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Separation Axioms in Diframes

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Abstract. Ditopological texture spaces are simultaneously generalizations of topological, bitopological and fuzzy topological spaces, and diframes are generalizations of ditopological texture spaces. In this paper we define and study the separation axioms in diframe setting.

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1. Introduction

The concept of ditopological texture spaces grew out of the study of the representation of lattice-valued topologies by bitopologies. However, as distinct from the theory of bitopological spaces based on the notion of open sets, it is a structure in which the open and closed sets play an equal role. Ditopologies are defined on a suitable subfamily $\mathcal{S} \subseteq \mathcal{P}(S)$ which is, in fact, a complete, completely distributive lattice with the relation of inclusion. Ever since the theory was first introduced by L.M. Brown [5], topological concepts, such as separation axioms, compactness and compactifications, have been studied in a series of papers by L.M. Brown and co-authors [2–4].

This work is a continuation of our previous paper [9]. In that paper, we defined the notion of diframe by replacing a texturing of a set with a lattice which is both a frame and a coframe. We also provided a link between the morphisms of the category of texture spaces (**drTex**) and the category of frames (**Frm**). This connection allows us to construct the category diFrm of diframes and diframe homomorphisms. There are at least two reasons why the theory of diframes is important. Dropping the complete distributivity condition, which makes the texture a spatial frame, (that is, a frame isomorphic to the lattice of open sets, $\Omega(X)$, of a set X), we obtain a larger family of lattices. Besides, diframe theory initiates the frame-theoretical perspective in the theory of ditopological spaces. It is well-known that the frame (locale) theory is an important area of research and it translates the (bi)topological concepts into the point-free language [1, 10].

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The rest of this paper is structured as follows. In the second section, some basic concepts and properties of ditopological texture spaces and frames are introduced to make the paper self-contained. In the third section, we define the separation axioms in the setting of diframes and we obtain equivalent characterizations of these axioms. Finally, the conclusion of this paper and some future works are discussed in Section 4.

2. Preliminaries

In this section, we recall some pertinent concepts of ditopological texture spaces, (co)frames and diframes. We refer to [2, 3] and [4] for ditopological texture spaces, and to [6] and [10] for lattice and frame theory.

Ditopological Texture Spaces: A *texturing* on a set S is a point separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with inclusion relation, which contains S and $\emptyset$ and for which arbitrary meet coincides with intersection and finite joins coincide with the union. The pair $(S, \mathcal{S})$ is known as a *texture space*, or shortly a *texture*.

A *dichotomous topology*, or *ditopology* for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of open sets τ satisfies

(T_1) $S, \emptyset \in \tau$,

(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$,

(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of closed sets κ satisfies

(CT_1) $S, \emptyset \in \kappa$,

(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$,

(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

A ditopology can be considered as a representation of lattice-valued topologies by bitopologies and one can simply infer that it is a structure in which the open and closed sets play an equal role.

(co)Frames and (co)Locales: Our notation for the theory of (co)frames and (co)locales is that of [10] and [9]. First we recall the following definitions for a lattice L:

Let L and M be posets. A pair (f, g) of monotone functions $f : L \to M$, $g : M \to L$ is called a Galois adjunction if, for all $x \in L$ and $y \in M$, $f(x) \leq y$ iff $x \leq g(y)$. In this case, f is called the left adjoint of g (denoted by $f = g^*$), and g is called the right adjoint of f (denoted by $g = f_*$).

Proposition 1. *Let (f, g) be a Galois adjunction. Then*

(i) f preserves arbitrary join, and g preserves arbitrary meet.

(ii) g is one-one iff f is onto.

(iii) If f is one-one then gf=id, if it is onto then fg=id.

Now let us recall the other required notions for the present paper:
L is called a *frame* if it is a complete lattice with the property

$$b \wedge (\bigvee A) = \bigvee\{b \wedge a : a \in A\}$$

for any $b \in L$ and any subset $A \subseteq L$.
Dually, M is called a *coframe* if it is a complete lattice with the property

$$b \vee (\bigwedge A) = \bigwedge\{b \vee a : a \in A\}$$

for any $b \in L$ and any subset $A \subseteq L$.

A frame (resp. coframe) homomorphism is a map between frames (resp. coframes) preserving arbitrary joins (resp. meets) and finite meets (resp. joins). The category of frames (resp. co-frames) and frame (resp. co-frame) homomorphisms is denoted by **Frm** (resp. **coFrm**), and the opposite category of **Frm** (resp. **coFrm**) is denoted by **Loc** (resp. **coLoc**).

A *Heyting algebra* is a bounded lattice L equipped with a binary operation $\rightarrow: L \times L \rightarrow L$ satisfying

$$c \leq a \rightarrow b \Leftrightarrow c \wedge a \leq b$$

for all $a, b, c \in L$.

A *coHeyting algebra* [11] is a bounded lattice M equipped with a binary operation $\leftarrow: M \times M \rightarrow M$ satisfying

$$a \leftarrow b \leq c \Leftrightarrow a \leq b \vee c$$

for all $a, b, c \in M$.

Every complete Boolean algebra is both a Heyting and a coHeyting algebra. The binary operations are defined by $x \rightarrow y = x^* \vee y$ and $x \leftarrow y = x \wedge y^*$, where the exponent * denotes the complement of an element. Both $x \rightarrow 0$ and $1 \leftarrow x$ coincide with the complement present in the Boolean algebra. Any (co)frame is a complete (co)Heyting algebra, and vice versa, hence each frame (coframe) carries a (co)Heyting operation. The (co)Heyting operation plays a crucial role in defining a sub(co)locale which is a subobject of a (co)locale L in the category of (co)Loc.

Given a frame L, a *subframe* is a subset $L' \subseteq L$ that is closed under arbitrary join and finite meets. Dually, a *subcoframe* is a subset $M' \subseteq M$ which is closed under arbitrary meet and finite joins.

According to [10], a *sublocale* is a subset $S \subseteq L$ with the following conditions:

($S1$) for all $N \subseteq S$, $\bigwedge N \in S$,

($S2$) $x \rightarrow s \in S$ for all $s \in S$ and $x \in L$.

Similarly, we define a *subcolocale* of a colocale M as a subset $S \subseteq M$ satisfying the following conditions:

$(cS1)$ $\bigvee N \in S$ for all $N \subseteq S$,

$(cS2)$ $s \leftarrow x \in S$ for all $s \in S$ and $x \in M$.

Observe that $S \subseteq M$ is a subcolocale if and only if S is a colocale with the induced order and the embedding $i_c : S \to M$ is a morphism of **Loc**.

The lattice of all sublocales of locale L and the lattice of all subcolocales of colocale M are denoted by $Sl(L)$ and $Scl(M)$, respectively. Note that these two lattices are both coframes and hence they satisfy de Morgan's second law stating that $(\bigwedge_{i\in I} a_i)^* = \bigvee_{i\in I} {a_i}^*$ whenever $\bigwedge_{i\in I} a_i$ exists. (Here ${a_i}^*$ denotes the pseudocomplement of a_i). All joins and meets of sublocales (resp. subcolocales) are taken in the lattice $Sl(L)$ (resp. $Scl(M)$).

Let L be a locale. Then the elements $\mathfrak{o}(a) = \{a \to x : x \in L\}$ and $\mathfrak{c}(a) = \uparrow a$ of $Sl(L)$ are referred to as *open* and *closed sublocales* corresponding $a \in L$, respectively. Dually, given a coframe M, we define the subcolocales $\mathfrak{o}_{\mathcal{C}}(k) = \{x \leftarrow k : k \in M\} = \{x \in M : x \leftarrow k = x\}$ and $\mathfrak{c}_{\mathcal{C}}(k) = \downarrow k$. The former is referred to as *open subcolocale* and the latter is referred to as *closed subcolocale* corresponding $k \in M$. Unlike subspaces, not every sublocale is complemented in the lattice of sublocales, however, $\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complementary pairs in $Sl(L)$. Similarly, $\mathfrak{o}_{\mathcal{C}}(k)$ is a complement of $\mathfrak{c}_{\mathcal{C}}(k)$ in $Scl(M)$.

There is another way of defining sublocales (resp. subcolocales) by using the notion of nuclei (resp. conuclei). A *nucleus* on a frame L is a closure operator $v : L \to L$ preserving finite meets. For a sublocale $S \subseteq L$, $v_S(a) = \bigwedge\{s \in S : a \leq s\}$ is a nucleus, and given a nucleus v on L, $S_v = v(L)$ is a sublocale. Further we have $v_{S_v} = v$ and $S_{v_S} = S$.

A *conucleus* on a coframe M is a kernel operator $t : M \to M$ preserving finite joins. The subcolocale generated by the conucleus $t : M \to M$ is $S_t = t(M)$. On the other hand, for a subcolocale $S \subseteq M$, the corresponding conuclei $t_S : M \to M$ is defined by $t_S(a) = i_{c*}(a) = \bigvee\{s \in S : s \leq a\}$. Moreover, there is a one-one correspondence between the subcolocales of M and the conuclei defined on M.

Proposition 2. *Let M be a coframe. Then*

(i) $a \leq b$ *iff* $\mathfrak{c}_{\mathcal{C}}(a) \subseteq \mathfrak{c}_{\mathcal{C}}(b)$ *iff* $\mathfrak{o}_{\mathcal{C}}(b) \subseteq \mathfrak{o}_{\mathcal{C}}(a)$.

(ii) $\bigcap_{i\in I} \mathfrak{c}_{\mathcal{C}}(a_i) = \mathfrak{c}_{\mathcal{C}}(\bigwedge_{i\in I} a_i)$.

(iii) $\mathfrak{c}_{\mathcal{C}}(a) \vee \mathfrak{c}_{\mathcal{C}}(b) = \mathfrak{c}_{\mathcal{C}}(a \vee b)$.

(iv) $\bigvee_{i\in I} \mathfrak{o}_{\mathcal{C}}(a_i) = \mathfrak{o}_{\mathcal{C}}(\bigwedge_{i\in I} a_i)$.

(v) $\mathfrak{o}_{\mathcal{C}}(a) \cap \mathfrak{o}_{\mathcal{C}}(b) = \mathfrak{o}_{\mathcal{C}}(a \vee b.)$

See [10, III 6.1.5] for the frame version of the proposition above.

Recall that a *diframe* is a triple $L = (L_e, L_{fr}, L_{cf})$ in which L_e is both a frame and a coframe, L_{fr} is a subframe and L_{cf} is a subcoframe of L_e.

A *diframe homomorphism* is a triple (φ, ψ) with the following properties:

(i) $\varphi : L_e \to M_e$ is a frame homomorphism and $\varphi[L_{fr}] \subseteq M_{fr}$,

(ii) $\psi : L_e \to M_e$ is a coframe homomorphism and $\psi[L_{cf}] \subseteq M_{cf}$.

The category of diframes and diframe homomorphisms is denoted by **diFrm**. The opposite category of **diFrm** is called the category of dilocales and denoted by **diLoc**.

The following examples will be useful in the sequel.

Example 1. *(i) Let us see the motivating example: Given a topological space X, denote by $\Omega(X)$ (resp. $\mathcal{C}(X)$) the lattice of open (resp. closed) sets of X. Then $(\mathcal{P}(X), \Omega(X), \mathcal{C}(X))$ is a diframe. For a continuous map $f : X \to Y$, the pair*

$$(f^{-1}, f^{-1}) : (\mathcal{P}(Y), \Omega(Y), \mathcal{C}(Y)) \to (\mathcal{P}(X), \Omega(X), \mathcal{C}(X))$$

is trivially a diframe homomorphism.

(ii) Let $\Omega_{reg}(\mathbb{R})$ be the complete Boolean algebra of regular open sets of $\mathbb{R}$ (with usual topology). Let $L_e = L_{cf} = \Omega_{reg}(\mathbb{R})$ and $L_{fr} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Then the triple $L = (L_e, L_{fr}, L_{cf})$ is a diframe.

(iii) Let $L_e = \Omega_{reg}(\mathbb{R})$, $L_{fr} = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and $L_{cf} = \{(a, \infty) : a, b \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Then $L = (L_e, L_{fr}, L_{cf})$ is a diframe.

(iv) If $(S, \mathcal{S}, \tau, \kappa)$ is a ditopological space then $(S, \mathcal{S}, \tau, \kappa)$ is a diframe.

Now recall the category **dfDitop** of ditopological texture spaces and bicontinuous difunctions [3]. We have the following functor $\mathfrak{E} :$ **dfDitop** $\to$ **diLoc**

$$\mathfrak{E}((S_1, \mathcal{S}_1, \tau_1, \kappa_1) \xrightarrow{(f,F)} (S_2, \mathcal{S}_2, \tau_2, \kappa_2) = (\mathcal{S}_1, \tau_1, \kappa_1) \xrightarrow{(\varphi_f, \psi_F)} (\mathcal{S}_2, \tau_2, \kappa_2),$$

where the arrow on the right represents the **diLoc** morphism corresponding to the **diFrm** morphism $(\mathcal{S}_2, \tau_2, \kappa_2) \xrightarrow{(\varphi_{F^{\leftarrow}}, \psi_{f^{\leftarrow}}) = ((\psi_F)^*, (\varphi_f)_*)} (\mathcal{S}_1, \tau_1, \kappa_1)$.

A *Hutton dispace* is a triple (L, τ, κ) where L is a complete, completely distributive lattice and (τ, κ) is a ditopology. Consider the mappings $\varphi : (L_1, \tau_1, \kappa_1) \to (L_2, \tau_2, \kappa_2)$ preserving arbitrary meets and joins and satisfying $\varphi[\tau_1] \subseteq \tau_2$, $\varphi[\kappa_1] \subseteq \kappa_2$. The resulting category is denoted by **diH**.

By **hdiFrm**, we shall denote the category of diframes and diframe homomorphism with $\varphi = \psi$. Obviously, **diH** is a full subcategory of **hdiFrm**, and **hdiFrm** is a non-full subcategory of **diFrm**.

Note that, due to the lack of space, the separation axioms for ditopological texture spaces is not repeated here. The reader is referred to [4] for a detailed discussion on this subject.

3. Separation Axioms

In this section, we define the separation axioms on diframes. We also give several characterizations of these axioms and discuss the relationship between them.

Definition 1. *A diframe $L = (L_e, L_{fr}, L_{cf})$ is said to be*

(i) *T_0 if, for all $a \in L_e$, there exists $c_i^j \in L_{fr} \cup L_{cf}$, $i \in I$, $j \in J$ such that $a = \bigvee_{j \in J} \bigwedge_{i \in I} c_i^j$.*

(ii) *co-T_0 if, for all $a \in L_e$, there exists $c_i^j \in L_{fr} \cup L_{cf}$, $i \in I$, $j \in J$ such that $a = \bigwedge_{j \in J} \bigvee_{i \in I} c_i^j$.*

Note that the axiom T_0 is not self-dual, and that T_0 and *co*-T_0 are equivalent if L_e is completely distributive.

Remark 1. *(i) We say $\mathcal{U} \subseteq L$ generates $\mathcal{V} \subseteq L$ if $\mathcal{V}$ is the smallest subset of L containing $\mathcal{U}$ and closed under arbitrary meet and join.*

(ii) In a diframe $L = (L_e, L_{fr}, L_{cf})$, L_e need not to be generated by $L_{fr} \cup L_{cf}$. If $L_e = \mathcal{P}(X)$, $L_{fr} = L_{cf} = \{\emptyset, X\}$, L_e is not generated by $L_{fr} \cup L_{cf}$. However, this property holds for T_0 or co-T_0 diframes. Indeed, if L is T_0, for all $a \in L_e$, $a = \bigvee_{j \in J} \bigwedge_{i \in I} c_i^j$ where $c_i^j \in L_{fr} \cup L_{cf}$. This means that a is an element of the set generated by $L_{fr} \cup L_{cf}$. The other inclusion is an immediate consequence of the fact that L_e is closed under arbitrary meets and joins.

(iii) If $(S, \mathcal{S}, \tau, \kappa)$ is T_0 as a diframe, it is not necessarily T_0 as a ditopological space.

Definition 2. *A diframe $L = (L_e, L_{fr}, L_{cf})$ is said to be*

(i) R_0 if every element of L_{fr} can be written as a join of elements from L_{cf}.

(ii) co-R_0 if every element of L_{cf} can be written as a meet of elements from L_{fr}.

(iii) T_1 if T_0 and R_0.

(iv) co-T_1 if co-T_0 and co-R_0.

For each property $\mathcal{P}$, the diframe $L = (L_e, L_{fr}, L_{cf})$ is said to be bi-$\mathcal{P}$ if it is $\mathcal{P}$ and *co*-$\mathcal{P}$.

Note that, Kopperman was studied R_0 in [8], under the name of "weak symmetry".

Example 2. *Consider the diframe $L = (L_e, L_{fr}, L_{cf})$ of Example 1 (ii). L is R_0 since $(-\infty, a) = \bigvee_{n \in \mathbb{N}} (a - n, a)$ for all $a \in \mathbb{R}$. However, L is not co-R_0 because the bounded intervals $(a, b) \in L_{cf}$ can not be expressed as a meet of elements from L_{fr}.*

Here are some statements equivalent to R_0 and *co*-R_0.

Proposition 3. *Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe.*

(i) The following are equivalent:

(*a*) *L is R_0.*

(*b*) *Every open sublocale associated with the elements of* L_{fr} *can be written as a join of the open sublocales associated with the elements of* L_{cf}*, that is,*

$$\mathfrak{o}(a) = \bigvee\{\mathfrak{o}(k) : k \in L_{cf} \text{ and } k \leq a\} \text{ for all } a \in L_{fr}.$$

(*c*) *Every closed sublocale associated with the elements of* L_{fr} *can be written as an intersection of the closed sublocales associated with the elements of* L_{cf}*, that is,*

$$\mathfrak{c}(a) = \bigcap\{\mathfrak{c}(k) : k \in L_{cf} \text{ and } k \leq a\} \text{ for all } a \in L_{fr}.$$

(*d*) $\forall a \in L_{fr}$, $\forall x, y \in L_e$, $a \nleq y \to x \Rightarrow k \in L_{cf};\ k \leq a,\ y \nleq k \to x$.

(ii) The following are equivalent:

(*a*) *L is co-*R_0.

(*b*) *Every open subcolocale associated with the elements of* L_{cf} *can be written as a join of the open subcolocales associated with the elements of* L_{fr}*, that is,*

$$\mathfrak{o}_{\mathcal{C}}(k) = \bigvee\{\mathfrak{o}_{\mathcal{C}}(a) : a \in L_{fr} \text{ and } k \leq a\} \text{ for all } k \in L_{cf}.$$

(*c*) *Every closed subcolocale associated with the elements of* L_{cf} *can be written as an intersection of the closed subcolocales associated with the elements of* L_{fr}*, that is,*

$$\mathfrak{c}_{\mathcal{C}}(k) = \bigcap\{\mathfrak{c}_{\mathcal{C}}(a) : a \in L_{fr} \text{ and } k \leq a\} \text{ for all } k \in L_{cf}.$$

(*d*) $\forall k \in L_{cf}$, $\forall x, y \in L_e$, $x \leftarrow y \nleq k \Rightarrow a \in L_{fr};\ k \leq a,\ x \leftarrow a \nleq y$.

Proof. (*ii*): (*a*) and (*b*) are equivalent since the equality $\bigvee_{i\in I} \mathfrak{o}_{\mathcal{C}}(a) = \mathfrak{o}_{\mathcal{C}}(\bigwedge_{i\in I} a_i)$ holds. Similarly, (*a*) and (*c*) are equivalent by the property $\bigcap_{i\in I} \mathfrak{c}_{\mathcal{C}}(a_i) = \mathfrak{c}_{\mathcal{C}}(\bigwedge_{i\in I} a_i)$.

For (*b*) implies (*d*), let $x \leftarrow y \nleq k$ for $k \in L_{cf}$ and $x, y \in L_e$. Then,

$$\mathfrak{o}_{\mathcal{C}}(k) = \bigvee\{\mathfrak{o}_{\mathcal{C}}(a) : a \in L_{fr} \text{ and } k \leq a\} \nsubseteq \mathfrak{o}_{\mathcal{C}}(x \leftarrow y)$$

and hence there exists an $a \in L_{fr}$ such that $k \leq a$ and $\mathfrak{o}_{\mathcal{C}}(a) \nsubseteq \mathfrak{o}_{\mathcal{C}}(x \leftarrow y)$, which implies the existence of an $a \in L_{fr}$ such that $k \leq a$ and $x \leftarrow a \nleq y$.

For the converse, assume contrary that $L = (L_e, L_{fr}, L_{cf})$ does not satisfy (*b*). Then there is a $k \in L_{cf}$ such that

$$\mathfrak{o}_{\mathcal{C}}(k) \nsubseteq \bigvee\{\mathfrak{o}_{\mathcal{C}}(a) : a \in L_{fr} \text{ and } k \leq a\}.$$

Thus, there exists an $x \in L_e$ such that $x \in \mathfrak{o}_{\mathcal{C}}(k)$ and $x \notin \mathfrak{o}_{\mathcal{C}}(a)$ for all $a \in L_{fr}$ satisfying $k \leq a$. Now we obtain $x \leftarrow k = x \neq x \leftarrow a$, and hence $x \leftarrow k \nleq x \leftarrow a$ since the converse inequality is always valid. Thereby, there exists a $y \in L_e$ such that $x \leftarrow a \leq y$ and $x \leftarrow k \nleq y$. We now obtain $x \leftarrow y \leq a$ and $x \leftarrow y \nleq k$ for all $a \in L_{fr}$ satisfying $k \leq a$, which contradicts with the assumption.

The proof of (*i*) is omitted since it can be proved in a similar way as above.

Remark 2. *The closure of an element $a \in L_e$ is given by $[a] = \bigwedge\{c \in L_{cf} : a \leq c\}$, and the interior by $]a[= \bigvee\{b \in L_{fr} : b \leq a\}$.*

Definition 3. *A diframe is said to be*

(i) R_1 *if, for all* $a \in L_{fr}$,

$$a = \bigvee_{j\in J} \bigwedge_{i\in I} c_i^j = \bigvee_{j\in J} \bigwedge_{i\in I} [c_i^j] \text{ where } c_i^j \in L_{fr}.$$

(ii) co-R_1 *if, for all* $k \in L_{cf}$,

$$k = \bigwedge_{j\in J} \bigvee_{i\in I} f_i^j = \bigwedge_{j\in J} \bigvee_{i\in I}]f_i^j[\text{ where } f_i^j \in L_{cf}.$$

(iii) T_2 *if* R_1 *and* T_0

(iv) co-T_2 *if* co-R_1 *and* co-T_0.

Note that, R_1 was also studied in [8], under the name "pseudo Hausdorff".

Proposition 4. *Every R_1 diframe is R_0. Dually, every co-R_1 diframe is co-R_0.*

Proof. Straightforward by definitions.

Remark 3. *As is well known, a bitopological space $(X, \mathcal{T}, \mathcal{T}^*)$ is regular if for all $G \in \mathcal{T}$ and $x \in G$, there exist a $\mathcal{T}$-open set H and a $\mathcal{T}^*$-closed set F such that $x \in H \subseteq F \subseteq G$, or equivalently, each $G \in \mathcal{T}$ can be expressed as follows:*

$$G = \bigcup\{H \in \mathcal{T} : \exists F\ \mathcal{T}^*\text{-closed};\ H \subseteq F \subseteq G\}$$

Similarly, the dual space $(X, \mathcal{T}^*, \mathcal{T})$ is regular if, for all $\mathcal{T}^*$-closed set F,

$$F = \bigcap\{K\ \mathcal{T}^*\text{-closed} : \exists G \in \mathcal{T};\ F \subseteq G \subseteq K\}.$$

Now define the relations $\prec_{fr}$ and $\prec_{cf}$ on $\mathcal{P}(X)$ by declaring that

$$H \prec_{fr} G \text{ iff there exists an } F \in \mathcal{C}(X) \text{ such that } H \subseteq F \subseteq G$$

and

$$F \prec_{cf} K \text{ iff there exists a } G \in \Omega(X) \text{ such that } F \subseteq G \subseteq K.$$

On the basis of the previous discussion, we introduce the following relations on L_e :
We say that a is fr-below b, in symbols $a \prec_{fr} b$, iff $a, b \in L_{fr}$ and there exists a $c \in L_{cf}$ such that $a \leq c \leq b$.

Dually, we say that f is cf-below k, in symbols $f \prec_{cf} k$, iff $f, k \in L_{cf}$ and there exists an $a \in L_{fr}$ such that $f \leq a \leq k$.

Proposition 5. *In a diframe L, the relations $\prec_{fr}$ and $\prec_{cf}$ satisfy the following conditions:*

(i) $0 \prec_{fr} a \prec_{fr} 1$ *for all* $a \in L_{fr}$*, and* $0 \prec_{cf} k \prec_{cf} 1$ *for all* $k \in L_{cf}$.

(ii) $a \prec_{fr} b$ *implies* $a \leq b$*, and* $f \prec_{cf} k$ *implies* $f \leq k$.

(iii) If $a \leq b \prec_{fr} c \leq d$ *then* $a \prec_{fr} d$*. If* $f \leq c \prec_{cf} d \leq k$ *then* $f \prec_{cf} k$.

(iv) For $i = 1, 2$ *if* $a_i \prec_{fr} b_i$ *then* $a_1 \vee a_2 \prec_{fr} b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec_{fr} b_1 \wedge b_2$*. Moreover, if* $f_i \prec_{cf} k_i$ *then* $f_1 \vee f_2 \prec_{cf} k_1 \vee k_2$ *and* $f_1 \wedge f_2 \prec_{cf} k_1 \wedge k_2$.

Clearly, $\prec_{fr}$ and $\prec_{cf}$ are auxiliary relations in the sense of definition I.1.9 in [6].

Definition 4. *A diframe is said to be*

(i) regular if

$$a = \bigvee\{x \in L_{fr} : x \prec_{fr} a\} \text{ for all } a \in L_{fr}.$$

(ii) co-regular if

$$c = \bigwedge\{x \in L_{cf} : c \prec_{cf} x\} \text{ for all } c \in L_{cf}.$$

(iii) T_3 *if regular and* T_0.

(iv) co-T_3 *if co-regular and co-*T_0.

The following proposition is immediate by definitions:

Proposition 6. *(i) A diframe L is regular iff* $a = \bigvee\{x \in L_{fr} : [x] \leq a\}$ *for all* $a \in L_{fr}$.
(ii) A diframe L is co-regular iff $c = \bigwedge\{x \in L_{cf} : c \leq]x[\}$ *for all* $c \in L_{cf}$.

Example 3. *Let* $\mathbb{I} = [0,1]$ *be the unit interval,* $L_e = \{[0,r], [0,r) : 0 \leq r \leq 1\}$, $L_{fr} = \{[0,r) : 0 \leq r \leq 1\} \cup \{\mathbb{I}\}$ *and* $L_{cf} = \{[0,r] : 0 \leq r \leq 1\} \cup \{\emptyset\}$*. Trivially, for* $[0,r), [0,s) \in L_{fr}$, $[0,r) \prec_{fr} [0,s)$ *iff* $r < s$.

For each $U = [0,r) \in L_{fr}$, $U = \bigvee\{[0, r - \frac{1}{n}) : [0, r - \frac{1}{n}) \prec_{fr} [0,r)\}$*. Thus,* $L = (L_e, L_{fr}, L_{cf})$ *is regular. Similarly, we can show the co-regularity of L.*

The proof of the next proposition is quite standard and will therefore be omitted.

Proposition 7. *If* $L = (L_e, L_{fr}, L_{cf})$ *is* R_0 *(*R_1*, regular) and* L'_{cf} *is a coframe with* $L_{cf} \subseteq L'_{cf}$ *then* $L' = (L_e, L_{fr}, L'_{cf})$ *is* R_0 *(*R_1*, regular). Dually, if* $L = (L_e, L_{fr}, L_{cf})$ *is co-*R_0 *(co-*R_1*, co-regular) and* L'_{fr} *is a frame with* $L_{fr} \subseteq L'_{fr}$ *then* $L' = (L_e, L'_{fr}, L_{cf})$ *is co-*R_0 *(co-*R_1*, co-regular).*

Proposition 8. *(i) A regular diframe is* R_1.
(ii) A co-regular diframe is co-R_1.

Proof. (i) Given $a \in L_{fr}$ we have $a = \bigvee_{i\in I}\{c_i \in L_{fr} : c_i \prec_{fr} a\}$ by regularity of L. Further, if $c_i \prec_{fr} a$ then there exists $k_i \in L_{cf}$ such that $c_i \le k_i \le a$. Setting $J = \{j\}$ and $c_i^j = c_i$, for all $i \in I$, we obtain

$$a = \bigvee_{i\in I}\bigwedge_{j\in J} c_i^j \le \bigvee_{i\in I}\bigwedge_{j\in J}[c_i^j] \le \bigvee_{i\in I}\bigwedge_{j\in J} k_i^j \le a$$

Thus $a = \bigvee_{i\in I}\bigwedge_{j\in J} c_i^j = \bigvee_{i\in I}\bigwedge_{j\in J}[c_i^j]$, showing that L is R_1.

Proposition 9. *(i) Every regular co-R_0 diframe is co-R_1.*
(ii) Every co -regular R_0 diframe is R_1.

Proof. (i) Let L be regular, co-R_0 and let $k \in L_{cf}$. First we have $a_i \in L_{fr}$ such that $k = \bigwedge_{i\in I} a_i$. Now, by regularity of L,

$$a_i = \bigvee_{j\in J}\{b_{ij} \in L_{fr} : \exists f_{ij} \in L_{cf}; b_{ij} \le f_{ij} \le a_i\}$$

for all $i \in I$. But then,

$$k \le \bigwedge_{i\in I}\bigvee_{j\in J} b_{ij} \le \bigwedge_{i\in I}\bigvee_{j\in J}]f_{ij}[\le \bigwedge_{i\in I}\bigvee_{j\in J} f_{ij} \le \bigwedge_{i\in I} a_i \le k$$

and hence $k = \bigwedge_{i\in I}\bigvee_{j\in J} f_{ij} = \bigwedge_{i\in I}\bigvee_{j\in J}]f_{ij}[$. Therefore, L is co-R_1.

(ii) Dual to (i), so we omit the details.

Note that complete regularity also has a counterpart in the theory of diframes. But first we need the following binary relations on L_e.

Remark 4. *Let $D = \{k/2^n : k, n \in \mathbb{N}, k = 0, \dots 2^n\}$ be the set of dyadic rationals. We can define a binary relation on L_e by setting $a \prec\!\!\prec_{fr} b$ iff $a, b \in L_{fr}$ and there exists $a_q \in L_{fr}$ ($q \in D$) satisfying*

$$a_0 = a,\ a_1 = b,\ \text{and}\ a_q \prec_{fr} a_r\ \text{for}\ q < r.$$

If $a \prec\!\!\prec_{fr} b$ then we say that a is completely fr-below b.

Similarly, the dual relation can be defined by setting $k \prec\!\!\prec_{cf} f$ iff $k, f \in L_{cf}$ and there exists $k_q \in L_{cf}$ ($q \in D$) satisfying

$$k_0 = k,\ k_1 = f,\ \text{and}\ k_q \prec_{cf} k_r\ \text{for}\ q < r.$$

If $k \prec\!\!\prec_{cf} f$ then we say that k is completely cf-below f.

The relations $\prec\!\!\prec_{fr}$ and $\prec\!\!\prec_{cf}$ have similar properties like those in Proposition 5.

Proposition 10. *The relations $\prec\!\!\prec_{fr}$ and $\prec\!\!\prec_{cf}$ on L_e satisfy the following properties:*

(i) $0 \prec\!\!\prec_{fr} a \prec\!\!\prec_{fr} 1$ for all $a \in L_{fr}$, and $0 \prec\!\!\prec_{cf} k \prec\!\!\prec_{fr} 1$ for all $k \in L_{cf}$.

(ii) $a \prec\!\!\prec_{fr} b$ *implies* $a \leq b$. *Moreover,* $f \prec\!\!\prec_{cf} k$ *implies* $f \leq k$.

(iii) If $a \leq b \prec\!\!\prec_{fr} c \leq d$ *then* $a \prec\!\!\prec_{fr} d$, *and if* $f \leq c \prec\!\!\prec_{cf} d \leq k$ *then* $f \prec\!\!\prec_{cf} k$.

(iv) If $a_i \prec\!\!\prec_{fr} b_i$ *for* $i = 1, 2$ *then* $a_1 \vee a_2 \prec\!\!\prec_{fr} b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec\!\!\prec_{fr} b_1 \wedge b_2$. *Similarly, if* $f_i \prec\!\!\prec_{cf} k_i$ *for* $i = 1, 2$ *then* $f_1 \vee f_2 \prec\!\!\prec_{cf} k_1 \vee k_2$ *and* $f_1 \wedge f_2 \prec\!\!\prec_{cf} k_1 \wedge k_2$.

(v) If $a \prec\!\!\prec_{fr} b$ *then there exists a* $c \in L_{fr}$ *with* $a \prec\!\!\prec_{fr} c \prec\!\!\prec_{fr} b$, *that is, the relation* $\prec\!\!\prec_{fr}$ *is interpolative. Moreover, it is the largest interpolative relation contained in* $\prec_{fr}$. *Similarly, the relation* $\prec\!\!\prec_{cf}$ *is interpolative and it is the largest interpolative relation contained in* $\prec_{cf}$.

Proof. The facts $(i) - (iv)$ are immediate consequences of the definitions.

(v) If $a \prec\!\!\prec_{fr} b$ then we have $a_q \in L_{fr}$ $(q \in D)$ with $a_0 = a, a_1 = b$ and $a_q \prec_{fr} a_r$ for $q < r$. Setting $c = a_{1/2}$ we obtain a sequence of elements such that $x_0 = a$, $x_1 = c$ and $x_{k/2^n} = a_{k/2^{n+1}}$. Clearly, $x_q \prec_{fr} x_r$ for $q < r$, and consequently $a \prec\!\!\prec_{fr} c$. Similarly, we can find a sequence of elements such that $y_0 = c$, $y_1 = b$ and $y_q \prec_{fr} y_r$ for $q < r$. Thus $c \prec\!\!\prec_{fr} b$ and hence the relation $\prec\!\!\prec_{fr}$ is interpolative.

Further, $\prec\!\!\prec_{fr}$ is obviously contained in $\prec_{fr}$. For the remaining assertion, let $\prec$ be any interpolative relation contained in $\prec_{fr}$. If $a \prec b$ for $a, b \in L_e$ then, by induction, we obtain a sequence of elements with $a_0 = a$, $a_1 = b$ and $a_q \prec a_r$ for $q < r$. We also have "$a_q \prec a_r \Rightarrow a_q \prec_{fr} a_r$ by assumption. Thus, $a \prec\!\!\prec_{fr} b$.

Definition 5. *A diframe is said to be*

(i) completely regular if

$$a = \bigvee \{x \in L_{fr} : x \prec\!\!\prec_{fr} a\} \textit{ for all } a \in L_{fr}.$$

(ii) completely co-regular if

$$c = \bigwedge \{x \in L_{cf} : c \prec\!\!\prec_{cf} x\} \textit{ for all } c \in L_{cf}.$$

(iii) $T_{3\frac{1}{2}}$ *if completely regular and* T_0.

(iv) co-$T_{3\frac{1}{2}}$ *if completely co-regular and co-*T_0.

As mentioned before, complete (co-) regularity is defined using bicontinuous difunctions in ditopological spaces. Here, we leave the following questions as open problems:
(1) Can we construct a diframe corresponding to the ditopological unit interval texture space $(\mathbb{I}, \mathcal{J}, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$?
(2) How do we characterize complete regularity by using diframe homomorphisms ?
(3) What is the relation between these two characterizations of completely regularity ?

Proposition 11. *(i) A completely regular diframe is regular.*

(ii) A completely co-regular diframe is co-regular.

Proof. This is an immediate consequence of the following facts: $a \prec\!\!\prec_{fr} b$ implies $a \prec_{fr} b$, and $a \prec\!\!\prec_{cf} b$ implies $a \prec_{cf} b$.

There is another way of characterizing complete regularity of a bitopological space in terms of a Urysohn relation due to Kopperman [8]. Now we will generalize this idea to diframes.

We start by recalling the definition of a Urysohn relation [7]. A binary relation $\lhd$ on a partially ordered set $(L, \leq)$ is called a *Urysohn relation* if it satisfies the following conditions:

$(U1)$ $a \lhd b$ implies $a \leq b$ for all $a, b \in L$,

$(U2)$ $a \leq b \lhd c \leq d$ implies $a \lhd d$ for all $a, b, c, d \in L$,

$(U3)$ $a \lhd b$ implies the existence of $c \in L$ such that $a \lhd c \lhd b$ for all $a, b \in L$ (that is, $\lhd$ is an interpolative relation).

If L is a lattice and $\lhd$ is a Urysohn relation on L, we call the pair $(L, \lhd)$ a Urysohn lattice. The following are some basic examples of Urysohn relations.

Example 4. *(i) Let X be a normal space. For $U, V \in \Omega(X)$, define a relation $\lhd$ by setting $U \lhd V$ iff $\overline{U} \subseteq V$. Then $\lhd$ is a Urysohn relation.*

(ii) The relations $\prec_{fr}$ and $\prec_{cf}$ are not Urysohn since the interpolation property does not hold. However, $\prec\!\!\prec_{fr}$ and $\prec\!\!\prec_{cf}$ are obviously Urysohn relations by Proposition 10.

Proposition 12. *Let $L = (L_e, L_{fr}, L_{cf})$ be a diframe.*

(i) L is completely regular if and only if there exists a Urysohn relation $\lhd$ on L_e satisfying the following conditions:

- (a) *$a \lhd b$ implies $[a] \leq]b[$,*
- (b) *for every $a \in L_{fr}$, $a = \bigvee\{x \in L_{fr} : x \lhd a\}$.*

(ii) L is completely co-regular if and only if there exists a Urysohn relation $\lhd$ on L_e satisfying the following conditions:

- (a) *$a \lhd b$ implies $[a] \leq]b[$,*
- (b) *for every $c \in L_{cf}$, $c = \bigwedge\{x \in L_{cf} : c \lhd x\}$.*

Proof. Here, we just prove (i), since (ii) can be proven similarly. If L is a completely regular diframe then $\prec\!\!\prec_{fr}$ is the desired relation. Indeed, as can be easily checked, it is a Urysohn relation. Further, the condition (b) is a direct result of the definition. Now let $a \prec\!\!\prec_{fr} b$. Then applying the definitions of $\prec\!\!\prec_{fr}$ and $\prec_{fr}$, respectively, we obtain $a_q \in L_{fr}$ and $c_q \in L_{cf}$ $(q \in D)$ such that

$$a \leq \ldots a_q \leq c_q \leq a_r \leq \ldots \leq b$$

and hence

$$[a] \leq \ldots \leq [a_q] \leq [c_q] = c_q \leq a_r \leq \ldots \leq b =]b[.$$

where $q < r$. Thus, the relation $\prec\!\!\prec_{fr}$ satisfies (a).

Conversely, suppose that we have a Urysohn relation $\lhd$ on L_e satisfying the conditions (a) and (b). Let $x \lhd a$ for $x, a \in L_{fr}$. By $(U3)$, there exists $y_q \in L_e$ $(q \in D)$ such that

$$x \lhd \ldots y_q \lhd y_r \ldots \lhd a$$

where $q < r$. Since $]y_q[\leq [y_q] \leq]y_r[$ by (a), we have

$$x \prec_{fr} \ldots \prec_{fr}]y_q[\prec_{fr}]y_r[\prec_{fr} \ldots \prec_{fr} a.$$

We now obtain $x \lhd a$ implies $x \prec\!\!\prec_{fr} a$. Therefore, for all $a \in L_{fr}$,

$$a = \bigvee \{x \in L_{fr} : x \lhd a\} \leq \bigvee \{x \in L_{fr} : x \prec\!\!\prec_{fr} a\} \leq a$$

and hence $L = (L_e, L_{fr}, L_{cf})$ is completely regular.

As is well known, normality is a separation axiom that can be defined purely in terms of the open and closed sets. In other words, its definition is not based on points, which makes it easier to discuss them in the point-free context.

Definition 6. *A diframe is said to be*

(i) normal if, for any $c \in L_{cf}$ and $a \in L_{fr}$ such that $c \leq a$, there exists a $b \in L_{fr}$ such that $c \leq b \leq [b] \leq a$.

(ii) T_4 if normal and T_1.

(iii) co-T_4 if normal and co-T_1.

Remark 5. *Normality is self-dual. Hence we can use the equivalent definition: "for any $c \in L_{cf}$ and $a \in L_{fr}$ such that $c \leq a$ there exists a $k \in L_{cf}$ such that $c \leq]k[\leq k \leq a$." This is easily obtained by setting $k = [b]$ in the definition of normality.*

Proposition 13. *Let $\lhd$ be a binary relation on L_e such that "$a \lhd b$ iff $[a] \leq]b[$". Then $L = (L_e, L_{fr}, L_{cf})$ is normal if and only if $\lhd$ is a Urysohn relation on L_e.*

Proof. Suppose L is a normal diframe. Then we claim that the relation $\lhd$ given in the proposition satisfies the properties $(U1) - (U3)$. We only prove $(U3)$ since $(U1)$ and $(U2)$ are straightforward.

Let $a \lhd b$. Then $[a] \leq]b[$ and hence, by normality, there exists a $c \in L_{fr}$ such that $[a] \leq]c[= c \leq [c] \leq]b[$. Thus we have $a \lhd c \lhd b$.

For the converse, let $c \leq a$ for any $c \in L_{cf}$ and $a \in L_{fr}$. Then $c \lhd a$ and hence, by (U3), there exists a $b \in L_e$ such that $c \lhd b \lhd a$. Now we have $c \leq [c] \leq]b[\leq b \leq [b] \leq]a[\leq a$. Setting $d =]b[$ we obtain $c \leq d \leq [d] \leq a$. Thus L is a normal diframe.

Example 5. *Normality does not imply regularity. Consider the diframe L of Example 1 (iii). L is normal: Let $C \in L_{cf}$, $A \in L_{fr}$ with $C \subseteq A$. Then there are three cases to consider: (i) $C = A = \emptyset$, (ii) $C = A = \mathbb{R}$, (iii) $C \neq \mathbb{R}$, $A = \mathbb{R}$. We may take $B = \emptyset$ in case (i), and $B = \mathbb{R}$ in cases (ii) and (iii), showing L is regular. However, L is obviously not normal.*

Proposition 14. *(i) Every normal R_0 diframe is regular.*
(ii) Every normal co-R_0 diframe is co-regular.

Proof. (i) Let $a \in L_{fr}$ and set $c = \bigvee\{b \in L_{fr} : b \prec_{fr} a\}$. Clearly, $c \leq a$. On the other hand, $\mathfrak{o}(a) = \bigvee\{\mathfrak{o}(k) : k \in L_{cf}$ and $k \leq a\}$ since L is R_0. Hence, to prove $a \leq c$, it is enough to show that $\mathfrak{o}(a) \subseteq \mathfrak{o}(c)$, that is, $\mathfrak{o}(k) \subseteq \mathfrak{o}(c)$ for all $k \in L_{cf}$ with $k \leq a$. So take an element $k \in L_{cf}$ such that $k \leq a$. Then, by normality, there exists a $b \in L_{fr}$ such that $k \leq b \leq [b] \leq a$, yielding $b \prec_{fr} a$ and $k \leq b$. Thus $k \leq b \leq c$, and hence $\mathfrak{o}(k) \subseteq \mathfrak{o}(c)$, as required.

Proposition 15. *(i) A normal R_0 diframe is completely regular.*
(ii) A normal co-R_0 diframe is completely co-regular.

Proof. We will just prove the first statement and leave the other statement to the reader. Since each normal R_0 diframe is regular it is enough to show that the relations $\prec_{fr}$ and $\prec\!\!\prec_{fr}$ coincide in a normal diframe. For this, we have to prove that $\prec_{fr}$ is interpolative. If $a \prec_{fr} b$ then there exists a $k \in L_{cf}$ such that $a \leq k \leq b$. Moreover, by normality, there is a $d \in L_{fr}$ such that $a \leq k \leq d \leq [d] \leq b$. Thus, $a \prec_{fr} d \prec_{fr} b$, which means that $\prec_{fr}$ is interpolative. Thus we have, by Proposition 10 (v), $\prec_{fr} = \prec\!\!\prec_{fr}$.

Corollary 1. *We have the following implications in a diframe:*

$$\textit{normal and } R_0 \Rightarrow \textit{completely regular} \Rightarrow \textit{regular} \Rightarrow R_0.$$

$$\textit{normal and co-}R_0 \Rightarrow \textit{completely co-regular} \Rightarrow \textit{co-regular} \Rightarrow \textit{co-}R_0.$$

$$(co\text{-})T_4 \Rightarrow (co\text{-})T_{3\frac{1}{2}} \Rightarrow (co\text{-})T_3 \Rightarrow (co\text{-})T_2 \Rightarrow (co\text{-})T_1 \Rightarrow (co\text{-})T_0.$$

We end this section by investigating the image of a diframe with a property $\mathcal{P}$ under a special kind of homomorphism.

Definition 7. *A diframe homomorphism $(\varphi, \psi) : L \to M$ is called*

- *(i) open (respectively, co-open) if $\psi^*(a) \in L_{fr}$ (resp. $\varphi_*(a) \in L_{fr}$) for all $a \in M_{fr}$.*
- *(ii) closed (respectively, co-closed) if $\psi^*(k) \in L_{cf}$ (resp. $\varphi_*(k) \in L_{cf}$) for all $k \in M_{cf}$.*

Proposition 16. *Let L and M be diframes and let $(\varphi, \psi) : L \to M$ be a one-one onto diframe homomorphism.*

- *(i) If (φ, ψ) is open (resp. co-open) then, for all $b \in M_{fr}$, there exists an $a \in L_{fr}$ such that $\psi(a) = b$ (resp. $\varphi(a) = b$).*

(ii) If (φ, ψ) is closed (resp. co-closed) then, for all $k \in M_{cf}$, there exists an $f \in L_{cf}$ such that $\psi(f) = k$ (resp. $\varphi(f) = k$).

Proof. Suppose $(\varphi, \psi) : L_e \to M_e$ is open and $b \in M_{fr}$. Since ψ is onto, there is an $a \in L_e$ with $\psi(a) = b$. Now, by Proposition 1, we have $\psi^*\psi(a) = a = \psi^*(b)$, and hence $a = \psi^*(b) \in L_{fr}$ by openness of (φ, ψ).

The other cases can be proved similarly.

Remark 6. *If φ is one-one and onto then, by Proposition 1 (iii), $\varphi_*\varphi = 1_{L_e}$ and $\varphi\varphi_* = 1_{M_e}$, that is, $\varphi^{-1} = \varphi_*$. Similarly, if ψ is one-one and onto then $\psi^{-1} = \psi^*$. Thus, if $(\varphi, \varphi) = \varphi : L \to M$ is a one-one onto **hdiFrm** homomorphism then $\varphi^* = \varphi_*$, and hence the concept of openness (resp., closedness) coincides with co-openness (resp. co-closedness).*

Definition 8. *If L and M are diframes, a **hdiFrm** homomorphism $(\varphi, \varphi) = \varphi : L \to M$ is called an isomorphism if it is one-one, onto, open and closed,.*

Proposition 17. *Let L, M be diframes and $\varphi : L \to M$ be a **hdiFrm** isomorphism. Then, L is bi-R_0 (respectively, bi-R_1, bi-regular, completely bi-regular, normal) if and only if M is bi-R_0 (respectively, bi-R_1, bi-regular, completely bi-regular, normal).*

Proof. We will just prove the regularity and the other axioms are left to the interested reader. Let L be regular and $b \in M_{fr}$. Then, by Proposition 16, there is an $a \in L_{fr}$ such that $\varphi(a) = b$ and, by regularity of L, $a = \bigvee\{x \in L_{fr} : x \prec_{fr} a\}$. Moreover, $x \prec_{fr} a$ implies $\varphi(x) \prec_{fr} b$ by definition of $\prec_{fr}$. Now we have

$$b = \varphi(a) = \varphi(\bigvee\{x \in L_{fr} : x \prec_{fr} a\}) \leq \bigvee\{\varphi(x) \in M_{fr} : \varphi(x) \prec_{fr} b)\} \leq b$$

and hence M is regular.

Conversely, suppose that M is regular and $a \in L_{fr}$. Then $\varphi(a) \in M_{fr}$ and hence, by regularity, $\varphi(a) = \bigvee\{x \in M_{fr} : x \prec_{fr} \varphi(a)\}$. Now if $x \prec_{fr} \varphi(a)$ then, by Proposition 1 together with the closedness of φ, we have $\varphi^*(x) \prec_{fr} a$. But then

$$a = \varphi^*\varphi(a) = \varphi^*(\bigvee\{x \in M_{fr} : x \prec_{fr} \varphi(a)\}) \leq \bigvee\{\varphi^*(x) \in L_{fr} : \varphi^*(x) \prec_{fr} a\} \leq a$$

and hence L is regular.

4. Conclusion

In this paper we have studied the separation axioms in diframes and examined the relations between them. We have defined new binary relations on a diframe and obtained a characterization of regularity and complete regularity by using these relations. As a future work, other topological and bitopological structures such as compactness, stability, join compactness and connectedness, etc. can be constructed on diframes.

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