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On semilattice congruences on hypersemigroups and on ordered hypersemigroups

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Abstract. We prove that if H is an hypersemigroup (resp. ordered hypersemigroup) and σ is a semilattice congruence (resp. complete semilattice congruence) on H, then there exists a family $\mathcal{A}$ of proper prime ideals of H such that σ is the intersection of the semilattice congruences σ_I, $I \in \mathcal{A}$ (σ_I is the known relation defined by $a\sigma_I b \Leftrightarrow a, b \in I$ or $a, b \notin I$). Furthermore, we study the relation between the semilattices of an ordered semigroup and the ordered hypersemigroup derived by the hyperoperations $a \circ b = \{ab\}$ and $a \circ b := \{t \in S \mid t \leq ab\}$. We introduce the concept of a pseudocomplete semilattice congruence as a semilattice congruence σ for which $\leq \subseteq \sigma$ and we prove, among others, that if $(S, \cdot, \leq)$ is an ordered semigroup, $(S, \circ, \leq)$ the hypersemigroup defined by $t \in a \circ b$ if and only if $t \leq ab$ and σ is a pseudocomplete semilattice congruence on $(S, \cdot, \leq)$, then it is a complete semilattice congruence on $(S, \circ, \leq)$. Illustrative examples are given.

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1. Introduction

Filters play an essential role in studying the structure of semigroups or ordered semigroups. For a semigroup S –especially for decompositions of a semigroup S– an important role is played by the relation $\mathcal{N}$ which is the least semilattice congruence on S and leads to several important results concerning the structure of semigroups (cf. [13]). Using our computer program, we have proved in [11] that for an ordered semigroup S, $\mathcal{N}$ is not the least semilattice congruence on S in general, we introduced the concept of the complete semilattice congruence and proved that $\mathcal{N}$ is the least complete semilattice congruence on S. We always use the terms "prime", "weakly prime" instead of "completely prime", "prime" considered by Petrich in [13]. The present paper is based on our papers in [3, 11] and its aim is to show how we pass from semigroups (ordered semigroups) to hypersemigroups (ordered hypersemigroups). The main result is that if H is an hypersemigroup (resp. ordered hypersemigroup) and σ a semilattice congruence (resp. complete semilattice congruence) on H, then there exists a family $\mathcal{A}$ of proper prime ideals of H such that $\sigma = \bigcap\limits_{I \in \mathcal{A}} \sigma_{\mathrm{I}}$, σ_I is the relation on H defined by $a\sigma_I \Leftrightarrow a, b \in I$ or $a, b \notin I$. Then

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we prove, among others, that if $(S, \cdot, \leq)$ is an ordered groupoid, and $(S, \circ, \leq)$ the ordered hypergroupoid defined by $a \circ b = \{ab\}$, then σ is a semilattice (resp. complete semilattice) congruence on $(S, \cdot, \leq)$ if and only if it is a semilattice (resp. complete semilattice) congruence on $(S, \circ, \leq)$. As an immediate consequence, in an hypersemigroup H, the complete semilattice congruence $\mathcal{N}$ defined by $x\mathcal{N}y \Leftrightarrow N(x) = N(y)$ (where $N(x)$ is the filter generated by the element x of H) cannot be the least semilattice congruence on H in general. For an ordered groupoid $(S, \cdot, \leq)$ we consider the hypergroupoid on S with the hyperoperation defined by $a \circ b =: \{t \in S \mid t \leq ab\}$ and we prove that if σ is a semilattice congruence on $(S, \circ, \leq)$, then it is a semilattice congruence on $(S, \cdot, \leq)$ but the converse statement does not hold in general. In addition, if σ is a complete semilattice congruence on $(S, \circ, \leq)$, then it is a complete semilattice congruence on $(S, \cdot, \leq)$. It is natural to ask if there are semilattice congruences on an ordered groupoid $(S, \cdot, \leq)$ that are semilattice congruences on $(S, \circ, \leq)$ as well. On this purpose, we introduce the concept of pseudocomplete semilattice congruences as the semilattice congruences σ such that $\leq \subseteq \sigma$, and we prove that the pseudocomplete semilattice congruences on an ordered groupoid $(S, \cdot, \leq)$ are complete semilattice congruences on $(S, \circ, \leq)$. We could finally mention the following: If $(S, \cdot)$ is a groupoid and "$\circ$" the hyperoperation on S defined by $a \circ b := \{ab\}$, then F is a filter of $(S, \cdot)$ if and and only if it is a filter of $(S, \circ)$; for an ordered groupoid $(S, \cdot, \leq)$ with the same hyperoperation, the filters of $(S, \cdot, \leq)$ and the filters of $(S, \circ, \leq)$ are also the same. If $(S, \cdot, \leq)$ is an ordered groupoid and "$\circ$" the hyperoperation on S defined by $t \in a \circ b \Leftrightarrow t \leq ab$, the filters of $(S, \circ, \leq)$ are also filters of $(S, \cdot, \leq)$ but the converse statement does not hold in general.

An *hypergroupoid* is a nonempty set H with an hyperoperation
$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \to a \circ b$ on H and an operation
$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \to A * B$ on $\mathcal{P}^*(H)$ (induced by the operation of H) such that $A * B = \bigcup\limits_{(a,b) \in A \times B} (a \circ b)$ for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H). A nonempty subset A of H is called a *subgroupoid* of H if $A * A \subseteq A$, equivalently if, for any $a, b \in A$, we have $a \circ b \subseteq A$. The following two properties, though clear, play an essential role in the theory of hypergroupoids:

(1) if $x \in A * B$, then $x \in a \circ b$ for some $a \in A$, $b \in B$ and

(2) if $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

Moreover, we have $\{x\} * \{y\} = x \circ y$ for any $x, y \in H$. An hypergroupoid $(H, \circ, *)$ is called *hypersemigroup* if $\{x\} * (y \circ z) = (x \circ y) * \{z\}$ for every $x, y, z \in H$.

2. Some results on hypergroupoids

If S is a groupoid or an ordered groupoid, an equivalence relation σ on S is called *right* (resp. *left*) *congruence* on S if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ (resp. $(ca, cb) \in \sigma$) for every $c \in S$. It is called a *congruence* on S if it is both a right and a left congruence on S. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for any

$a, b \in S$ [3,13]. These concepts can be naturally transferred to hypergroupoids by replacing the multiplication "$\cdot$" of the groupoid by the hyperoperation "$\circ$" of the hypergroupoid. But while for a groupoid ac is an element, in case of an hypergroupoid where $a \circ c$ is a set, we have to declare what the $(a \circ c, b \circ c) \in \sigma$ means. We can define it as "for every $u \in a \circ c$ and every $v \in b \circ c$ we have $(u, v) \in \sigma$" or "for every $u \in a \circ c$ there exists $v \in b \circ c$ such that $(u, v) \in \sigma$" and get two different definitions of the left congruence, two different definitions for the right congruence; and so two different definitions of a congruence or a semilattice congruence. Although there is one between them that implies the other (it can be easily proved) and so they could be named differently (like congruence–weak congruence; complete congruence–congruence; strong congruence–congruence, for example), we will define as "congruence" both of them and, according to our investigation it will be clear which of them we use. This have been said, we give the Definitions 2.2 and 2.3 below. We first have to introduce the following notation:

Notation 2.1. If H is an hypergroupoid, σ an equivalence relation on H and A, B two nonempty subsets of H, then we write $(A, B) \in \sigma$ if for every $a \in A$ and every $b \in B$, we have $(a, b) \in \sigma$.
We write (A, b) instead of $(A, \{b\})$ and (a, B) instead of $(\{a\}, B)$. So (A, b) means that, for every $a \in A$, we have $(a, b) \in \sigma$. If it is convenient we write, for short, $A * c$ instead of $A * \{c\}$ $(A \subseteq H,\ c \in H)$.

Definition 2.2. Let H be an hypergroupoid. An equivalence relation σ on H is called *right congruence* if $(a, b) \in \sigma$ implies $(a \circ c, b \circ c) \in \sigma$ for every $c \in H$. It is called *left congruence* if $(a, b) \in \sigma$ implies $(c \circ a, c \circ b) \in \sigma$ for every $c \in H$. By a *congruence* on H we mean a relation on H which is both a right and a left congruence on H.

Definition 2.3. Let H be an hypergroupoid. A congruence σ on H is called *semilattice congruence* if, for any $a, b \in H$, we have

$$(a \circ a, a) \in \sigma \text{ and } (a \circ b, b \circ a) \in \sigma.$$

If $(S, \cdot)$ is a groupoid, a nonempty subset F of S is called a filter of S [13] if the following assertions are satisfied: (1) if $a, b \in F$, then $ab \in F$ and (2) if $a, b \in S$ such that $ab \in F$, then $a \in F$ and $b \in F$; in other words, if it is a subgroupoid of S satisfying the property (2). A nonempty subset A of S is called an ideal of S [13] if $AS \subseteq A$ and $SA \subseteq A$, that is if $a \in A$ and $s \in S$ implies $as \in A$ and $sa \in A$. If $(S, \cdot, \leq)$ is an ordered groupoid, a subset F of S is called a filter of S if it is a filter of $(S, \cdot)$ and, in addition if $a \in F$ and $S \in b \geq a$ implies $b \in F$ [1]; it is called an ideal of $(S, \cdot, \leq)$ if it is an ideal of $(S, \cdot)$ and, in addition if $a \in A$ and $S \ni b \leq a$ implies $b \in A$ [2]. A subset T of a groupoid (or ordered groupoid) S is said to be prime [3,13] if $a, b \in S$ such that $ab \in T$ implies $a \in T$ or $b \in T$. It is well known that a nonempty subset F of a groupoid (or an ordered groupoid) S is a filter of S if and only if the complement of F to S is either empty or a prime ideal of S [3,13] and, when we pass from groupoids to hypergroupoids the corresponding result should be satisfied. To manage it, a new condition should be added to the corresponding conditions of the filter and of prime ideals of groupoids we already have. And the concept of the filter of groupoids can be naturally transferred to hypergroupoids in the definition below; the prime subsets can be defined in a similar way –adding a new condition.

Definition 2.4. (cf. also [7]) Let H be an hypergroupoid. A nonempty subset F of H is called a *filter* of H if the following assertions are satisfied:
(1) if $x, y \in F$, then $x \circ y \subseteq F$;
(2) if $x, y \in H$ such that $x \circ y \subseteq F$, then $x \in F$ and $y \in F$; and
(3) for any $x, y \in H$, we have $x \circ y \subseteq F$ or $(x \circ y) \cap F = \emptyset$.
That is, a filter of H is a subgroupoid of H satisfying the relations (2) and (3).

Definition 2.5. Let H be an hypergroupoid. A nonempty subset T of H is called a *prime subset* of H if the following assertions are satisfied:
(1) if $a, b \in H$ such that $a \circ b \subseteq T$, then $a \in T$ or $b \in T$ and
(2) for every $a, b \in H$, we have $a \circ b \subseteq T$ or $(a \circ b) \cap T = \emptyset$.

As we see, we keep the definitions of filters and prime subsets of groupoids in which we add condition (3) in case of filters and condition (2) in case of prime subsets. If a subset T of an hypergroupoid satisfies only the condition (2) of Definition 2.5, then we call it *half prime* subset of H.

Remark 2.6. If H is an hypergroupoid, I a half prime subset of H and $a, c \notin I$, then $a \circ c \nsubseteq I$. So $a \notin I$ implies $a \circ a \nsubseteq I$.

As in groupoids, for an element x of H, we denote by $N(x)$ the filter of H generated by x, and by $\mathcal{N}$ the equivalence relation on H defined by

$$\mathcal{N} := \{(x, y) \in H \times H \mid N(x) = N(y)\}.$$

Using the Definitions 2.4 and 2.5, with the usual changes we pass from groupoids to hypergroupoids. In an ordered semigroup S, the relation $\mathcal{N}$ is a semilattice congruence on S [3], and the same holds for groupoids as well. By a modification of that proof, we have the following proposition; for the sake of completeness we will give its proof.

Proposition 2.7. (see also [3; the Proposition]) *If H is an hypergroupoid, then the equivalence relation $\mathcal{N}$ is a semilattice congruence on H.*

Proof. Let $(x, y) \in \mathcal{N}$ and $z \in H$. Then $(z \circ x, z \circ y) \in \mathcal{N}$. In fact: Let $u \in z \circ x$ and $v \in z \circ y$. Then $(u, v) \in \mathcal{N}$. Indeed: Since $u \in N(u)$ and $u \in z \circ x$, we have $(z \circ x) \cap N(u) \neq \emptyset$. Since $N(u)$ is a filter of H, we have $z \circ x \subseteq N(u)$, and $z, x \in N(u)$. Since $x \in N(u)$, we have $N(x) \subseteq N(u)$, then $y \in N(u)$. Since $z, y \in N(u)$, we have $z \circ y \subseteq N(u)$, then $v \in N(u)$, and $N(v) \subseteq N(u)$. By symmetry, we get $N(u) \subseteq N(v)$, so we have $N(u) = N(v)$, and $(u, v) \in \mathcal{N}$. Thus $\mathcal{N}$ is a left congruence on H. In a similar way we prove that $\mathcal{N}$ is a right congruence on H, so $\mathcal{N}$ is a congruence on H. Let $x \in H$. Then $(x \circ x, x) \in \mathcal{N}$. In fact: Let $u \in x \circ x$. Then $(u, x) \in \mathcal{N}$. Indeed: Since $u \in N(u)$, we have $(x \circ x) \cap N(u) \neq \emptyset$. Since $N(u)$ is a filter of H, we have $x \circ x \subseteq N(u)$, then $x \in N(u)$, and $N(x) \subseteq N(u)$. On the other hand, since $x \in N(x)$ and $N(x)$ is a filter of H, we have $x \circ x \subseteq N(x)$. Then $u \in N(x)$, so $N(u) \subseteq N(x)$. Thus we have $N(u) = N(x)$, and $(u, x) \in \mathcal{N}$. Let $x, y \in H$. Then $(x \circ y, y \circ x) \in \mathcal{N}$. In fact: Let $u \in x \circ y$ and $v \in y \circ x$.

Then $(u, v) \in \mathcal{N}$. Indeed: Since $u \in N(u)$, we have $(x \circ y) \cap N(u) \neq \emptyset$, then $x \circ y \subseteq N(u)$, $x, y \in N(u)$, and $y \circ x \subseteq N(u)$. Then $v \in N(u)$, and $N(v) \subseteq N(u)$. By symmetry, we get $N(u) \subseteq N(v)$, then $N(u) = N(v)$, and $(u, v) \in \mathcal{N}$. $\square$

Notation 2.8. For a subset I of H, we denote by σ_I the equivalence relation on H defined by:

$$\sigma_I := \{(a, b) \in H \times H \mid a, b \in I \text{ or } a, b \notin I\}$$

(i.e. a, b both belong to I or a, b both do not belong to I).

If S is a semigroup or an ordered semigroup and I a prime ideal of S, then the relation σ_I is a semilattice congruence on S [3,13] (and the same holds if we replace the work "semigroup" by "groupoid"). Recall that for ordered groupoids the semilattice congruences are defined exactly as in groupoids. In an attempt to show the way we pass from semigroups to Γ-semigroups, we transferred this result to Γ-semigroups in [5]. Here we do the same for hypergroupoids using the following proposition.

Proposition 2.9. *Let H be an hypergroupoid, $a, b, c \in H$ and $I \subseteq H$. Then we have the following:*

(1) *if $a \circ c, b \circ c \subseteq I$, then $(a \circ c, b \circ c) \in \sigma_I$.*

Suppose now that, for every $a, b \in H$, we have $a \circ b \subseteq I$ or $(a \circ b) \cap I = \emptyset$ $\quad (*)$

Then the following two conditions are satisfied:

(2) *if $a \circ c, b \circ c \nsubseteq I$, then $(a \circ c, b \circ c) \in \sigma_I$.*

(3) if $a \notin I$ and $a \circ a \nsubseteq I$, then $(a, a \circ a) \in \sigma_I$.

Proof. (1) Let $a \circ c, b \circ c \subseteq I$, $u \in a \circ c$ and $v \in b \circ c$. Then $u, v \in I$, so $(u, v) \in \sigma_I$.

(2) Let $a \circ c, b \circ c \nsubseteq I$, $u \in a \circ c$ and $v \in b \circ c$. If $u, v \in I$, then $(u, v) \in \sigma_I$. If $u \notin I$, then $v \notin I$. Indeed, if $v \in I$, then $v \in (b \circ c) \cap I$. Since $(b \circ c) \cap I \neq \emptyset$, by $(*)$, we have $b \circ c \subseteq I$ which is impossible. So $u, v \notin I$, and $(u, v) \in \sigma_I$. If $v \notin I$, in a similar way we get $u \notin I$, so again $(u, v) \in \sigma_I$.

(3) Let $a \notin I$, $a \circ a \nsubseteq I$ and $u \in a \circ a$. If $u \in I$, then $(a \circ a) \cap I \neq \emptyset$ and, by $(*)$, $a \circ a \subseteq I$ which is impossible. Thus we have $u \notin I$. Since $a, u \notin I$, we have $(a, u) \in \sigma_I$. $\square$

If H is an hypergroupoid, a nonempty subset A of H is called an ideal of H if $A * H \subseteq A$ and $H * A \subseteq A$, equivalently if $a \in A$ and $h \in H$, then $a \circ h \subseteq A$ and $h \circ a \subseteq A$ [8]. By a *prime ideal* of H we clearly mean an ideal of H which is at the same time a prime subset of H.

Corollary 2.10. (cf. also [3] and [5; Proposition 2.2]) *Let H be an hypergroupoid and I a prime ideal of H. Then the equivalence relation σ_I is a semilattice congruence on H.*

Proof. Let $(a, b) \in \sigma_I$ and $c \in H$. Then $(a \circ c, b \circ c) \in \sigma_I$. In fact: Since $(a, b) \in \sigma_I$, we have $a, b \in I$ or $a, b \notin I$. Let $a, b \in I$. Since I is an ideal of H, we have $a \circ c, b \circ c \subseteq I$. Then, by Proposition 2.9(1), we have $(a \circ c, b \circ c) \in \sigma_I$. Let $a, b \notin I$. If $c \in I$ then, since I is an ideal of H, we have $a \circ c, b \circ c \subseteq I$, then $(a \circ c, b \circ c) \in \sigma_I$. Let $c \notin I$. Since $a, b, c \notin I$, by Remark 2.6, we have $a \circ c, b \circ c \nsubseteq I$. Then, by Proposition 2.9(2), we have $(a \circ c, b \circ c) \in \sigma_I$. Thus σ_I is a right congruence on H. In a similar way we can prove that σ_I is a left congruence on H and so it is a congruence on H. Let $a \in H$. Then

$(a \circ a, a) \in \sigma_I$. In fact: Let $u \in a \circ a$. If $a \in I$ then, since I is an ideal of H, we have $a \circ a \subseteq I$, then $u \in I$; since $u, a \in I$, we have $(u, a) \in \sigma_I$. If $a \notin I$, then $a \circ a \nsubseteq I$. Then, by Proposition 2.9(3), we have $(a, a \circ a) \in \sigma_I$. Let $a, b \in H$. Then $(a \circ b, b \circ a) \in \sigma_I$. Indeed: if $a \circ b \subseteq I$ then, since I is a prime ideal of H, we have $a \in I$ or $b \in I$. Since I is an ideal of H, we have $b \circ a \subseteq I$. Since $a \circ b, b \circ a \subseteq I$, by Lemma 2.9(1), we have $(a \circ b, b \circ a) \in \sigma_I$. If $a \circ b \nsubseteq I$ then $b \circ a \nsubseteq I$. This is because if $b \circ a \nsubseteq I$ then, since I is a prime ideal of H, we have $b \in I$ or $a \in I$ and, since I is an ideal of H, we have $a \circ b \subseteq I$ which is impossible. Since $a \circ b, b \circ a \nsubseteq I$, by Proposition 2.9(2), we have $(a \circ b, b \circ a) \in \sigma_I$. $\square$

We have the following:

(1) if $(x, A) \in \sigma$ and $\emptyset \neq B \subseteq A$, then $(x, B) \in \sigma$;

(2) if $(A, B) \in \sigma$, then $(B, A) \in \sigma$;

(3) if $(A, B) \in \sigma$, $(B, C) \in \sigma$ and $B \neq \emptyset$, then $(A, C) \in \sigma$; indeed, let $a \in A$, $c \in C$. Take an element $b \in B$ $(B \neq \emptyset)$. Since $(a, b) \in \sigma$ and $(b, c) \in \sigma$, we have $(a, c) \in \sigma$.

Proposition 2.11. *Let H be an hypergroupoid, σ a congruence on H and A, B, C, D nonempty subsets of H. If $(A, B) \in \sigma$ and $(C, D) \in \sigma$, then $(A * C, B * D) \in \sigma$ and $(C * A, D * B) \in \sigma$.*

Proof. Let $(A, B) \in \sigma$, $u \in A * C$ and $v \in B * D$. We have $u \in a \circ c$ for some $a \in A$, $c \in C$ and $v \in b \circ d$ for some $b \in B$, $d \in D$. Since $a \in A$, $b \in B$ and $(A, B) \in \sigma$, we have $(a, b) \in \sigma$ and, since σ is a right congruence on H, we have $(a \circ c, b \circ c) \in \sigma$. Since $(C, D) \in \sigma$, $c \in C$ and $d \in D$, we have $(c, d) \in \sigma$ and, since σ is a left congruence on H, we have $(b \circ c, b \circ d) \in \sigma$. By the transitivity relation, we have $(a \circ c, b \circ d) \in \sigma$. Since $u \in a \circ c$ and $v \in b \circ d$, we have $(u, v) \in \sigma$. Similarly we get $(C * A, D * B) \in \sigma$. $\square$

Lemma 2.12. *Let H be an hypergroupoid, A, B nonempty subsets of H and $c \in H$. If σ a right congruence on H and $(A, B) \in \sigma$, then $(A * c, B * c) \in \sigma$.*

Proof. Since $c \in H$ and σ is a reflexive relation on H, we have $(\{c\}, \{c\}) \in \sigma$. Since $(A, B) \in \sigma$ and $(\{c\}, \{c\}) \in \sigma$, by Proposition 2.11, we have $(A * c, B * c) \in \sigma$.
An independent proof is the following: Let $u \in A * c$ and $v \in B * c$. Then $u \in a \circ c$ for some $a \in A$ and $v \in b \circ c$ for some $b \in B$. Since $a \in A$, $b \in B$ and $(A, B) \in \sigma$, we have $(a, b) \in \sigma$. Since σ is a right congruence on H, we have $(a \circ c, b \circ c) \in \sigma$. Since $u \in a \circ c$ and $v \in b \circ c$, we get $(u, v) \in \sigma$ and so $(A * c, B * c) \in \sigma$. $\square$
In a similar way we have the following lemma.

Lemma 2.13. *Let H be an hypergroupoid, A, B nonempty subsets of H and $c \in H$. If σ a left congruence on H and $(A, B) \in \sigma$, then $(c * A, c * B) \in \sigma$.*

As in groupoids, the following proposition holds and one can prove it as a modification of the proof of the corresponding result in [3].

Proposition 2.14. (cf. also [3; the Lemma]) *Let H be an hypergroupoid. If H is a filter of H, then the property $(*)$ is satisfied:*

($*$) *either* $H\backslash F = \emptyset$ *or* $H\backslash F$ *is a prime ideal of* H.
In particular, any nonempty subset F *of* H *satisfying* ($*$) *is a filter of* H.

An ideal I of H is called *proper* if $I \neq H$.

3. Main result

Theorem 3.1. *Let* H *be an hypersemigroup and* σ *be a semilattice congruence on* H. *Then there exists a family* $\mathcal{A}$ *of proper prime ideals of* H *such that*

$$\sigma = \bigcap_{I \in \mathcal{A}} \sigma_{\mathrm{I}}.$$

Proof. Let $x \in H$. We consider the set

$$A_x := \{y \in H \mid (x, x \circ y) \in \sigma\}.$$

The set A_x is a filter of H. Indeed: Since $x \in H$ and σ is a semilattice congruence on H, we have $(x, x \circ x) \in \sigma$, thus $x \in A_x$ and A_x is a nonempty subset of H.

Let $y, z \in A_x$. Then $y \circ z \subseteq A_x$. In fact: Let $u \in y \circ z$. Then $u \in A_x$, that is $(x, x \circ u) \in \sigma$. Indeed: Let $v \in x \circ u$. Since $y \in A_x$, we have $(x, x \circ y) \in \sigma$. Then, by Lemma 2.12, we have $\Big(x \circ z, (x \circ y) * \{z\}\Big) \in \sigma$. Since $z \in A_x$, we have $(x, x \circ z) \in \sigma$ and, by the transitivity relation, we have $\Big(x, \{x\} * (y \circ z)\Big) \in \sigma$. Since $v \in x \circ u \subseteq \{x\} * (y \circ z)$, we have $(x, v) \in \sigma$.

Let $y, z \in H$ such that $y \circ z \subseteq A_x$. Then $y \in A_x$ and $z \in A_x$. In fact:
Since $y \circ z \subseteq A_x$, we have $\Big(x, \{x\} * (y \circ z)\Big) \in \sigma$ (1)
Indeed: if $u \in \{x\} * (y \circ z)$, then $u \in x \circ t$ for some $t \in y \circ z \subseteq A_x$. Since $t \in A_x$, we have $(x, x \circ t) \in \sigma$. Then, since $u \in x \circ t$, we obtain $(x, u) \in \sigma$, so property (1) is satisfied.
By (1) and Lemma 2.12, we have $\Big(x \circ z,\ \{x\} * (y \circ z) * \{z\}\Big) \in \sigma$ (2)
On the other hand, since $(z, z \circ z) \in \sigma$, we have $\Big((x \circ y) * \{z\},\ (x \circ y) * (z \circ z)\Big) \in \sigma$ (3)
In fact: Since $(z, z \circ z) \in \sigma$, by Lemma 2.13, we have $\Big(y \circ z, \{y\} * (z \circ z\Big) \in \sigma$; again by Lemma 2.13, we have $\Big(\{x\} * (y \circ z), (x \circ y) * (z \circ z)\Big) \in \sigma$ and (3) holds.
By (1),(2) and (3), we obtain $(x, x \circ z) \in \sigma$, and so $z \in A_x$. It remains to prove that $y \in A_x$. Since $z \in A_x$, we have $(x, x \circ z) \in \sigma$. By Lemma 2.12, we have $\Big(x \circ y, (x \circ z) * \{y\}\Big) \in \sigma$. Since $(y \circ z, z \circ y) \in \sigma$, by Lemma 2.13, we have $\Big(\{x\} * (y \circ z), \{x\} * (z \circ y)\Big) \in \sigma$. Then, by (1), we get $(x, x \circ y) \in \sigma$, and so $y \in A_x$.

Let $y, z \in H$. Then $y \circ z \subseteq A_x$ or $(y \circ z) \cap A_x = \emptyset$. In fact:
Let $y \circ z \nsubseteq A_x$ and $(y \circ z) \cap A_x \neq \emptyset$. Let $u \in y \circ z$ such that $u \notin A_x$, $v \in y \circ z$ and $v \in A_x$. Then we have

$$u \in y \circ z,\ (x, x \circ u) \notin \sigma,\ v \in y \circ z,\ (x, x \circ v) \in \sigma.$$

On the other hand,

$$(x, x \circ v) \in \sigma \text{ and } v \in y \circ z \text{ implies } \Big(x, \{x\} * (y \circ z)\Big) \in \sigma.$$

Indeed: Let $a \in \{x\} * (y \circ z)$. Then $a \in x \circ d$ for some $d \in y \circ z$. Since $(y \circ z, y \circ z) \in \sigma$, $v \in y \circ z$ and $d \in y \circ z$, we have $(v, d) \in \sigma$, then $(x \circ v, x \circ d) \in \sigma$. Since $(x, x \circ v) \in \sigma$ and $(x \circ v, x \circ d) \in \sigma$, we have $(x, x \circ d) \in \sigma$. Since $a \in x \circ d$, we have $(x, a) \in \sigma$.
We have $x \circ u \subseteq \{x\} * (y \circ z)$ and $\Big(x, \{x\} * (y \circ z)\Big) \in \sigma$, thus we have $(x, x \circ u) \in \sigma$ which is impossible.

Since A_x is a filter of H, by Proposition 2.14, we have $H \backslash A_x = \emptyset$ or $H \backslash A_x$ is a prime ideal of H. Then $H \backslash A_x = \emptyset$ or $H \backslash A_x$ is a proper prime ideal of H (indeed, if $H \backslash A_x = H$ then, since $A_x \subseteq H$, we have $A_x = \emptyset$ which is not possible).
We consider the set

$$\{H \backslash A_z \mid z \in H,\ H \backslash A_z \text{ proper prime ideal of } H\}.$$

We have $\sigma = \bigcap\limits_{z \in H} \sigma_{H \backslash A_z}$. In fact: Let $(x, y) \in \sigma$ and $z \in H$. Then $(x, y) \in \sigma_{H \backslash A_z}$. Indeed: Since $x \in H$, we have $x \in H \backslash A_z$ or $x \notin H \backslash A_z$.

(a) If $x \notin H \backslash A_z$, then $x \in A_z$, so $(z, z \circ x) \in \sigma$. Since $(x, y) \in \sigma$, we have $(z \circ x, z \circ y) \in \sigma$. Then $(z, z \circ y) \in \sigma$, and $y \in A_z$, so $y \notin H \backslash A_z$.

(b) Let $x \in H \backslash A_z$. If $y \notin H \backslash A_z$, then in a similar way as in (a), we prove that $x \notin H \backslash A_z$ which is impossible. Thus we have $y \in H \backslash A_z$.
Since both x and y belong to $H \backslash A_z$ or both do not belong to $H \backslash A_z$, we have $(x, y) \in \sigma_{H \backslash A_z}$.

Let now $(x, y) \in \sigma_{H \backslash A_z}$ for every $z \in H$. Then $(x, y) \in \sigma$. Indeed: Since $x \in A_x$, we have $x \notin H \backslash A_x$ and, since $(x, y) \in \sigma_{H \backslash A_x}$, we have $y \notin H \backslash A_x$, so $y \in A_x$, that is $(x, x \circ y) \in \sigma$. Since $y \in A_y$, we have $y \notin H \backslash A_y$, then $x \notin H \backslash A_y$, so $x \in A_y$, thus $(y, y \circ x) \in \sigma$. Since σ is a semilattice congruence on H, we have $(x \circ y, y \circ x) \in \sigma$. Since $(x, x \circ y) \in \sigma$, $(x \circ y, y \circ x) \in \sigma$ and $(y \circ x, y) \in \sigma$, we have $(x, y) \in \sigma$. □

The following proposition holds for hypergroupoids and its proof is exactly the same as the proof of the corresponding result in [3] (no change is needed).

Proposition 3.2. (cf. also [3; the Proposition]) *Let H be an hypergroupoid and $\mathcal{I}(H)$ the set of prime ideals of H. Then we have*

$$\mathcal{N} = \bigcap_{I \in \mathcal{I}(H)} \sigma_I.$$

Corollary 3.3. *If H is an hypersemigroup, then the relation $\mathcal{N}$ is the least semilattice congruence on H.*

Proof. Let σ be a semilattice congruence on H. Then $\mathcal{N} \subseteq \sigma$. In fact: By Theorem 3.1, there exists a family $\mathcal{A}$ of proper prime ideals of H such that $\sigma = \bigcap\limits_{I \in \mathcal{A}} \sigma_{\text{I}}$. By Proposition 3.2, $\mathcal{N} = \bigcap\limits_{I \in \mathcal{I}(H)} \sigma_I$, where $\mathcal{I}(H)$ is the set of prime ideals of H. On the other hand,

$\bigcap\limits_{I\in\mathcal{A}} \sigma_{\rm I} \supseteq \bigcap\limits_{I\in\mathcal{I}(H)} \sigma_{\rm I}$. Indeed, if $(x,y) \in \sigma_I$ for every prime ideal of H, then clearly $(x,y) \in \sigma_I$ for every proper prime ideal of H, and so for the elements of $\mathcal{A}$ as well. Hence we obtain $\mathcal{N} \subseteq \sigma$. $\square$

Proposition 3.4. *Let $(S,\cdot)$ be a groupoid and "$\circ$" the hypergroupoid with the hyperoperation "$\circ$" defined by*

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (a,b) \to a \circ b := \{ab\}.$$

Then F is a filter of $(S,\cdot)$ if and only if it is a filter of $(S,\circ)$.

Proof. $\Longrightarrow$. Let F be a filter of $(S,\cdot)$. If $a,b \in F$ and $x \in a \circ b$, then $x = ab \in F$ and so $a \circ b \subseteq F$. If $a,b \in S$ such that $a \circ b \subseteq F$, then $ab \in a \circ b \subseteq F$, $ab \in F$ and so $a \in F$ and $b \in F$. Let $(a \circ b) \cap F \neq \emptyset$, $u \in a \circ b$ and $u \in F$. Then $u = ab$ and $u \in F$, then $ab \in F$, then $a \circ b = \{ab\} \subseteq F$ and so $a \circ b \subseteq F$.
$\Longleftarrow$. Let F be a filter of $(S,\circ)$. If $a,b \in F$, then $\{ab\} = a \circ b \subseteq F$, thus $ab \in F$. If $a,b \in S$ such that $ab \in F$, then $a \circ b = \{ab\} \subseteq F$, so $a \circ b \subseteq F$, then $a \in F$ and $b \in F$ and so F is a filter of $(S,\cdot)$. $\square$

4. Complete semilattice congruences on ordered hypersemigroups

Let us consider now the case of ordered hypergroupoids. For the necessary definitions and notations on ordered hypergroupoids we refer to [6] and [9]. For an ordered hypergroupoid, the semilattice congruence is defined exactly as in hypergroupoids. The concept of complete semilattice congruences of ordered groupoids introduced by Kehayopulu and Tsingelis in [11] can be naturally transferred to ordered hypergroupoids by the following definition.

Definition 4.1. If $(S,\circ,\leq)$ is an ordered hypergroupoid, a semilattice congruence σ on S is called *complete* if, for every $a,b \in S$, the relation $a \leq b$ implies $(a, a \circ b) \in \sigma$.

Proposition 4.2. *Let $(S,\cdot,\leq)$ be an ordered groupoid and "$\circ$" the hyperoperation on S defined by $a \circ b := \{ab\}$. Then $(S,\circ,\leq)$ is an ordered hypergroupoid. The relation σ is a semilattice (resp. complete semilattice) congruence on $(S,\cdot,\leq)$ if and only if it is semilattice (resp. complete semilattice) congruence on $(S,\circ,\leq)$. If $(S,\cdot,\leq)$ is an ordered semigroup, then $(S,\circ,\leq)$ is an ordered hypersemigroup as well.*

Proof. If $(S,\cdot,\leq)$ is an ordered groupoid, $a \leq b$, $c \in S$ and $u \in a \circ c$, then $u = ac \leq bc$, so for the element $v := bc \in b \circ c$ we have $u \leq v$; similarly $c \circ a \preceq c \circ b$ and so $(S,\circ,\leq)$ is an ordered hypergroupoid. Let σ be a semilattice congruence on $(S,\cdot,\leq)$. If $(a,b) \in \sigma$ and $c \in S$, then $(a \circ c, b \circ c) \in \sigma$. Indeed, if $u \in a \circ c$ and $v \in b \circ c$, then $u = ac$, $v = bc$ and $(ac,bc) \in \sigma$, so $(u,v) \in \sigma$. Similarly σ is a left congruence on $(S,\circ,\leq)$. Let $a,b \in S$. Then $(a \circ a, a) \in \sigma$. Indeed, if $u \in a \circ a$, then $u = a^2$ and $(a^2,a) \in \sigma$, thus we get $(u,a) \in \sigma$. We have $(a \circ b, b \circ a) \in \sigma$. Indeed, if $u \in a \circ b$ and $v \in b \circ a$, then $u = ab$, $v = ba$ and $(ab,ba) \in \sigma$, thus we get $(u,v) \in \sigma$ and so σ is a semilattice congruence on $(S,\circ,\leq)$. Let σ be a complete semilattice congruence on $(S,\cdot,\leq)$, $a \leq b$, and $u \in a \circ b$. Since $u = ab$ and

$a \le b$, we have $(a, ab) \in \sigma$, then $(a, u) \in \sigma$ and so σ is a complete semilattice congruence on $(S, \circ, \le)$.

Let σ be a semilattice congruence on $(S, \circ, \le)$. If $(a, b) \in \sigma$ and $c \in S$ then, since $(a \circ c, b \circ c) \in \sigma$, $ac \in a \circ c$ and $bc \in b \circ c$, we have $(ac, bc) \in \sigma$. Similarly σ is a left congruence on $(S, \cdot, \le)$. If $a \in S$, then $(a \circ a, a) \in \sigma$ and, since $a^2 \in a \circ a$, we have $(a^2, a) \in \sigma$. If $a, b \in S$, then $(a \circ b, b \circ a) \in \sigma$ and, since $ab \in a \circ b$ and $ba \in b \circ a$, we have $(ab, ba) \in \sigma$. Hence σ is a semilattice congruence on $(S, \cdot, \le)$. Let now σ be a complete semilattice congruence on $(S, \circ, \le)$ and $a \le b$. Since $(a, a \circ b) \in \sigma$ and $ab \in a \circ b$, we have $(a, ab) \in \sigma$, so σ is a complete semilattice congruence on $(S, \cdot, \le)$.

Let now $(S, \cdot, \le)$ be an ordered semigroup, $a, b, c \in S$ and $x \in \{a\} * (b \circ c)$. Then $x \in a \circ u$ for some $u \in b \circ c$, then $x = au$ and $u = bc$. Then we have

$$x = a(bc) \in \{a(bc)\} = \{(ab)c\} = (ab) \circ c \subseteq \{ab\} * \{c\} = (a \circ b) * \{c\},$$

then $\{a\} * (b \circ c) \subseteq (a \circ b) * \{c\}$. Similarly $(a \circ b) * \{c\} \subseteq \{a\} * (b \circ c)$ and so $(S, \circ, \le)$ is an ordered hypersemigroup. $\square$

Definition 4.3. Let $(S, \circ, \le)$ be an ordered hypergroupoid. A subset F of S is called a *filter* of $(S, \circ, \le)$ if it is a filter of the hypergroupoid $(S, \circ)$ and, in addition,

$$\text{if } a \in F \text{ and } S \ni b \ge a \text{ implies } b \in F.$$

By Proposition 3.4, we have the following

Proposition 4.4. *Let $(S, \cdot, \le)$ be an ordered groupoid and "$\circ$" the hyperoperation on S defined by $a \circ b := \{ab\}$. Then F is a filter of $(S, \cdot, \le)$ if and only if it is a filter of $(S, \circ, \le)$.*

Remark 4.5. According to Proposition 4.4, if the hyperoperation "$\circ$" is defined by $a \circ b = \{ab\}$, then the filters of the groupoid $(S, \cdot, \le)$ and the filters of the hypergroupoid $(S, \circ, \le)$ are the same. Also, by Proposition 4.2, the semilattice congruences on $(S, \cdot, \le)$ and the semilattice congruences on $(S, \circ, \le)$ are the same. As we have seen in [11], in an ordered hypersemigroup S, the relation $\mathcal{N}$ is not the least semilattice congruence on S in general, so according to Proposition 4.2, in an ordered hypersemigroup, the relation $\mathcal{N}$ cannot be the least semilattice congruence as well, in general. Let us see it in the following example.

Example 4.6. We get the ordered semigroup defined in [11] with the following table and figure.

$\cdot$	a	b	c	d	f	g
a	b	b	a	d	a	a
b	b	b	b	d	b	b
c	a	b	c	d	c	c
d	d	d	d	d	d	d
f	a	b	c	d	c	c
g	a	b	c	d	f	g

Table 1.

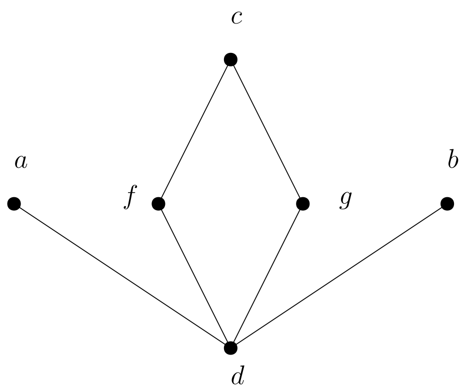

Figure 1.

For this semigroup,

$$N(a) = N(b) = \{a, b, c, f, g\},\ N(c) = N(f) = N(g) = \{c, f, g\} \text{ and } N(d) = S.$$

We consider the semilattice congruences on S. They are eight and they are the following:

$$\begin{aligned}
\sigma_1 &= \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,f),(d,d),(f,c),(f,f),(g,g)\}.\\
\sigma_2 &= \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,f),(c,g),(d,d),(f,c),(f,f),\\
&\quad (f,g),(g,c),(g,f),(g,g)\} = \mathcal{N}.\\
\sigma_3 &= \{(a,a),(a,b),(a,d),(b,a),(b,b),(b,d),(c,c),(c,f),(d,a),(d,b),\\
&\quad (d,d),(f,c),(f,f),(g,g)\}.\\
\sigma_4 &= \{(a,a),(a,b),(a,c),(a,f),(b,a),(b,b),(b,c),(b,f),(c,a),(c,b),\\
&\quad (c,c),(c,f),(d,d),(f,a),(f,b),(f,c),(f,f),(g,g)\}.\\
\sigma_5 &= \{(a,a),(a,b),(a,d),(b,a),(b,b),(b,d),(c,c),(c,f),(c,g),(d,a),\\
&\quad (d,b),(d,d),(f,c),(f,f),(f,g),(g,c),(g,f),(g,g)\}.\\
\sigma_6 &= \{(a,a),(a,b),(a,c),(a,d),(a,f),(b,a),(b,b),(b,c),(b,d),(b,f),\\
&\quad (c,a),(c,b),(c,c),(c,d),(c,f),(d,a),(d,b),(d,c),(d,d),(d,f),\\
&\quad (f,a),(f,b),(f,c),(f,d),(f,f),(g,g)\}.\\
\sigma_7 &= \{(a,a),(a,b),(a,c),(a,f),(a,g),(b,a),(b,b),(b,c),(b,f),(b,g),\\
&\quad (c,a),(c,b),(c,c),(c,f),(c,g),(d,d),(f,a),(f,b),(f,c),(f,f),\\
&\quad (f,g),(g,a),(g,b),(g,c),(g,f),(g,g)\}.\\
\sigma_8 &= S \times S.
\end{aligned}$$

σ_1 is the least semilattice congruence on S, the relations σ_2, σ_5, σ_7, σ_8 are complete semilattice congruences on S, $\mathcal{N}\,(=\sigma_2) \subseteq \sigma_5, \sigma_7, \sigma_8$, that is $\mathcal{N}$ is the least complete semilattice congruence on S and $\sigma_1 \neq \mathcal{N}$.

According to Proposition 4.2, the ordered hypersemigroup $(S, \circ, \leq)$ defined by the table below and the same Figure 1 is an ordered hypersemigroup and the semilattice congruences on $(S,\cdot,\leq)$ and on $(S,\circ,\leq)$ are the same, so σ_1 is the least semilattice congruence on $(S,\circ,\leq)$, $\mathcal{N}$ is the least complete semilattice congruence on $(S,\circ,\leq)$ and $\mathcal{N}$ is different than σ_1.

$\circ$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{f\}$	$\{g\}$
a	$\{b\}$	$\{b\}$	$\{a\}$	$\{d\}$	$\{a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{d\}$	$\{b\}$	$\{b\}$
c	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{c\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
f	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{c\}$	$\{c\}$
g	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{f\}$	$\{g\}$

Table 2.

In the above example the ordered semigroup has been found using our computer program. Let us give another example which is easier to check by hand.

Example 4.7. (cf. [4; Example 1]) The ordered semigroup given by the multiplication "·" and the figure below is an example of an ordered semigroup for which the relation $\mathcal{N}$ is not the least semilattice congruence on S.

$\cdot$	a	b	c	d	e
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	b	b	b
d	a	b	b	d	d
e	a	b	c	d	e

Table 3.

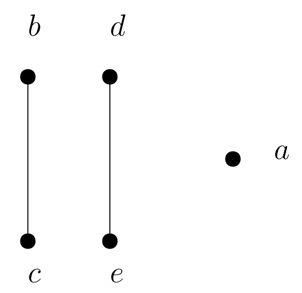

Figure 2.

We have $N(a) = N(b) = N(c) = S$ and $N(d) = N(f) = \{d,e\}$.

We give all the semilattice congruences on S. They are four and they are the following:

$$\sigma_1 = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),$$

$$\begin{aligned}
& (d,d),(e,e)\}. \\
\sigma_2 &= \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c),(d,d) \\
& (d,e),(e,d),(e,e)\} = \mathcal{N}. \\
\sigma_3 &= \{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b), \\
& (c,c),(c,d),(d,a),(d,b),(d,c),(d,d),(e,e)\}. \\
\sigma_4 &= S \times S.
\end{aligned}$$

The relation σ_1 is the least semilattice congruence on S, the relation $\mathcal{N}$ is the least complete semilattice congruence on S, and $\sigma_1 \neq \mathcal{N}$.

According to Proposition 4.2, the ordered semigroup $(S, \cdot, \leq)$ with the operation "$\circ$" on S defined by $a \circ b := \{ab\}$ is an ordered hypersemigroup and the semilattice congruences on $(S, \cdot, \leq)$ and on $(S, \circ, \leq)$ coincide. In other words, the set S with the multiplication given by the table

$\circ$	a	b	c	d	e
a	$\{b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{a\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
d	$\{a\}$	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$
e	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$

Table 4.

and the same order as in $(S, \cdot, \leq)$ (: Figure 2) is an ordered hypersemigroup, the relation σ_1 is the least semilattice congruence on S and it is different than $\mathcal{N}$.

In [4] there are also examples of ordered semigroups S for which the complete semilattice congruence $\mathcal{N}$ is at the same time the least semilattice congruence on S. Let us get one of them and pass from the ordered semigroup to ordered hypersemigroup.

Example 4.8. If we take the ordered hypersemigroup $(S, \cdot, \leq)$ given in the Example 2 in [4] and define the hyperoperation as $a \circ b := \{ab\}$, then we get the ordered hypersemigroup defined by the table and the figure below.

$\circ$	a	b	c	d	f
a	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{d\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
f	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c\}$

Table 5.

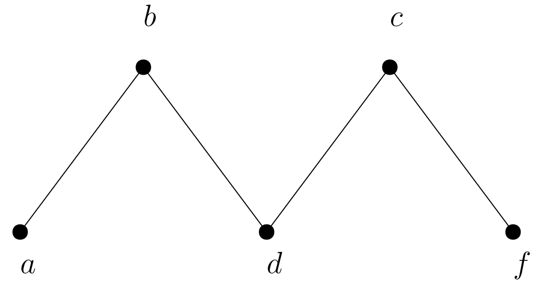

Figure 3.

We have $N(a) = N(b) = \{a, b\}$, $N(c) = N(f) = \{c, f\}$ and $N(d) = S$.

There are four semilattice congruences on $(S, \circ \leq)$ and they are the following

$$\begin{aligned}
\sigma_1 &= \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,f),(d,d),(f,c),(f,f)\} = \mathcal{N}. \\
\sigma_2 &= \{(a,a),(a,b),(a,d),(b,a),(b,b),(b,d),(c,c),(c,f),(d,a),(d,b), \\
&\quad (d,d),(f,c),(f,f)\}. \\
\sigma_3 &= \{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(c,f),(d,c),(d,d),(d,f), \\
&\quad (f,c),(f,d),(f,f)\}. \\
\sigma_4 &= S \times S.
\end{aligned}$$

The relation σ_1 is the least semilattice congruence on $(S, \circ, \leq)$ and at the same time the least complete semilattice congruence on $(S, \circ, \leq)$.

In the Example 3 in [4] and in its "dual" given immediately after the Example 3, the relation σ_1 mentioned in them is at the same time the least semilattice congruence and the least complete semilattice congruence; as a consequence it is so in the corresponding ordered hypersemigroup defined by the hyperoperation $a \circ b = \{ab\}$.

Proposition 4.9. (cf. also [12; Remark 1]) *If $(H, \circ, \leq)$ is an ordered hypergroupoid, then the semilattice congruence $\mathcal{N}$ is a complete semilattice congruence on H.*

Proof. Let $a \leq b$. Then $(a, a \circ b) \in \mathcal{N}$. In fact: Let $u \in a \circ b$. Then $(a, u) \in \mathcal{N}$, that is $N(a) = N(u)$. Indeed: Since $N(a) \ni a \leq b$, we have $b \in N(a)$. Since $a, b \in N(a)$, we have $a \circ b \subseteq N(a)$, then $u \in N(a)$, and so $N(u) \subseteq N(a)$. On the other hand, since $u \in a \circ b$ and $u \in N(u)$, we have $(a \circ b) \cap N(u) \neq \emptyset$, then $a \circ b \subseteq N(u)$, then $a \in N(u)$, and $N(a) \subseteq N(u)$. Hence we obtain $N(u) = N(a)$ and the proof is complete. $\square$

Theorem 4.10. (cf. also [11]) *Let H be an ordered hypersemigroup and σ a complete semilattice congruence on H. Then there exists a family $\mathcal{A}$ of proper prime ideals of H such that*

$$\sigma = \bigcap_{I \in \mathcal{A}} \sigma_{\mathrm{I}}.$$

Proof. Following Theorem 3.1, it remains to prove that for the set

$$A_x := \{y \in H \mid (x, x \circ y) \in \sigma\}$$

the following property is satisfied

$$\text{if } y \in A_x \text{ and } H \ni z \geq y, \text{ then } z \in A_x.$$

Indeed: Since $y \in A_x$, we have $(x, x \circ y) \in \sigma$ then, by Lemma 2.12, $\Big(x \circ z, (x \circ y) * \{z\}\Big) \in \sigma$, that is $\Big((x \circ y) * \{z\}, x \circ z\Big) \in \sigma$. Since $y \leq z$ and σ is a complete semilattice congruence on H, we have $(y, y \circ z) \in \sigma$ and, by Lemma 2.13, $\Big(x \circ y, \{x\} * (y \circ z)\Big) \in \sigma$. Hence we obtain $(x, x \circ z) \in \sigma$ that is, $z \in A_x$. □

By Proposition 3.2, Proposition 4.9 and Theorem 4.10, we have the following

Corollary 4.11. (cf. also [11; the Proposition]) *If H is an ordered hypersemigroup, then the relation $\mathcal{N}$ is the least complete semilattice congruence on H.*

If $(S, \cdot, \leq)$ is an ordered groupoid and "$\circ$" the hyperoperation on S defined by: $\circ : S \times S \to \mathcal{P}^*(S) \mid (a, b) \to a \circ b$, where $a \circ b := \{t \in S \mid t \leq ab\}$, then $(S, \circ, \leq)$ is an ordered hypergroupoid [10; Lemma 1].

Proposition 4.12. *Let $(S, \cdot, \leq)$ be an ordered groupoid and "$\circ$" the hyperoperation on S defined by*

$$\circ : S \times S \to \mathcal{P}^*(S) \mid (a, b) \to a \circ b := \{t \in S \mid t \leq ab\}.$$

If F is a filter of $(S, \circ, \leq)$, then it is a filter of $(S, \cdot, \leq)$ as well. The converse statement does not hold in general.

Proof. Let $a, b \in F$. Since F is a filter of $(S, \circ, \leq)$, we have $a \circ b \subseteq F$. Since $ab \in a \circ b$, we have $ab \in F$. Let now $a, b \in S$ such that $ab \in F$. Since $ab \in a \circ b$ and $ab \in F$, we have $(a \circ b) \cap F \neq \emptyset$, then $a \circ b \subseteq F$, and then $a, b \in F$.

For the converse statement, consider the ordered semigroup $(S, \cdot, \leq)$ of the Example 4.6 given by Table 1 and Figure 1 and the ordered hypersemigroup defined by the same order and the hyperoperation $x \circ y := \{t \in S \mid t \leq xy\}$ in the following table.

$\circ$	a	b	c	d	f	g
a	$\{b, d\}$	$\{b, d\}$	$\{a, d\}$	$\{d\}$	$\{a, d\}$	$\{a, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{a, d\}$	$\{b, d\}$	$\{c, d, f, g\}$	$\{d\}$	$\{c, d, f, g\}$	$\{c, d, f, g\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
f	$\{a, d\}$	$\{b, d\}$	$\{c, d, f, g\}$	$\{d\}$	$\{c, d, f, g\}$	$\{c, d, f, g\}$
g	$\{a, d\}$	$\{b, d\}$	$\{c, d, f, g\}$	$\{d\}$	$\{d, f\}$	$\{d, g\}$

Table 6.

The set $\{b, c, f\}$ is a filter of $(S, \cdot, \leq)$, but it is not a filter of $(S, \circ, \leq)$. Indeed, for example, $b \circ c = \{d, b\} \nsubseteq \{b, c, f\}$. □

Proposition 4.13. *Let $(S, \cdot, \leq)$ be an ordered groupoid and $(S, \circ, \leq)$ the ordered hypergroupoid defined by the hyperoperation $a \circ b := \{t \in S \mid t \leq ab\}$. If σ is a semilattice* *(resp. complete semilattice) congruence on $(S, \circ, \leq)$, then it is a semilattice (resp. complete semilattice) congruence on $(S, \cdot, \leq)$. If σ is a semilattice congruence on $(S, \cdot, \leq)$, then it is not a semilattice congruence on $(S, \circ, \leq)$ in general.*

Proof. Let σ be a semilattice congruence on $(S, \circ, \leq)$. Let $(a, b) \in \sigma$ and $c \in S$. Since $(a \circ c, b \circ c) \in \sigma$, $ac \in a \circ c$, and $bc \in b \circ c$, we have $(ac, bc) \in \sigma$. Similarly we get $(ca, cb) \in \sigma$, and σ is a congruence on $(S, \cdot, \leq)$. Let now $a, b \in S$. Since $(a \circ a, a) \in \sigma$ and $a^2 \in a \circ a$, we have $(a^2, a) \in \sigma$. Since $(a \circ b, b \circ a) \in \sigma$, $ab \in a \circ b$ and $ba \in b \circ a$, we have $(ab, ba) \in \sigma$, so σ is a semilattice congruence on $(S, \cdot, \leq)$. Let σ be a complete semilattice congruence on $(S, \circ, \leq)$ and $a \leq b$. Since $(a, a \circ b) \in \sigma$ and $ab \in a \circ b$, we have $(a, ab) \in \sigma$, thus σ is a complete semilattice congruence on $(S, \cdot, \leq)$.

For the converse statement, consider the ordered semigroup of the Example 4.6 given by Table 1 and Figure 1 and the ordered hypersemigroup defined by the same order and the hyperoperation $x \circ y := \{t \in S \mid t \leq xy\}$ in Table 6. As we have seen, the relation

$$\sigma_1 = \{(a, a), (a, b), (b, a), (b, b), (c, c), (c, f), (d, d), (f, c), (f, f), (g, g)\}$$

is a semilattice congruence on $(S, \cdot, \leq)$. On the other site, σ_1 is not a semilattice congruence on $(S, \circ, \leq)$. It is enough to observe that $(a, b) \in \sigma_1$ but $(a \circ c, b \circ c) \notin \sigma_1$, since $a \in a \circ c$, $d \in b \circ c$ but $(a, d) \notin \sigma_1$. $\square$

The following question is natural. Under what restrictions a semilattice congruence on $(S, \cdot, \leq)$ is a semilattice congruence on $(S, \circ, \leq)$? For this purpose, we introduce the concept of pseudocomplete semilattice congruences as follows:

Definition 4.14. Let $(S, \cdot, \leq)$ be an ordered groupoid. A semilattice congruence σ on S is called *pseudocomplete* if $\leq \subseteq \sigma$.

Example 4.15. The relation $\sigma_2\, (= \mathcal{N})$ in the example 4.7 is an example of a pseudocomplete semilattice congruence on $(S, \cdot, \leq)$. There is no proper pseudocomplete semilattice congruence on $(S, \cdot, \leq)$ in the Example 4.6, in fact the only pseudocomplete semilattice congruence on $(S, \cdot, \leq)$ is the set $S \times S$.

Proposition 4.16. *Let $(S, \cdot, \leq)$ be an ordered groupoid and σ a semilattice congruence on S. If σ is pseudocomplete, then it is complete.*

Proof. Let $a \leq b$. Then $(a, ab) \in \sigma$. Indeed: Since $a \leq b$ and σ is pseudocomplete, we have $(a, b) \in \sigma$. Since σ is a semilattice congruence, we have $(a^2, ab) \in \sigma$ and $(a, a^2) \in \sigma$, thus we get $(a, ab) \in \sigma$. $\square$

Proposition 4.17. *Let $(S, \cdot, \leq)$ be an ordered groupoid and "$\circ$" the hyperoperation on S defined by*

$$\circ : S \times S \to S \mid (a, b) \to a \circ b := \{t \in S \mid t \leq ab\}.$$

If σ is a pseudocomplete semilattice congruence on $(S, \cdot, \leq)$, then it is a complete semilattice congruence on $(S, \circ, \leq)$.

Proof. Let $(a, b) \in \sigma$ and $c \in S$. Then $(a \circ c, b \circ c) \in \sigma$ and $(c \circ a, c \circ b) \in \sigma$. Indeed: Let $u \in a \circ c$ and $v \in b \circ c$. Then $u \leq ac$ and $v \leq bc$. Since σ is pseudocomplete, we have

$(u, ac) \in \sigma$ and $(v, bc) \in \sigma$. Moreover $(ac, bc) \in \sigma$, thus we get $(u, v) \in \sigma$, and σ is a right congruence on $(S, \circ, \leq)$. Similarly σ is a left congruence on $(S, \circ, \leq)$. Let $a \in S$. Then $(a \circ a, a) \in \sigma$. In fact: Let $u \in a \circ a$. Then $u \leq a^2$, thus $(u, a^2) \in \sigma$. Moreover $(a^2, a) \in \sigma$, and then $(u, a) \in \sigma$. Let $a, b \in S$. Then $(a \circ b, b \circ a) \in \sigma$. In fact: Let $u \in a \circ b$ and $v \in b \circ a$. Then $u \leq ab$, $v \leq ba$, from which $(u, ab) \in \sigma$, $(v, ba) \in \sigma$. Moreover $(ab, ba) \in \sigma$, thus we get $(u, v) \in \sigma$. Let $a \leq b$. Then $(a, a \circ b) \in \sigma$. Indeed: Let $u \in a \circ b$. Then $u \leq ab$, so $(u, ab) \in \sigma$. Since $a \leq b$, we have $(a, b) \in \sigma$, then $(a^2, ab) \in \sigma$. Moreover $(a, a^2) \in \sigma$, and then $(a, u) \in \sigma$. Hence σ is a complete semilattice congruence on $(S, \circ, \leq)$. □

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References

[1] N. Kehayopulu. On weakly commutative *poe*-semigroups. *Semigroup Forum* 34(3):367–370, 1987.

[2] N. Kehayopulu. On weakly prime ideals of ordered semigroups. *Math. Japon.* 35(6):1051–1056, 1990.

[3] N. Kehayopulu. Remark on ordered semigroups. *Math. Japon.* 35(6):1061–1063, 1990.

[4] N. Kehayopulu. On intra-regular ordered semigroups. *Semigroup Forum* 46(3):271–278, 1993.

[5] N. Kehayopulu. Green's relations and the relation $\mathcal{N}$ in Γ-semigroups. *Quasigroups Related Systems* 22(1):89–96, 2014.

[6] N. Kehayopulu. Left regular and intra-regular ordered nypersemigroups in terms of semiprime and fuzzy semiprime subsets. *Sci. Math. Jpn.* 80(3):295–305, 2017.

[7] N. Kehayopulu. Fuzzy sets in $\leq$-hypergroupoids. *Sci. Math. Jpn.* 80(3):307–314, 2017.

[8] N. Kehayopulu. How we pass from semigroups to hypersemigroups. *Lobachevskii J. Math.* 39(1):121–128, 2018.

[9] N. Kehayopulu. On ordered hypersemigroups with idempotent ideals, prime or weakly prime ideals. *Eur. J. Pure Appl. Math.* 11(1):10–22, 2018.

[10] N. Kehayopulu. On ordered hypersemigroups given by a table of multiplication and a figure. *Turkish J. Math.*, submitted.

[11] N. Kehayopulu, M. Tsingelis. Remark on ordered semigroups. In: Partitions and holomorphic mappings of semigroups (Russian) 50–55, *"Obrazovanie"*, St. Petersburg 1992.

[12] N. Kehayopulu, M. Tsingelis. On the decomposition of prime ideals of ordered semigroups into their $\mathcal{N}$-classes. *Semigroup Forum* 47(3):393–395, 1993.

[13] M. Petrich. Introduction to Semigroups, *Charles E. Merrill Publishing Company, A Bell & Howell Company.* Columbus, Ohio 1973. viii+198 pp.