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Left and right magnifying elements in generalized semigroups of transformations by using partitions of a set

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Abstract. An element a of a semigroup S is called left [right] magnifying if there exists a proper subset M of S such that $S = aM$ [$S = Ma$]. Let X be a nonempty set and $T(X)$ be the semigroup of all transformations from X into itself under the composition of functions. For a partition $P = \{X_\alpha \mid \alpha \in I\}$ of the set X, let $T(X,P) = \{f \in T(X) \mid (X_\alpha)f \subseteq X_\alpha \text{ for all } \alpha \in I\}$. Then $T(X,P)$ is a subsemigroup of $T(X)$ and if $P = \{X\}, T(X,P) = T(X)$. Our aim in this paper is to give necessary and sufficient conditions for elements in $T(X,P)$ to be left or right magnifying. Moreover, we apply those conditions to give necessary and sufficient conditions for elements in some generalized linear transformation semigroups.

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1. Introduction and Preliminaries

The notions of left and right magnifying elements of semigroups were introduced by Ljapin [7]. An element a of a semigroup S is called left [right] magnifying if there exists a proper subset M of S such that $S = aM$ [$S = Ma$]. Minimal subsets associated with the magnifying element, were introduced and studied by Migliorini in [9] and [10]. In [2], Catino and Migliorini gave necessary and sufficient conditions for any semigroup to contain left or right magnifying elements. In [8], Magill, Jr. gave necessary and sufficient

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conditions for elements in transformation semigroups to be left or right magnifying and applied those conditions for elements in linear transformation semigroups and semigroups of all continuous selfmaps of topological spaces to be left or right magnifying. Gutan studied semigroups with strong and nonstrong magnifying elements in [3] and showed that every semigroup containing magnifying elements is factorizable in [4]. In [5], semigroups with magnifiers admitting minimal subsemigroups were studied by Gutan. Semigroups with good and bad magnifying were investigated by Gutan and Kisielewicz in [6]. Let X be a nonempty set and let $T(X)$ be the set of all transformations from X into itself, that is, $T(X) = \{f : X \to X \mid f \text{ is a function}\}$. It is well-known that $T(X)$ is a semigroup under the composition of functions and it is called the full transformation semigroup on X. Transformation semigroups play an important role in semigroup theory since it is well-known that every semigroup is isomorphic to a subsemigroup of a suitable full transformation semigroup. We will write functions from the right, $(x)f$ rather than $f(x)$ and compose from the left to the right, $(x)(fg)$ rather than $(g \circ f)(x)$, for $f, g \in T(X)$ and $x \in X$. For a partition $P = \{X_\alpha \mid \alpha \in I\}$ of a set X, consider the semigroup $T(X, P) = \{f \in T(X) \mid (X_\alpha)f \subseteq X_\alpha \text{ for all } \alpha \in I\}$. We have that $T(X, P)$ is a subsemigroup of $T(X)$ and if $P = \{X\}$, then $T(X, P) = T(X)$. In 2015, Araujo, Bentz, Mitchelll and Schneider [1] solved the problem of finding the minimum size of the generating sets of $T(X, P)$, when P is an arbitrary partition. Next, in 2016, Purisang and Rakbud investigated the regularity of transformation semigroups which defined by a partition in [11]. These are our motivation to do this research. Our aim in this paper is to give necessary and sufficient conditions for elements in $T(X, P)$ to be left or right magnifying.

2. Left magnifying elements of $T(X, P)$

Our purpose in this section is to give necessary and sufficient conditions for elements in $T(X, P)$ to be left magnifying.

Lemma 1. *If a function f is left magnifying of $T(X, P)$, then f is one-to-one.*

Proof. Assume that f is a left magnifying element of $T(X, P)$. Then there exists a proper subset M of $T(X, P)$ such that $fM = T(X, P)$. Let id_X be an identity function on X. Clearly, $id_X \in T(X, P)$. So there exists a function $h \in M$ such that $fh = id_X$. This implies that f is one-to-one.

Lemma 2. *If $f \in T(X, P)$ is bijective, then f is not left magnifying of $T(X, P)$.*

Proof. Suppose that f is left magnifying of $T(X, P)$. Then there exists a proper subset M of $T(X, P)$ such that $fM = T(X, P)$. This implies that $fM = fT(X, P)$. Since f is bijective, its inverse function f^{-1} exists and $f^{-1} \in T(X, P)$. So $M = f^{-1}fM = f^{-1}fT(X, P) = T(X, P)$, a contradiction. Therefore, f is not left magnifying of $T(X, P)$.

Lemma 3. *If $f \in T(X, P)$ is one-to-one but not onto, then f is left magnifying of $T(X, P)$.*

Proof. Assume that f is one-to-one but not onto. Let $M = \{h \in T(X,P) \mid (x)h = x$ for all $x \notin ran\, f\}$. Then M is a proper subset of $T(X,P)$. Claim that $fM = T(X,P)$. Let g be any function in $T(X,P)$. Define a function $h \in T(X,P)$ by for all $x \in X$,

$$(x)h = \begin{cases} (x')g & \text{if } x \in ran\, f \text{ and } (x')f = x, \\ x & \text{if } x \notin ran\, f. \end{cases}$$

Let $x', x \in X$ be such that $(x')f = x$ and $x \in X_\alpha$ for some $\alpha \in I$. Clearly, $x' \in X_\alpha$. Therefore, $(x)h = (x')g \in X_\alpha$. Then $h \in M$. For all $x \in X$, we have

$$(x)fh = ((x)f)h = (x)g.$$

Then $fh = g$, this implies that $fM = T(X,P)$. Hence f is left magnifying of $T(X,P)$.

Example 1. Consider $X = \mathbb{N}$ and $P = \{\{x \mid x \text{ is odd}\}, \{x \mid x \text{ is even}\}\}$.
Let $f \in T(X,P)$ by

$$(x)f = \begin{cases} x+2 & \text{if } x \text{ is even}, \\ x & \text{if } x \text{ is odd}, \end{cases}$$

that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & \cdots \end{pmatrix}.$$

Then $f \in T(X,P)$ and f is one-to-one but not onto because $2 \notin ran\, f$. Let $M = \{h \in T(X,P) \mid (2)h = 2\}$. Let $g \in T(X,P)$ be any function. Define a function $h \in T(X,P)$ by

$$(x)h = \begin{cases} 2 & \text{if } x = 2, \\ (x-2)g & \text{if } x \text{ is even and } x > 2, \\ (x)g & \text{if } x \text{ is odd}. \end{cases}$$

So $h \in M$. If x is odd, we have $(x)fh = ((x)f)h = (x)h = (x)g$. If x is even, we have $(x)fh = ((x)f)h = (x+2)h = (x)g$. Then $fh = g$. For example, if $g \in T(X,P)$ such that

$$(x)g = \begin{cases} 2x & \text{if } x \text{ is even}, \\ x & \text{if } x \text{ is odd}, \end{cases}$$

that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 3 & 8 & 5 & 12 & 7 & 16 & \cdots \end{pmatrix}.$$

Define a function $h \in T(X,P)$ by $(2)h = 2, (2x+2)h = (2x)g = 4x$ and $(2x-1)h = (2x-1)g = 2x-1$ for all $x \in X$, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 8 & 7 & 12 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$fh = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 8 & 7 & 12 & \cdots \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 4 & 3 & 8 & 5 & 12 & 7 & 16 & \cdots \end{pmatrix} = g.$$

The following theorem is the main result in this section.

Theorem 1. *Let $P = \{X_\alpha \mid \alpha \in I\}$ be a partition of a set X.*

(1) *A semigroup $T(X, P)$ has a left magnifying element if and only if X_α is infinite for some $\alpha \in I$.*

(2) *A function f is left magnifying of $T(X, P)$ if and only if f is one-to-one but not onto.*

Proof. This follows by Lemma 1, Lemma 2 and Lemma 3.

Example 2. Let $X = \mathbb{N}$ and $P = \{\{1, 2\}, \{3, 4, 5\}, \{x \mid x > 5\}\}$. By Theorem 1(1), $T(X, P)$ has a left magnifying element. Let $f \in T(X, P)$ by

$$(x)f = \begin{cases} x & \text{if } x \le 6, \\ x + 1 & \text{if } x > 6, \end{cases}$$

that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & \cdots \end{pmatrix}.$$

Then f is one-to-one but not onto. By Theorem 1(2), f is left magnifying of $T(X, P)$.

Corollary 1. *The following statements hold for a semigroup $T(X)$.*

(1) *A semigroup $T(X)$ has a left magnifying if and only if X is infinite.*

(2) *A function f is left magnifying of $T(X)$ if and only if f is one-to-one but not onto.*

Proof. This follows by Theorem 1 by using $P = \{X\}$.

3. Right magnifying elements of $T(X, P)$

In this section, we give necessary and sufficient conditions for elements in $T(X, P)$ to be right magnifying.

Lemma 4. *If f is a right magnifying element of $T(X, P)$, then f is onto.*

Proof. Assume that f is a right magnifying element of $T(X, P)$. Then there exists a proper subset M of $T(X, P)$ such that $Mf = T(X, P)$. Since $id_X \in T(X, P)$, there exists a function $h \in M$ such that $hf = id_X$. This implies that f is onto.

Lemma 5. *If $f \in T(X,P)$ is bijective, then f is not right magnifying of $T(X,P)$.*

Proof. Assume that f is bijective. Then its inverse function f^{-1} exists and $f^{-1} \in T(X,P)$. Suppose that f is a right magnifying element of $T(X,P)$. Then there exists a proper subset M of $T(X,P)$ such that $Mf = T(X,P)$. Hence $Mf = T(X,P)f$ and $M = Mff^{-1} = T(X,P)ff^{-1} = T(X,P)$, a contradiction. Therefore, f is not right magnifying of $T(X,P)$.

Lemma 6. *Let $f \in T(X,P)$ be onto but not one-to-one. Then f is right magnifying of $T(X,P)$.*

Proof. Assume that f is onto but not one-to-one. Let $M = \{h \in T(X,P) \mid h \text{ is not onto}\}$. Then $M \neq T(X,P)$. Let g be any function in $T(X,P)$. Since f is onto, there exists for each $x \in X_\alpha$, an element $y_x \in X_\alpha$ such that $(y_x)f = (x)g$ (if $(x_1)g = (x_2)g$, we must choose $y_{x_1} = y_{x_2}$). Define a function $h \in T(X,P)$ by $(x)h = y_x$ for all $x \in X$. We claim that h is not onto. Since f is not one-to-one, there exist an element $y' \in X$ and distinct elements $y_1, y_2 \in X$ such that $(y_1)f = (y_2)f = y'$. If $y' \notin ran\, g$, we have $y_1, y_2 \notin ran\, h$. If $y' \in ran\, g$, there is at most one between y_1 and y_2 in $ran\, h$. Then h is not onto. Hence $h \in M$ and for all $x \in X$, we have

$$(x)hf = (y_x)f = (x)g.$$

Then $hf = g$, hence $Mf = T(X,P)$. Therefore, f is right magnifying of $T(X,P)$.

Example 3. Consider $X = \mathbb{N}$ and $P = \{\{x \mid x \text{ is odd}\}, \{x \mid x \text{ is even}\}\}$.
Let $f \in T(X,P)$ by $f(1) = 1, f(2) = 2$ and $(x)f = x - 2$ for all positive integer $x > 2$, that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix}.$$

Then $f \in T(X,P)$ and f is onto but not one-to-one. Let $M = \{h \in T(X,P) \mid h \text{ is not onto}\}$. Let g be any function in $T(X,P)$. By Lemma 6, there exists $h \in M$ such that $hf = g$. For example, if $g \in T(X,P)$ is such that $(x)g = x + 2$ for all $x \in X$, that is,

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix}.$$

Define a function $h \in T(X,P)$ by $(x)h = x + 4$ for all positive integer x, that is,

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix}.$$

So $h \in M$ and we have

$$\begin{aligned} hf &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \end{pmatrix} = g. \end{aligned}$$

Our main result in this section is the following theorem.

Theorem 2. *Let $P = \{X_\alpha \mid \alpha \in I\}$ be a partition of a set X.*

(1) *A semigroup $T(X, P)$ has a right magnifying element if and only if X_α is infinite for some $\alpha \in I$.*

(2) *A function f is right magnifying of $T(X, P)$ if and only if f is onto but not one-to-one.*

Proof. This follows by Lemma 4, Lemma 5 and Lemma 6.

Example 4. Let $X = \mathbb{N}$ and $P = \{\{1, 2\}, \{3, 4, 5\}, \{x \mid x > 5\}\}$. By Theorem 2(1), $T(X, P)$ has a right magnifying element. Let $f \in T(X, P)$ by

$$(x)f = \begin{cases} x & \text{if } x \leq 6, \\ x - 1 & \text{if } x > 6, \end{cases}$$

that is,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 & \cdots \end{pmatrix}.$$

Then f is onto but not one-to-one. By Theorem 2(2), f is right magnifying of $T(X, P)$.

Corollary 2. *The following statements hold for a semigroup $T(X)$.*

(1) *A semigroup $T(X)$ has a right magnifying if and only if X is infinite.*

(2) *A function f is right magnifying of $T(X)$ if and only if f is onto but not one-to-one.*

Proof. This follows by Theorem 2 by using $P = \{X\}$.

4. Application to left and right magnifying elements of some generalized transformation semigroup

Let V_1 and V_2 be subspaces of a vector space V over a field F such that $V = V_1 \oplus V_2$. This mean that $V = V_1 + V_2$ and $V_1 \cap V_2 = \{0\}$. Let $L(V)$ be the semigroup of all linear transformations from V into itself under the composition of functions and $L_P(V) = \{f \in L(V) \mid (V_1)f \subseteq V_1 \text{ and } (V_2)f \subseteq V_2\}$. Then $L_P(V)$ is a subsemigroup of $L(V)$. If $V = V_1$ and $V_2 = \{0\}$, then $L_P(V) = L(V)$. Our purpose in this section is to give necessary and sufficient condition for elements in $L_P(V)$ to be right or left magnifying.

Lemma 7. *If a function f is left magnifying of $L_P(V)$, then f is one-to-one.*

Proof. This is similar to the proof of Lemma 1.

Lemma 8. *If $f \in L_P(V)$ is bijective, then f is not left magnifying of $L_P(V)$.*

Proof. This is similar to the proof of Lemma 2.

Lemma 9. *If $f \in L_P(V)$ is one-to-one but not onto, then f is left magnifying of $L_P(V)$.*

Proof. Assume that f is one-to-one but not onto. Let $M = \{h \in L_P(V) \mid (v)h = 0$ for all $v \notin ran\, f\}$. Claim that $fM = L_P(V)$. Let g be any linear transformation in $L_P(V)$. Let B_1 and B_2 be bases of V_1 and V_2, respectively. Clearly, $B_1 + B_2$ is a basis of V. Define a linear transformation $h \in L_P(V)$ by for all $v \in B_1 \cup B_2$,

$$(v)h = \begin{cases} (v')g & \text{if } v \in ran\, f \text{ and } (v')f = v, \\ 0 & \text{if } v \notin ran\, f. \end{cases}$$

Let $v', v \in V$ be such that $(v')f = v$. Assume that $v \in B_1$. Clearly, $v' \in V_1$. Therefore, $(v)h = (v')g \in V_1$. Similarly, if $v \in B_2$, then $(v)h = (v')g \in V_2$. Thus $h \in M$ and $fh = g$, this implies that $fM = L_P(V)$. Hence f is left magnifying of $L_P(V)$.

Theorem 3. *The following statements hold for a semigroup $L_P(V)$.*

(1) *A semigroup $L_P(V)$ has a left magnifying element if and only if $dim\, V_1$ is infinite or $dim\, V_2$ is infinite.*

(2) *A linear transformation f is left magnifying of $L_P(V)$ if and only if f is one-to-one but not onto.*

Proof. This follows by Lemma 7, Lemma 8 and Lemma 9.

Corollary 3. *Let $L(V)$ be the linear transformation semigroup on a vector space V.*

(1) *A semigroup $L(V)$ has a left magnifying if and only if $dim\, V$ is infinite.*

(2) *A linear transformation f is left magnifying of $L(V)$ if and only if f is one-to-one but not onto.*

Proof. This follows by Theorem 3 by using $V = V_1$ and $V_2 = \{0\}$.

Lemma 10. *If f is a right magnifying element of $L_P(V)$. Then f is onto.*

Proof. This is similar to the proof of Lemma 4.

Lemma 11. *If $f \in L_P(V)$ is bijective, then f is not right magnifying of $L_P(V)$.*

Proof. This is similar to the proof of Lemma 5.

Lemma 12. *Let $f \in L_P(V)$ be onto but not one-to-one, then f is right magnifying of $L_P(V)$.*

Proof. Assume that f is onto but not one-to-one. Let $M = \{f \in L_P(V) \mid f$ is not onto$\}$. Then $M \neq L_P(V)$. Let g be any linear transformation in $L_P(V)$. Let B_1 and B_2 be bases of V_1 and V_2, respectively. Since f is onto, there exists for each $v \in B_1$, an element $u_v \in B_1$ such that $(u_v)f = (v)g$ and there exists for each $v \in B_2$, an element $u_v \in B_2$ such that $(u_v)f = (v)g$. Define a linear transformation $h \in L_P(V)$ by $(v)h = u_v$ for all $v \in B_1 \cup B_2$. Then $h \in L_P(V)$. Since f is not one-to-one, h is not onto, and so $h \in M$. Then $hf = g$, and hence $Mf = L_P(V)$. Therefore, f is right magnifying of $L_P(V)$.

Theorem 4. *The following statements hold for a semigroup $L_P(V)$.*

(1) *A semigroup $L_P(V)$ has a right magnifying element if and only if $\dim V_1$ is infinite or $\dim V_2$ is infinite.*

(2) *A linear transformation f is right magnifying of $L_P(V)$ if and only if f is onto but not one-to-one.*

Proof. This follows by Lemma 10, Lemma 11 and Lemma 12.

Corollary 4. *Let $L(V)$ be the linear transformation semigroup on a vector space V.*

(1) *A semigroup $L(V)$ has a right magnifying if and only if $\dim V$ is infinite.*

(2) *A linear transformation f is right magnifying of $L(V)$ if and only if f is onto but not one-to-one.*

Proof. This follows by Theorem 4 by using $V = V_1$ and $V_2 = \{0\}$.

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