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Multiplicative (generalized) reverse derivations on semiprime ring

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Abstract. Let R be a semiprime ring. A mapping $F : R \to R$ (not necessarily additive) is called a multiplicative (generalized) reverse derivation if there exists a map $d : R \to R$ (not necessarily a derivation nor an additive map) such that $F(xy) = F(y)x + yd(x)$ for all $x, y \in R$. In this paper we investigate some identities involving multiplicative (generalized) reverse derivation and prove some theorems in which we characterize these mappings.

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1. Introduction

Let R be an associative ring. The centre of R is denoted by $Z(R)$. For $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$ and a ring R is semiprime if for any $a \in R$, $aRa = 0$ implies $a = 0$. An additive map $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. The concept of derivation was extended to generalized derivation by Bresar [2]. An additive map $F : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Daif [4] introduced the concept of multiplicative derivation. A map $D : R \to R$ is said to be a multiplicative derivation if it satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. Daif and Tammam El-Sayiad [5] extended the concept of multiplicative derivation to multiplicative generalized derivation. A mapping $F : R \to R$ is said to be a multiplicative generalized derivation if there exists a derivation d on R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In this definition if we take d to be a mapping not necessarily a derivation nor an additive map, then F is said to be a multiplicative (generalized)-derivation which was introduced by Dhara and Ali [7]. Dhara and Ali [7] studied the following identities related

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on multiplicative (generalized) derivation on a semiprime ring: $(i)F(xy) \pm xy \in Z(R)$, $(ii)F(xy) \pm yx \in Z(R)$, $(iii)F(x)F(y) \pm xy \in Z(R)$ and $(iv)F(x)F(y) \pm yx \in Z(R)$ for all x, y in some suitable subset of a semiprime ring R.

The concept of reverse derivation was first time introduced by Herstein [8]. A mapping $d : R \to R$ which satisfies $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$ is called a reverse derivation. Further Bresar and Vukman [3] studied the reverse derivation. Aboubakr and gonzalez [1] generalized the notion of reverse derivation by introducing generalized reverse derivation. An additive map $F : R \to R$ is said to be a generalized reverse derivation if $F(xy) = F(y)x + yd(x)$ holds for all $x, y \in R$, where d is a reverse derivation of R. Very recently Tiwari et al [11] defined multiplicative (generalized) reverse derivation. A map $F : R \to R$ is said to be a multiplicative (generalized) reverse derivation if $F(xy) = F(y)x + yd(x)$ holds for all $x, y \in R$, where d is any map on R.

In the mentioned paper they proved commutativity of semiprime ring admitting a multiplicative (generalized) reverse derivation satisfying one of the following conditions:
$(i)F(x)F(y) \pm xy = 0$, $(ii)F(x)F(y) \pm yx = 0$,
$(iii)F(xy) \pm [x, y] = 0$, $(iv)F(xy) \pm (x \circ y) = 0$,
$(v)F(xy) \pm F(x)F(y) = 0$,
for all $x, y \in I$, a two sided ideal in a semiprime ring R.
As a line of investigation we study the following situations:
$(i)F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$,
$(ii)F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$,
$(iii)F(xy) \pm [x, y] \in Z(R)$ for all $x, y \in I$,
$(iv)F(xy) \pm x \circ y \in Z(R)$ for all $x, y \in I$,
$(v)F(xy) \pm F(y)F(x) \in Z(R)$ for all $x, y \in I$,
$(vi)[F(x), y] \pm xy \in Z(R)$ for all $x, y \in I$,
$(vii)F(x) \circ y \pm xy \in Z(R)$ for all $x, y \in I$,
$(viii)[F(x), y] \pm yx \in Z(R)$ for all $x, y \in I$,
$(ix)F(x) \circ y \pm yx \in Z(R)$ for all $x, y \in I$.
where I is a non zero ideal in a semiprime ring R.

2. Preliminaries

Let R be a ring, we need the following basic identities which will be used in the proof of our results. For any $x, y, z \in R$,

$$\begin{aligned} [xy, z] &= x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z. \\ x \circ (yz) &= (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z. \\ (xy) \circ z &= x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]. \end{aligned}$$

For any subset S of R, we will denote $r_R(S)$ the right annihilator of S in R, that is $r_R(S) = \{x \in R | Sx = 0\}$ and by $l_R(S)$ the left annihilator of S in R, that is, $l_R(S) = \{x \in R | xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator of R and is written as $ann_R(S)$. For given $x, y \in R$, set $[x, y]_0 = x$, $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$.

Moreover, we shall require the following known results.

Lemma 2.1. [9, Corollary 1] If R is a semiprime ring and I is an ideal of R, then $r_R(I) = l_R(I)$.

Lemma 2.2. [9, Corollary 2] If R is a semiprime ring and I is an ideal of R, then $I \cap ann_R(I) = 0$.

Lemma 2.3. [6, Fact-4] Let R be a semiprime ring, d a nonzero derivation of R such that $x[[d(x), x], x] = 0$ for all $x \in R$, then d maps R into its centre.

3. Main Results

Theorem 3.1. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(x)F(y) \pm xy \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Proof By the hypothesis, we have

$$F(x)F(y) + xy \in Z(R) \text{ for all } x, y \in I. \tag{3.1}$$

Substituting zy for y in (3.1) and using the definition of multiplicative (generalized) reverse derivation, we find

$$F(x)F(y)z + F(x)yd(z) + xzy - xyz + xyz \in Z(R) \text{ for all } x, y, z \in I.$$

This implies that

$$(F(x)F(y) + xy)z + F(x)yd(z) + x[z, y] \in Z(R). \tag{3.2}$$

Commuting (3.2) with z and using (3.1), we get

$$[F(x)yd(z), z] + [x[z, y], z] = 0 \text{ for all } x, y, z \in I. \tag{3.3}$$

Replacing x by zx in (3.3), we obtain

$$[(F(x)z + xd(z))yd(z), z] + [zx[z, y], z] = 0.$$

Therefore we get

$$[F(x)zyd(z), z] + [xd(z)yd(z), z] + [zx[z, y], z] = 0. \tag{3.4}$$

Replacing y by zy in (3.3), we get

$$[F(x)zyd(z), z] + [xz[z, y], z] = 0 \text{ for all } x, y, z \in I. \tag{3.5}$$

Subtracting (3.5) from (3.4), we obtain

$$[xd(z)yd(z), z] + [[z, x][z, y], z] = 0. \tag{3.6}$$

Substituting yz for y in (3.6), we obtain

$$[xd(z)yzd(z), z] + [[z, x][z, y], z]z = 0. \tag{3.7}$$

Right multiplying (3.6) by z and subtracting from (3.7), we get

$$[xd(z)y[d(z), z], z] = 0 \text{ for all } x, y, z \in I. \tag{3.8}$$

Replacing x by $d(z)x$ in (3.8) and using (3.8), we find

$$[d(z), z]xd(z)y[d(z), z] = 0 \text{ for all } x, y, z \in I. \tag{3.9}$$

Substituting zy for y in (3.9), we obtain

$$[d(z), z]xd(z)zy[d(z), z] = 0 \text{ for all } x, y, z \in I. \tag{3.10}$$

Substituting xz for x in (3.9), we get

$$[d(z), z]xzd(z)y[d(z), z] = 0 \text{ for all } x, y, z \in I. \tag{3.11}$$

Subtracting (3.11) from (3.10), we have

$$[d(z), z]x[d(z), z]y[d(z), z] = 0 \text{ for all } x, y, z \in I.$$

This implies that $(I[d(z), z])^3 = 0$. Since a semiprime ring has no non-zero nilpotent left ideal, we get $I[d(z), z] = 0$ for all $z \in I$. Therefore, we get $[d(z), z] \in I \cap ann_R(I)$. By Lemma 2.2, we get $[d(z), z] = 0$ for all $z \in I$. By using similar argument, we arrive at the same conclusion for $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$.

Theorem 3.2. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(x)F(y) \pm yx \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Proof By assumption, we have

$$F(x)F(y) + yx \in Z(R) \text{ for all } x, y \in I. \tag{3.12}$$

Replacing y by zy in (3.12), we get

$$F(x)(F(y)z + yd(z)) + zyx + yxz - yxz \in Z(R) \text{ for all } x, y, z \in I.$$

This implies that

$$(F(x)F(y) + yx)z + F(x)yd(z) + [z, yx] \in Z(R) \tag{3.13}$$

Commuting (3.13) with z and using (3.12), we obtain

$$[F(x)yd(z), z] + [[z, yx], z] = 0 \text{ for all } x, y, z \in I.$$

This implies that

$$[F(x)yd(z), z] + [y[z, x], z] + [[z, y]x, z] = 0. \tag{3.14}$$

Replacing x by zx in (3.14), we obtain

$$[F(x)zyd(z), z] + [xd(z)yd(z), z] + [yz[z, x], z] + [[z, y]zx, z] = 0. \tag{3.15}$$

Substituting zy for y in (3.14), we obtain

$$[F(x)zyd(z), z] + [zy[z, x], z] + [z[z, y]x, z] = 0. \tag{3.16}$$

Subtracting (3.16) from (3.15), we get

$$[xd(z)yd(z), z] + [[y, z][z, x], z] + [[[z, y], z]x, z] = 0. \tag{3.17}$$

Replacing x by xz in (3.17), we obtain

$$[xzd(z)yd(z), z] + [[y, z][z, x], z]z + [[[z, y], z]x, z]z = 0 \tag{3.18}$$

Right multiplying (3.17) by z and subtracting from (3.18), we get

$$[x[d(z)yd(z), z], z] = 0 \text{ for all } x, y, z \in I. \tag{3.19}$$

This implies that

$$[x, z][d(z)yd(z), z] + x[[d(z)yd(z), z], z] = 0. \tag{3.20}$$

Substituting ux for x in (3.20), we get

$$u[x, z][d(z)yd(z), z] + [u, z]x[d(z)yd(z), z] + ux[[d(z)yd(z), z], z] = 0. \text{ for all } x, y, z, u \in I. \tag{3.21}$$

Left multiplying (3.20) by u and subtracting from (3.21), we obtain

$$[u, z]x[d(z)yd(z), z] = 0 \text{ for all } x, y, z, u \in I. \tag{3.22}$$

Putting $u = d(z)yd(z)$ in (3.22), we get

$$(I[d(z)yd(z), z])^2 = 0 \text{ for all } y, z \in I. \tag{3.23}$$

Since a semiprime ring has no non-zero nilpotent left ideal, therefore we get $I[d(z)yd(z), z] = 0$, this implies that $[d(z)yd(z), z)] \in I \cap ann_R(I)$. By Lemma 2.2, we get $[d(z)yd(z), z] = 0$. This implies that

$$d(z)yd(z)z - zd(z)yd(z) = 0 \text{ for all } y, z \in I. \tag{3.24}$$

Replacing y by $yd(z)w$ in (3.24), we get

$$d(z)yd(z)wd(z)z - zd(z)yd(z)wd(z) = 0 \text{ for all } y, z, w \in I. \tag{3.25}$$

Using (3.24), (3.25) gives

$$d(z)yzd(z)wd(z) - d(z)yd(z)zwd(z) = 0.$$

This implies that

$$d(z)y[d(z), z]wd(z) = 0 \text{ for all } y, z, w \in I.$$

This yields that $[d(z), z]y[d(z), z]w[d(z), z] = 0$ for all $y, z, w \in I$. That is $(I[d(z), z])^3 = 0$ for all $z \in I$. Since a semiprime ring has no non-zero nilpotent left ideal, we get $I[d(z), z] = 0$ which implies that $[d(z), z] \in I \cap ann_R(I)$. By Lemma 2.2, we get $[d(z), z] = 0$. Similarly we can prove the result for the case $F(x)F(y) - yx \in Z(R)$ for all $x, y \in I$.

Theorem 3.3. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(xy) \pm [x, y] \in Z(R)$ for all $x, y \in I$, then $x[d(x), x]_2 = 0$ for all $x \in I$.*

Proof By the hypothesis, we have

$$F(xy) + [x, y] \in Z(R) \text{ for all } x, y \in I. \tag{3.26}$$

Replacing x by zx in (3.26), we get

$$F(xy)z + xyd(z) + z[x, y] + [z, y]x \in Z(R) \text{ for all } x, y, z \in I.$$

This implies that

$$(F(xy) + [x, y])z + xyd(z) + [z, [x, y]] + [z, y]x \in Z(R). \tag{3.27}$$

Commuting (3.27) with z and using (3.26), we get

$$[xyd(z), z] + [[z, [x, y]], z] + [[z, y]x, z] = 0 \text{ for all } x, y, z \in I. \tag{3.28}$$

Replacing y by zy in (3.28), we get

$$[xzyd(z), z] + [[z, [x, zy]], z] + [[z, zy]x, z] = 0.$$

$$[xzyd(z), z] + [[z, z[x, y] + [x, z]y], z] + [z[z, y]x, z] = 0.$$

$$[xzyd(z), z] + [[z, z[x, y]], z] + [[z, [x, z]y], z] + z[[z, y]x, z] = 0.$$

This implies that

$$[xzyd(z), z] + z[[z, [x, y]], z] + [[x, z][z, y], z] + [[z, [x, z]]y, z] + z[[z, y]x, z] = 0. \tag{3.29}$$

Left multiplying (3.28) by z and subtracting from (3.29), we obtain

$$[[z, x]yd(z), z] - [[x, z][z, y], z] - [[z, [x, z]]y, z] = 0 \text{ for all } x, y, z \in I. \tag{3.30}$$

Replacing y by yz in (3.30), we get

$$[[z,x]yzd(z),z]-[[x,z][z,y],z]z-[[z,[x,z]]y,z]z=0. \tag{3.31}$$

Right multiplying (3.30) by z and subtracting from (3.31), we obtain

$$[[z,x]y[d(z),z],z]=0 \text{ for all } x,y,z\in I. \tag{3.32}$$

Substituting $zd(z)$ for x in (3.32), we get

$$[z[z,d(z)]y[d(z),z],z]=0 \text{ for all } y,z\in I. \tag{3.33}$$

Replacing y by yz in (3.33), we obtain

$$[z[d(z),z]yz[d(z),z],z]=0 \text{ for all } y,z\in I.$$

This implies that

$$z[d(z),z]yz[d(z),z]z-z^2[d(z),z]yz[d(z),z]=0. \tag{3.34}$$

Replacing y by $yz[d(z),z]w$ in (3.34), we get

$$z[d(z),z]yz[d(z),z]wz[d(z),z]z-z^2[d(z),z]yz[d(z),z]wz[d(z),z]=0 \text{ for all } y,z,w\in I. \tag{3.35}$$

Using (3.34) in (3.35), (3.35) yields that

$$z[d(z),z]yz^2[d(z),z]wz[d(z),z]-z[d(z),z]yz[d(z),z]zwz[d(z),z]=0.$$

This implies that

$$z[d(z),z]y[z[d(z),z],z]wz[d(z),z]=0 \text{ for all } y,z,w\in I.$$

After a simple calculation this gives that

$$[z[d(z),z],z]y[z[d(z),z],z]w[z[d(z),z],z]=0. \tag{3.36}$$

This implies that

$$(I[z[d(z),z],z])^3=0 \text{ for all } z\in I.$$

Since a semiprime ring has no non-zero nilpotent left ideal, we get $I[z[d(z),z],z]=0$ for all $z\in I$. This implies that $[z[d(z),z],z]\in I\cap ann_R(I)$. By Lemma 2.2, we get that $[z[d(z),z],z]=0$ that is $z[d(z),z]_2=0$ for all $z\in I$. By using similar argument we can get the result for the case $F(xy)-[x,y]\in Z(R)$ for all $x,y\in I$.

Theorem 3.4. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(xy)\pm x\circ y\in Z(R)$ for all $x,y\in I$, then $x[d(x),x]_2=0$ for all $x\in I$.*

Proof By the hypothesis, we have

$$F(xy) + x \circ y \in Z(R) \text{ for all } x, y \in I. \tag{3.37}$$

Replacing x by zx in (3.37), we get

$$F(xy)z + xyd(z) + z(x \circ y) - [z, y]x + (x \circ y)z - (x \circ y)z \in Z(R) \text{ for all } x, y, z \in I.$$

This implies that

$$(F(xy) + x \circ y)z + xyd(z) + [z, x \circ y] - [z, y]x \in Z(R). \tag{3.38}$$

Commuting (3.38) with z and using (3.37) we get

$$[xyd(z), z] + [[z, x \circ y], z] - [[z, y]x, z] = 0 \text{ for all } x, y, z \in I. \tag{3.39}$$

Substituting zy for y in (3.39), we obtain

$$[xzyd(z), z] + [[z, x \circ zy], z] - [z[z, y]x, z] = 0.$$

This implies that

$$[xzyd(z), z] + [[z, z(x \circ y) + [x, z]y], z] - [z[z, y]x, z] = 0.$$

A simple calculation yields that

$$[xzyd(z), z] + z[[z, x \circ y], z] + [[x, z][z, y], z] + [[z, [x, z]]y, z] - z[[z, y]x, z] = 0. \tag{3.40}$$

Left multiplying (3.39) by z and subtracting from (3.40), we obtain

$$[[z, x]yd(z), z] - [[x, z][z, y], z] - [[z, [x, z]]y, z] = 0. \tag{3.41}$$

This is the same as (3.30) in Theorem 3.3. Then by the same argument as we have used in Theorem 3.3, we get the result. In the similar manner the conclusion can be obtained for the case $F(xy) - x \circ y \in Z(R)$ for all $x, y \in I$.

Theorem 3.5. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(xy) \pm F(y)F(x) \in Z(R)$ for all $x, y \in I$, then $x[d(x), x]_2 = 0$ for all $x \in I$.*

Proof By the hypothesis, we have

$$F(xy) + F(y)F(x) \in Z(R) \text{ for all } x, y \in I. \tag{3.42}$$

Replacing x by zx in (3.42), we get

$$F(xy)z + xyd(z) + F(y)F(x)z + F(y)xd(z) \in Z(R) \text{ for all } x, y, z \in I. \tag{3.43}$$

Commuting (3.43) with z and using (3.42), we have

$$[xyd(z), z] + [F(y)xd(z), z] = 0 \text{ for all } x, y, z \in I. \tag{3.44}$$

Replacing y by z^2 and applying definition of F in (3.44), we get

$$[xz^2d(z), z] + [F(z)zxd(z), z] + [zd(z)xd(z), z] = 0. \tag{3.45}$$

Replacing x by zx and y by z in (3.44), we get

$$[zxzd(z), z] + [F(z)zxd(z), z] = 0 \text{ for all } x, z \in I. \tag{3.46}$$

Subtracting (3.46) from (3.45), we obtain

$$[[x, z]zd(z), z] + [zd(z)xd(z), z] = 0 \text{ for all } x, z \in I. \tag{3.47}$$

Substituting zx for x in (3.47), we get

$$z[[x, z]zd(z), z] + [zd(z)zxd(z), z] = 0. \tag{3.48}$$

Left multiplying (3.47) by z and subtracting from (3.48), we obtain

$$[z[d(z), z]xd(z), z] = 0 \text{ for all } x, z \in I. \tag{3.49}$$

Substituting xz for x in (3.49), we get

$$[z[d(z), z]xzd(z), z] = 0 \text{ for all } x, z \in I. \tag{3.50}$$

Right multiplying (3.49) by z and subtracting from (3.50), we get

$$[z[d(z), z]x[d(z), z], z] = 0 \text{ for all } x, z \in I. \tag{3.51}$$

This is the same as (3.33) in Theorem 3.3. Then by using the same technique as we have used in Theorem 3.3, we get the result. In a similar manner we can prove the result for the case $F(xy) - F(y)F(x) \in Z(R)$ for all $x, y \in I$.

An immediate consequence of above Theorems and together with Lemma 2.3 we have the following corollary:

Corollary 3.1. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a non-zero derivation $d : R \to R$. Then d maps R into $Z(R)$, the centre of R if for all $x, y \in R$, one of the following holds:*
(i)$F(xy) \pm [x, y] \in Z(R)$.
(ii)$F(xy) \pm x \circ y \in Z(R)$.
(iii)$F(xy) \pm F(y)F(x) \in Z(R)$.

Theorem 3.6. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $[F(x), y] \pm xy \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Proof By the hypothesis, we have

$$[F(x), y] + xy \in Z(R) \text{ for all } x, y \in I. \tag{3.52}$$

Replacing x by zx in (3.52), we get

$$[F(x)z + xd(z), y] + zxy + xyz - xyz \in Z(R) \text{ for all } x, y, z \in I.$$

This gives that

$$F(x)[z, y] + [F(x), y]z + x[d(z), y] + [x, y]d(z) + xyz + [z, xy] \in Z(R). \tag{3.53}$$

Commuting (3.53) with z and using (3.52), we obtain

$$[F(x)[z, y], z] + [x[d(z), y], z] + [[x, y]d(z), z] + [x[z, y] + [z, x]y, z] = 0.$$

This implies that

$$[F(x)[z, y], z] + [x[d(z), y], z] + [[x, y]d(z), z] + [x[z, y], z] + [[z, x]y, z] = 0. \tag{3.54}$$

Substituting yz for y in (3.54), we get

$$\begin{aligned} &[F(x)[z, y], z]z + [xy[d(z), z], z] + [x[d(z), y], z]z + [y[x, z]d(z), z] \\ &\qquad + [[x, y]zd(z), z] + [x[z, y], z]z + [[z, x]y, z]z = 0. \end{aligned} \tag{3.55}$$

Right multiplying (3.54) by z and subtracting from (3.55), we obtain

$$[[x, y]d(z)z, z] - [xy[d(z), z], z] - [y[x, z]d(z), z] - [[x, y]zd(z), z] = 0. \tag{3.56}$$

This implies that

$$[xyd(z)z - yxd(z)z - xyd(z)z + xyzd(z), z] - [yxzd(z) - yzxd(z), z] - [xyzd(z) - yxzd(z), z] = 0.$$

After a simple calculation this yields that

$$[y[xd(z), z], z] = 0 \text{ for all } x, y, z \in I. \tag{3.57}$$

Replacing x by $d(z)x$ in (3.57), we get

$$[y[d(z)xd(z), z], z] = 0 \text{ for all } x, y, z \in I. \tag{3.58}$$

This is the same as (3.19) in Theorem 3.2. We can complete the proof by using similar technique as we have used in Theorem 3.2. In the similar manner the conclusion can be obtained for $[F(x), y] - xy \in Z(R)$ for all $x, y \in I$.

Using similar technique with some necessary variations, we can prove the following:

Theorem 3.7. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(x) \circ y \pm xy \in Z(R)$ for all $x, y \in I$, then $[d(x), x] = 0$ for all $x \in I$.*

Theorem 3.8. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $[F(x), y] \pm yx \in Z(R)$ for all $x, y \in I$, then $[F(x), x]_2 x = 0$ and $x[d(x), x]_2 = 0$ for all $x \in I$.*

Proof We begin with the situation

$$[F(x), y] + yx \in Z(R) \text{ for all } x, y \in I. \tag{3.59}$$

Replacing y by yz in (3.59), we get

$$y[F(x), z] + [F(x), y]z + yzx + yxz - yxz \in Z(R) \text{ for all } x, y, z \in I.$$

This implies that

$$([F(x), y] + yx)z + y[F(x), z] + y[z, x] \in Z(R). \tag{3.60}$$

Commuting (3.60) with z and applying (3.59), we obtain

$$[y[F(x), z], z] + [y[z, x], z] = 0 \text{ for all } x, y, z \in I. \tag{3.61}$$

Substituting zx for x in (3.61), we obtain

$$[y[F(x)z + xd(z), z], z] + [yz[z, x], z] = 0 \text{ for all } x, y, z \in I.$$

This gives that

$$[y[F(x), z], z]z + [y[xd(z), z], z] + [yz[z, x], z] = 0. \tag{3.62}$$

Replacing y by ry in (3.62), we obtain

$$r[y[F(x), z], z]z + [r, z]y[F(x), z]z + r[y[xd(z), z], z] + [r, z]y[xd(z), z] + r[yz[z, x], z] + [r, z]yz[z, x] = 0. \tag{3.63}$$

Left multiplying (3.62) by r and subtracting from (3.63), we get

$$[r, z]y\{[F(x), z]z + [xd(z), z] + z[z, x]\} = 0 \text{ for all } x, y, z \in I \text{ and } r \in R. \tag{3.64}$$

In particular take $x = z$ in (3.64), we obtain

$$[r, z]y\{[F(z), z]z + z[d(z), z]\} = 0 \text{ for all } y, z \in I \text{ and } r \in R.$$

This implies that

$$[r, z]y[F(z)z + zd(z), z] = 0 \text{ for all } y, z \in I \text{ and } r \in R.$$

Since F is a multiplicative (generalized) reverse derivation, we conclude that

$$[r,z]y[F(z^2),z]=0 \text{ for all } y,z\in I \text{ and } r\in R. \quad (3.65)$$

Putting $r=F(z^2)$ and using semiprimeness of R, we have

$$[F(z^2),z]=0 \text{ for all } z\in I.$$

Replacing yx in place of x in (3.59), we have

$$[F(x)y+xd(y),y]+y^2x\in Z(R) \text{ for all } x,y\in I.$$

That is

$$([F(x),y]+yx)y+[xd(y),y]+y[y,x]\in Z(R) \text{ for all } x,y\in I. \quad (3.66)$$

Commuting (3.66) with y and using (3.59), we have

$$[[xd(y),y],y]+y[[y,x],y]=0 \text{ for all } x,y\in I. \quad (3.67)$$

In particular for $y=x$ in (3.67), we have $x[d(x),x]_2=0$ for all $x\in I$. Since $[F(x^2),x]=0$. It implies that

$$[F(x),x]x+x[d(x),x]=0 \text{ for all } x\in I. \quad (3.68)$$

This can be written as $[F(x),x]_2x+x[d(x),x]_2=0$ for all $x\in I$. By using $x[d(x),x]_2=0$. This implies that $[F(x),x]_2x=0$ for all $x\in I$, which is our desired result. In the similar manner the conclusion can be obtained for the case $[F(x),y]-yx\in Z(R)$ for all $x,y\in I$.

Using similar technique with some necessary variations, we can prove the following:

Theorem 3.9. *Let R be a semiprime ring and F be a non-zero multiplicative (generalized) reverse derivation associated with a map d, I be a non-zero ideal of R. If $F(x)\circ y\pm yx\in Z(R)$ for all $x,y\in I$, then $[F(x),x]_2x=0$ and $x[d(x),x]_2=0$ for all $x\in I$.*

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