EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 882-892
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

C-Tychonoff and L-Tychonoff Topological Spaces

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Abstract. A topological space X is called *C-Tychonoff* if there exist a one-to-one function f from X onto a Tychonoff space Y such that the restriction $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. We discuss this property and illustrate the relationships between C-Tychonoffness and some other properties like submetrizability, local compactness, L-Tychonoffness, C-normality, C-regularity, epinormality, σ-compactness, pseudocompactness and zero-dimensional.

2010 Mathematics Subject Classifications: 54D10, 54D15, 54C10

Key Words and Phrases: Tychonoff, C-Tychonoff, C-normal, C-regular, epinormal, submetrizable, L-Tychonoff

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1. Introduction

We define a new topological property called *C-Tychonoff*. Unlike C-normality[2], we prove that C-Tychonoffness is a topological property which is multiplicative and hereditary. We show that C-Tychonoff and C-normal are independent. Also we investigate the function witnesses the C-Tychonoffness when it is continuous and when it is not. We introduce the notion of L-Tychonoffness. Throughout this paper, we denoted of the set of positive integers by $\mathbb{N}$, and an order pair by $\langle x, y \rangle$. An ordinal γ is the set of all ordinal α, with $\alpha < \gamma$, we denoted the first infinite ordinal by ω_0 and the first uncountable ordinal by ω_1. A T_3 space is a T_1 regular space, a Tychonoff ($T_{3\frac{1}{2}}$) space is a T_1 completely regular space, and a T_4 space is a T_1 normal space. For a subset B of a space X, int$B$ denote the interior of B and $\overline{B}$ denote the closure of B. A space X is locally compact if for each $y \in X$ and each open neighborhood U of y there exists an open neighborhood V of y such that $y \in V \subseteq \overline{V} \subseteq U$ and $\overline{V}$ is compact, we do not assume T_2 in the definition of local compactness.

DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3253

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2. C-Tychonoffness

Definition 1. A topological space X is called *C-Tychonoff* if there exist a one-to-one function f from X onto a Tychonoff space Y such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$.

Recall that a topological space $(X, \mathcal{T})$ is called *submetrizable* if there exists a metric d on X such that the topology $\mathcal{T}_d$ on X generated by d is coarser than $\mathcal{T}$, i.e., $\mathcal{T}_d \subseteq \mathcal{T}$, see [10].

Theorem 1. Every submetrizable space is C-Tychonoff.

Proof. Let $\mathcal{T}'$ be a metrizable topology on X such that $\mathcal{T}' \subseteq \mathcal{T}$. Then $(X, \mathcal{T}')$ is Tychonoff and the identity function $id_X : (X, \mathcal{T}) \longrightarrow (X, \mathcal{T}')$ is a bijective and continuous. If K is any compact subspace of $(X, \mathcal{T})$, then $id_X(K)$ is Hausdorff being a subspace of the metrizable space $(X, \mathcal{T}')$, and the restriction of the identity function on K onto $id_X(K)$ is a homeomorphism by [8, 3.1.13].

Since any Hausdorff locally compact space is Tychonoff, then we have the following theorem.

Theorem 2. Every Hausdorff locally compact space is C-Tychonoff.

The converse of Theorem 1 is not true in general. For example, the Tychonoff Plank $((\omega_1+1)\times(\omega_0+1))\setminus\{\langle\omega_1,\omega_0\rangle\}$ is C-Tychonoff being Hausdorff locally compact, but it is not submetrizabl, because if it was, then $(\omega_1+1)\times\{0\} \subseteq ((\omega_1+1)\times(\omega_0+1))\setminus\{\langle\omega_1,\omega_0\rangle\}$ is submetrizabl, because submetrizablity is hereditary, but $((\omega_1+1)\times\{0\} \cong \omega_1+1$ and ω_1+1 is not submetrizabl.
The converse of Theorem 2 is not true in general as the Dieudonné Plank [16] is Tychonoff, hence C-Tychonoff but not locally compact. Hausdorffness is essential in Theorem 2. Here is an example of a locally compact space which is neither C-Tychonoff nor Hausdorff.

Example 1. The particular point topology $\mathcal{T}_{\sqrt{2}}$ on $\mathbb{R}$, see [16], is not C-Tychonoff. It is well-known that $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is neither T_1 nor Tychonoff. If $B \subseteq \mathbb{R}$, then $\{\{x, \sqrt{2}\} : x \in B\}$ is an open cover for B, thus a subset B of $\mathbb{R}$ is compact if and only if it is finite. To show that $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is not C-Tychonoff, suppose that $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is C-Tychonoff. Let Z be a Tychonoff space and $f : \mathbb{R} \longrightarrow Z$ be a bijective function such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace K of $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$. Take $K = \{x, \sqrt{2}\}$, such that $x \neq \sqrt{2}$, hence K is a compact subspace of $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$. By assumption $f_{|_K} : K \longrightarrow f(K) = \{f(x), f(\sqrt{2})\}$ is a homeomorphism. Because $f(K)$ is a finite subspace of Z and Z is T_1, then $f(K)$ is discrete subspace of Z. Therefore,we obtain that $f_{|_K}$ is not continuous and this a contradiction as $f_{|_K}$ is a homeomorphism. Thus $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is not C-Tychonoff. ■

By the definition, it is clear that a compact C-Tychonoff space must be Tychonoff see Theorem 3 below. Obviously, any Tychonoff space is C-Tychonoff, just by taking $Y = X$ and f to be the identity function, but the converse is not true in general. For example, the Half-Disc space [16] is C-Tychonoff which is not Tychonoff. It is C-Tychonoff because it is submetrizable. C-Tychonoffness does not imply Tychonoffness even with first countability. For example, Smirnov's deleted sequence topology [16] is first countable and C-Tychonoff being submetrizabl but not Tychonoff.

Theorem 3. If X is a compact non-Tychonoff space, then X cennot be C-Tychonoff.

We conclude that from the above theorem, $\mathbb{R}$ with the finite complement topology is not C-Tychonoff.

Theorem 4. If X is a T_1-space such that the only compact subspace are the finite subspace, then X is C-Tychonoff.

Proof. Let $Y = X$ and consider Y with the discrete topology. Then the identity function from X onto Y is a bijective function. If K is any compact subspace of $(X, \mathcal{T})$, then by assumption K is a finite subspace. Because any finite set in a T_1-space is discrete, hence the restriction of the identity function on K onto K is a homeomorphism since both of the domain and the codomain are discrete and have the same cardinality.

If X is C-Tychonoff and $f : X \longrightarrow Y$ is a witness of the C-Tychonoffness of X, then f may not be continuous. Here is an example.

Example 2. Consider $\mathbb{R}$ with the countable complement topology $\mathcal{CC}$ [16]. Since the only compact subspace are the finite subspaces and $(\mathbb{R}, \mathcal{CC})$ is T_1, then the compact subspace are discrete. Hence $\mathbb{R}$ with the discrete topology and the identity function will give the C-Tychonoffness, see Theorem 4. Observe that the identity function in this case is not continuous. ■

Recall that a space X is *Fréchet* if for any subset B of X and any $x \in \overline{B}$ there exist a sequence $(b_n)_{n\in\mathbb{N}}$ of points of B such that $b_n \longrightarrow x$, see [8].

Theorem 5. If X is C-Tychonoff and Fréchet, then any function witnesses its C-Tychonoffness is continuous.

Proof. Let X be C-Tychonoff and Fréchet. Let $f : X \longrightarrow Y$ be a witness of the C-Tychonoffness of X. Take $B \subseteq X$ and pick $y \in f(\overline{B})$. There is a unique $x \in X$ such that $f(x) = y$, thus $x \in \overline{B}$. Since X is Fréchet, then there exists a sequence $(b_n) \subseteq B$ such that $b_n \longrightarrow x$. The sequence $K = \{x\} \cup \{b_n : n \in \mathbb{N}\}$ of X is compact since it is a convergent sequence with its limit, thus $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism. Let

$W \subseteq Y$ be any open neighborhood of y. Then $W \cap f(K)$ is open in the subspace $f(K)$ containing y. Since $f(\{b_n : n \in \mathbb{N}\}) \subseteq \underline{f(K)} \cap \overline{f(B)}$ and $W \cap f(K) \neq \emptyset$, then we have $W \cap f(B) \neq \emptyset$. Hence $y \in \overline{f(B)}$ and $f(\overline{B}) \subseteq \overline{f(B)}$. Thus f is continuous.

Since any first countable space is Fréchet [8], we conclude the following corollary:

Corollary 1. If X is C-Tychonoff first countable and $f : X \longrightarrow Y$ witnessing the C-Tychonoffness of X, then f is continuous.

Corollary 2. Any C-Tychonoff Fréchet space is Urysohn.

Proof. Let $(X, \mathcal{T})$ be any C-Tychonoff Fréchet space. We may assume that X has more than one element. Pick a Tychonoff space $(Y, \mathcal{T}')$ and a bijection function $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ such that $f_{|A} : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace A of X. Since X is Fréchet, then f is continuous. Define a topology $\mathcal{T}^\star$ on X as follows: $\mathcal{T}^\star = \{ f^{-1}(U) : U \in \mathcal{T}' \}$. It clear that $\mathcal{T}^\star$ is a topology on X coarser that $\mathcal{T}$ such that $f : (X, \mathcal{T}^\star) \longrightarrow (Y, \mathcal{T}')$ is continuous. If $W \in \mathcal{T}^\star$, then W is of the form $W = f^{-1}(U)$ where $U \in \mathcal{T}'$. So, $f(W) = f(f^{-1}(U)) = U$ which gives that f is open, hence homeomorphism. Thus $(X, \mathcal{T}^\star)$ is Tychonoff. Pick distinct $a, b \in X$. Using T_2 of $(X, \mathcal{T}^\star)$, choose $G, H \in \mathcal{T}^\star$ such that $a \in G, b \in H$, and $G \cap H = \emptyset$. Using regularity of $(X, \mathcal{T}^\star)$, choose $U, V \in \mathcal{T}^\star$ such that $a \in U \subseteq \overline{U}^{\mathcal{T}^\star} \subseteq G$ and $b \in V \subseteq \overline{V}^{\mathcal{T}^\star} \subseteq H$. We have that $U, V \in \mathcal{T}$ and since $\overline{B}^{\mathcal{T}} \subseteq \overline{B}^{\mathcal{T}^\star}$ for any $B \subseteq X$, we get $\overline{U}^{\mathcal{T}} \cap \overline{V}^{\mathcal{T}} = \emptyset$. Therefore, $(X, \mathcal{T})$ is Urysohn.

So, we conclude that any first countable C-Tychonoff space is Hausdorff.

Recall that a space X is a *k-space* if X is T_2 and it is a quotient image of a locally compact space [8]. By the theorem: " *a function f from a k-space X into a space Y is continuous if and only if $f_{|Z} : Z \longrightarrow Y$ is continuous for each compact subspace Z of X*", [8, 3.3.21]. We conclude the following:

Corollary 3. If X is a C-Tychonoff k-space and $f : X \longrightarrow Y$ witnessing the C-Tychonoffness of X, then f is continuous.

Recall that a topological space X is called *C-normal* if there exist a one-to-one function f from X onto a normal space Y such that the restriction $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$[2].

Theorem 6. Every C-Tychonoff Fréchet Lindelöf space is C-normal.

Proof. Let X be any C-Tychonoff Fréchet Lindelöf space. Pick a Tychonoff space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. By Theorem 5, f is continuous. Since the continuous image of a Lindelöf space is Lindelöf [8, 3.8.7], we conclude that Y is Lindelöf, hence normal as any regular Lindelöf space is normal [8, 3.8.2]. Therefore, X is C-normal.

C-normality and C-Tychonoffness are independent from each other. Here is an example of a C-normal which is not C-Tychonoff.

Example 3. Consider $\mathbb{R}$ with its *right ray topology* $\mathcal{R}$ [16]. So, $\mathcal{R} = \{\emptyset, \mathbb{R}\} \cup \{ (x, \infty) : x \in \mathbb{R} \}$. Since any two non-empty closed sets must intersect, then $(\mathbb{R}, \mathcal{R})$ is normal, hence C-normal [2]. Now, suppose that $(\mathbb{R}, \mathcal{R})$ is C-Tychonoff. Pick a Tychonoff space Y and a bijective function $f : \mathbb{R} \longrightarrow Y$ such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq \mathbb{R}$. It is well-known that a subspace K of $(\mathbb{R}, \mathcal{R})$ is compact if and only if K has a minimal element. Thus $[2, \infty)$ is compact, hence $f_{|_{[2,\infty)}} : [2, \infty) \longrightarrow f([2, \infty)) \subset Y$ is a homeomorphism. i.e. $f([2, \infty))$ as a subspace of $(\mathbb{R}, \mathcal{R})$ is regular which is a contradiction as $[2, 3]$ is closed in $[2, \infty)$ and $5 \notin [2, 3]$ and any non-empty open sets in $[2, \infty)$ must intersect. Therefore, $(\mathbb{R}, \mathcal{R})$ cannot be C-Tychonoff. ■

Here is an example of a C-Tychonoff space which is not C-normal.

Example 4. Consider the infinite Tychonoff product space $G = D^{\omega_1} = \prod_{\alpha \in \omega_1} D$, where $D = \{0, 1\}$ considered with the discrete topology. Let H be the subspace of G consisting of all points of G with at most countably many non-zero coordinates. Put $M = G \times H$. Raushan Buzyakova proved that M cannot be mapped onto a normal space Z by a bijective continuous function [7]. Using Buzyakova's result and the fact that M is a k-space, we conclude that M is a Tychonoff space which is not C-normal [13]. Since M is Tychonoff, then it is C-Tychonoff. ■

Theorem 7. C-Tychonoffness is a topological property.

Proof. Let X be a C-Tychonoff space and $X \cong Y$. Let Z be a Tychonoff space and let $f : X \longrightarrow Z$ be a bijective function such that the restriction $f_{|_K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. Let $h : Y \longrightarrow X$ be a homeomorphism. Then Z and $f \circ h : Y \longrightarrow Z$ satisfies the requirement.

Theorem 8. C-Tychonoffness is an additive property.

Proof. Let X_s be a C-Tychonoff space for each $s \in S$. We prove that their sum $\oplus_{s\in S} X_s$ is C-Tychonoff. For each $s \in S$, pick a Tychonoff space Y_s and a bijective function $f_s : X_s \longrightarrow Y_s$ such that $f_{s_{|K_s}} : K_s \longrightarrow f_s(K_s)$ is a homeomorphism for each compact subspace K_s of X_s. Because Y_s is Tychonoff for each $s \in S$, then the sum $\oplus_{s\in S} Y_s$ is Tychonoff, [8, 2.2.7]. Consider the function sum [8, 2.2.E] $f = \oplus_{s\in S} f_s : \oplus_{s\in S} X_s \longrightarrow \oplus_{s\in S} Y_s$ defined by $f(x) = f_s(x)$ if $x \in X_s, s \in S$. A subspace $K \subseteq \oplus_{\alpha\in\Lambda} X_\alpha$ is compact if and only if the set $S_0 = \{s \in S : K \cap X_s \neq \emptyset\}$ is finite and $K \cap X_s$ is compact in X_s for each $s \in S_0$. If $K \subseteq \oplus_{s\in S} X_s$ is compact. then $(\oplus_{s\in S} f_s)_{|K}$ is a homeomorphism since $f_{s_{|K\cap X_s}}$ is a homeomorphism for each $s \in S_0$.

Theorem 9. C-Tychonoffness is a multiplicative property.

Proof. Let X_s be a C-Tychonoff space for each $s \in S$. Pick a Tychonoff space Y_s and a bijective function $f_s : X_s \longrightarrow Y_s$ such that $f_{s_{|K_s}} : K_s \longrightarrow f_s(K_s)$ is a homeomorphism for each compact subspace K_s of X_s. Since Y_s is Tychonoff for each $s \in S$, then the Cartesian product $\prod_{s\in S} Y_s$ is Tychonoff [8, 2.3.11]. Define $f : \prod_{s\in S} X_s \longrightarrow \prod_{s\in S} Y_s$ by $f((x_s : s \in S)) = (f_s(x_s) : s \in S)$ for each $s \in S$, then f is bijective. Let $K \subseteq \prod_{s\in S} X_s$ be any compact subspace and let p_s be the usual projection, then $p_s(K) \subseteq X_s$ is compact. Now, $K \subseteq \prod_{s\in S} p_s(K) = K^\star$ is compact, by the Tychonoff theorem. Hence $f_{|K^\star} = \prod_{s\in S} f_s{}_{|_{p_s(K)}}$ is a homeomorphism. Thus $f_{|K}$ is a homeomorphism, because the restriction of a homeomorphism is a homeomorphism.

Theorem 10. C-Tychonoffness is a hereditary property.

Proof. Let A be any non empty subspace of C-Tychonoff space X. Pick a bijective function f from X onto a Tychonoff space Y such that $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$. Let $B = f(A) \subseteq Y$. Then B is Tychonoff being a subspace of a Tychonoff space Y. Now, we have $f_{|A} : A \longrightarrow B$ is a bijective function. Since any compact subspace of A is compact in X and $f_{|A_{|K}} = f_{|K}$, we conclude that A is C-Tychonoff.

Frome Theorem 9 and Theorem 10, we conclude the following corollary.

Corollary 4. $\prod_{s\in S} X_s$ is C-Tychonoff if and only if X_s is C-Tychonoff $\forall s \in S$.

3. L-Tychonoffness and Other Properties

We introduce another new topological property called *L-Tychonoff* .

Definition 2. A topological space X is called *L-Tychonoff* if there exist a one-to-one function f from X onto a Tychonoff space Y such that the restriction $f_{|_L} : L \longrightarrow f(L)$ is a homeomorphism for each Lindelöf subspace $L \subseteq X$.

By the definition it is clear that a Lindelöf L-Tychonoff space must be Tychonoff. Since any compact space is Lindelöf, then any L-Tychonoff space is C-Tychonoff. The converse is not true in general. Obviously, no Lindelöf non-Tychonoff space is L-Tychonoff. So, no countable complement topology on uncountable set X is L-Tychonoff, but it is C-Tychonoff, see Example 2. An example of an L-Tychonoff space which is not Tychonoff.

Example 5. Consider ω_2, the successor cardinal number of the cardinal number ω_1. Let $X = \omega_2 \cup \{i, j\}$ where $\{i, j\} \cap \omega_2 = \emptyset$, so $i \notin \omega_2$ and $j \notin \omega_2$. Generate a topology on X as follows: Each $\alpha \in \omega_2$ is isolated. A basic open neighborhood of i is of the form $U = \{i\} \cup (\omega_2 \setminus E)$ where $E \subset \omega_2$ with $|E| = \omega_1$. Similarly, a basic open neighborhood of j is of the form $V = \{j\} \cup (\omega_2 \setminus F)$ where $F \subset \omega_2$ with $|F| = \omega_1$. Then X is not T_2 as i and j cannot be separated by disjoint open sets. X is not Lindelöf as the open cover $\{\{i\} \cup (\omega_2 \setminus \omega_1), \{j\} \cup (\omega_2 \setminus \omega_1), \{\alpha\} : \alpha \in \omega_1\}$ of X has no countable subcover. Also, if C is any countable subspace of X, then C is discrete as a subspace because if $i \in C$, then $U = \{i\} \cup (\omega_2 \setminus (\omega_1 \cup (C \setminus \{j\})))$ is an open neighborhood of i in X such that $U \cap C = \{i\}$. Similarly, if $j \in C$. It is clear that if C is countable, then C is Lindelöf. Assume that C is uncountable. Then $|C| \geq \omega_1$. Suppose that $\{i, j\} \subset C$. Partition C into three partitions C_1, C_2, and C_3 such that $i \in C_1$ with $|C_1| = \omega_1$, $j \in C_2$ with $|C_2| = \omega_1$, and $|C_3| \geq \omega_1$. The open cover $\{\{i\} \cup (\omega_2 \setminus ((C_1 \cup C_2) \setminus \{i, j\})), \{j\} \cup (\omega_2 \setminus ((C_1 \cup C_2) \setminus \{i, j\}))), \{\alpha\} : \alpha \in C_1 \cup C_2\}$ of C has no countable subcover. If C contains either i or j, we do the same idea but for just two partitions. Thus a subspace C of X is Lindelöf if and only if C is countable. Thus X is L-Tychonoff which is not Tychonoff.

A function $f : X \longrightarrow Y$ witnessing the L-Tychonoffness of X need not be continuous. But it will be if X is of countable tightness. Recall that a space X is of *countable tightness* if for each subset B of X and each $x \in \overline{B}$, there exists a countable subset B_0 of B such that $x \in \overline{B_0}$ [8].

Theorem 11. If X is L-Tychonoff and of countable tightness and $f : X \longrightarrow Y$ is a witness of the L-Tychonoffness of X, then f is continuous.

Proof. Let A be any non-empty subset of X. Let $y \in f(\overline{A})$ be arbitrary. Let $x \in X$ be the unique element such that$f(x) = y$. Then $x \in \overline{A}$. Pick a countable subset $A_0 \subseteq A$ such that $x \in \overline{A_0}$. Let $B = \{x\} \cup A_0$; then B is a Lindelöf subspace of X and hence $f_{|_B} : B \longrightarrow f(B)$ is a homeomorphism. Now, let $V \subseteq Y$ be any open neighborhood of y; then $V \cap f(B)$ is open in the subspace $f(B)$ containing y. Thus $f^{-1}(V) \cap B$ is open in the subspace B containing x. Thus $(f^{-1}(V) \cap B) \cap A_0 \neq \emptyset$. So $(f^{-1}(V) \cap B) \cap A \neq \emptyset$. Hence $\emptyset \neq f((f^{-1}(V) \cap B) \cap A) \subseteq f(f^{-1}(V) \cap A) = V \cap f(A)$. Thus $y \in \overline{f(A)}$. Therefore, f is continuous.

Recall that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X, then the *convergency set of* (x_n) is defined by $C(x_n) = \{x \in X : x_n \longrightarrow x\}$ and a topological space X is *sequential* if for any $A \subseteq X$ we have that A is closed if and only if $C(x_n) \subseteq A$ for any sequence $(x_n) \subseteq A$, see [8]. We have the following implications, see [8, 1.6.14, 1.7.13].

First countability ⇒ Fréchet ⇒ Sequential ⇒ Countable tightness.

Corollary 5. If X is L-Tychonoff and first countable (Fréchet, Sequential) and $f : X \longrightarrow Y$ is a witness of the L-Tychonoffness of X, then f is continuous.

Theorem 12. L-Tychonoffness is a topological property.

Theorem 13. L-Tychonoffness is an additive property.

Theorem 14. L-Tychonoffness is a multiplicative property.

Theorem 15. L-Tychonoffness is a hereditary property.

Theorem 16. If any countable subspace of a space X is discrete and the only Lindelöf subspaces are the countable subspaces, then X is L-Tychonoff.

Proof. Let $Y = X$ and consider Y with the discrete topology. Then the identity function from X onto Y is a bijective function. If K is any Lindelöf subspace of X, then, by assumption, K is countable and discrete, hence the restriction of the identity function on K onto K is a homeomorphism.

Theorem 17. If X is C-Tychonoff space such that each Lindelöf subspace is contained in a compact subspace, then X is L-Tychonoff.

Proof. Assume that X is C-Tychonoff and if L is any Lindelöf subspace of X, then there exists a compact subspace K with $L \subseteq K$. Let f be a bijective function from X onto a Tychonoff space Y such that the restriction $f_{|C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace C of X. Now, let L be any Lindelöf subspace of X. Pick a compact subspace K of X where $L \subseteq K$, then $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism, thus $f_{|_L} : L \longrightarrow f(L)$ is a homeomorphism as $(f_{|K})_{|_L} = f_{|_L}$.

Now, we study some relationships between C-Tychonoffness and some other properties.

Recall that a topological space X is called *C-regular* if there exist a one-to-one function f from X onto a regular space Y such that the restriction $f_{|K} : K \longrightarrow f(K)$ is a homeomorphism for each compact subspace $K \subseteq X$ [5]. Any C-Tychonoff space is C-regular space, but the converse is not true in general. For example, any indiscrete space which has more than one element is an example of C-regular space which is not C-Tychonoff by Theorem 3.

Recall that a topological space $(X, \mathcal{T})$ is called *epinormal* if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4 [3]. By a similar proof as that of Theorem 1 above, we can prove the following corollary:

Corollary 6. Any epinormal space is C-Tychonoff.

$\mathbb{R}$ with the countable complement topology $\mathcal{CC}$ [16], is an example of C-Tychonoff space which is not epinormal because $(\mathbb{R}, \mathcal{CC})$ is not T_2 and any epinormal space is T_2 [3].

Let X be any Hausdorff non-k-space. Let $kX = X$. Define a topology on kX as follows: a subset of kX is open if and only if its intersection with any compact subspace C of the space X is open in C. kX with this topology is Hausdorff and k-space such that X and kX have the same compact subspace and the same topology on these subspace [6], we conclude the following:

Theorem 18. If X is Hausdorff but not k-space, then X is C-Tychonoff if and only if kX is C-Tychonoff.

C-Tychonoffness and σ-compactness are independent from each other. For example the rational sequence space [16] is C-Tychonoff being Tychonoff, but not σ-compact. $\mathbb{R}$ with the finite complement topology is not C-Tychonoff by Theorem 3, but it is σ-compact being compact. Any pseudocompact is C-Tychonoff being Tychonoff, but the converse is not true, for example Sorgenfrey line square topology [16], it is C-Tychonoff being Tychonoff but not pseudocompact. Also any zero-dimensional space is C-Tychonoff, but the converse is not true, for example Niemytzki's tangent disc topology [16], it is C-Tychonoff being Tychonoff but not zero-dimensional because it is connected.

Let X be any topological space. Let $X' = X \times \{a\}$. Note that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, a \rangle$ in X' by x' and for a subset $E \subseteq X$ let $E' = \{x' : x \in E\} = E \times \{a\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of X*. Similar proof as in [2], we get the following theorem.

Theorem 19. If X is C-Tychonoff, then its Alexandroff Duplicate $A(X)$ is also C-Tychonoff.

Also a similar proof as in [15], we get the following theorem.

Theorem 20. If X is L-Tychonoff, then its Alexandroff Duplicate $A(X)$ is also L-Tychonoff.

Acknowledgements

The authors wish to express their sincere thanks to the referee for his/her helpful comments and valuable suggestions.

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