal{G} = (0 * x^{n-1})\,\mathcal{G} \subseteq x\mathcal{G}.$$

Using (5), we have

$$(0 * x^{n-2})\,\mathcal{G} \subseteq ((0 * x^{n-2}) * x)\,\mathcal{G} \cup x\mathcal{G} \subseteq x\mathcal{G}.$$

Continuing this process, we have $(0 * x)\mathcal{G} \subseteq x\mathcal{G}$ for all $x \in X$. Therefore $\mathcal{G}$ is hesitant closed.

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