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On the Irreducibility of Fourth Dimensional Tuba's Representation of the Pure Braid Group on Three Strands

Hasan A. Haidar[1], Mohammad N. Abdulrahim[1,*]

[1] *Department of Mathematics, Faculty of Science, Beirut Arab University, P.O. Box: 11-5020, Beirut, Lebanon*

Abstract. We consider Tuba's representation of the pure braid group, P_3, given by the map $\phi : P_3 \longrightarrow GL(4, F)$, where F is an algebraically closed field. After, specializing the indeterminates used in defining the representation to non- zero complex numbers, we find sufficient conditions that guarantee the irreducibility of Tuba's representation of the pure braid group P_3 with dimension $d = 4$. Under further restriction for the complex specialization of the indeterminates, we get a necessary and sufficient condition for the irreducibility of ϕ.

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1. Introduction

Let B_n be the braid group on n strands. There exists a surjective group homomorphism $\pi : B_n \longrightarrow S_n$. The kernel of π is referred to as the pure braid group P_n with $\frac{n(n-1)}{2}$ generators. In 2001, a representation of B_3 was defined by I. Tuba and H. Wenzl , namely $\rho : B_3 \longrightarrow GL(V)$, which is irreducible on the dimensional vector space V over an algebraically closed field F. A complete classification of irreducible representations of the braid group B_3 was given by Tuba and Wenzl, for dimensions $d \leq 5$ (see [7]). This was done by assuming a certain triangular form of the matrices of the generators of B_3. Albeverio has found a class of representations of B_3 in every dimension n, which depends on n parameters [1]. The author in that work uses a deformation of pascal's triangle connected with qshifted factorials to get the representations, and this generalizes the work of Tuba and Wenzl who classified all irreducible representations of B_3 for dimensions $d \leq 5$ [7] . This is also a generalization of the results of Humphries, who constructed the representations of the braid group B_3 in arbitrary dimension using the classical pascal triangle [3]. Le

*Corresponding author.
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Email addresses: hah339@student.bau.edu.lb (H. A. Haidar), mna@bau.edu.lb (M. N. Abdulrahim)

Bruyn in [4] proved that all the components of n-dimensional irreducible representations of B_3 are densely parametrized by rational quiver varieties and the explicit parametrizations are given for $n < 12$. Then Le Bruyn in [5] extended all this by establishing such parametrizations for all finite dimensions n, which also generalizes the work of Tuba and Wenzl. Also, researchers gave a great value for representations of the pure braid group P_n, the normal subgroup of B_n.
Recently, N. Maanna and M. Abdulrahim gave a necessary and sufficient condition for the irreducibility of the Tuba's representation of pure braid group P_3 for dimensions 2 and 3 (see [6]). In our work, we mainly consider the irreducibility criteria of Tuba's representation of the pure braid group P_3, with dimension four. Our main result is Theorem 11, which determines sufficient conditions for the irreducibility of Tuba's representation of P_3 with dimension $d = 4$. Under further restriction on the indeterminates used in defining Tuba's representation of dimension 4, we get a necessary and sufficient condition for the irreducibility of the representation. This will be corollary 12.

2. Preliminaries

Definition 1. *[2] The braid group on n strings, B_n, is the abstract group with presentation B_n =$\{\sigma_1, ..., \sigma_{n-1}; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, for $i = 1, 2, ..., n-2$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \succ 1\}$.*

The generators $\sigma_1, ..., \sigma_{n-1}$ are called the standard generators of B_n.

Definition 2. *[2] The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \longrightarrow S_n$, defined by $\sigma_i \longrightarrow (i, i+1), 1 \leq i \leq n-1$. It has the following generators:*

$$A_{ij} = \sigma_{j-1}\sigma_{j-2}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}, 1 \leq i, j \leq n$$

Definition 3. *A representation is a map $\gamma : G \longrightarrow GL(V)$, where G is a group and $GL(V)$ is the group of $n \times n$ invertible matrices over the algebraically closed field V.*

Definition 4. *A representation $\gamma : G \longrightarrow GL(V)$ is said to be irreducible if it has no non trivial proper invariant subspaces.*

3. Tuba's Representation of B_3

Imre Tuba and Hans Wenzl gave a complete classification of all simple representations of B_3 with dimensions $d \leq 5$ by assuming a certain triangular form for the invertible $d \times d$ matrices A and B of the generators of B_3 that satisfy $ABA = BAB$. In particular, they proved that a simple $d-$ dimensional representation $\varphi : B_3 \longrightarrow GL(V)$ is determined, up to isomorphism, by the eigenvalues $\lambda_1, ...\lambda_d$ of the images of the generators σ_1 and σ_2 of B_3. For more details, see [7]. Below,we write the explicit matrices in the case $d = 4$.

Proposition 1. [7, p.500] *Tuba′s representaion of B_3 of dimension $d = 4$ is defind as follows:*

$$\sigma_1 \longrightarrow \begin{pmatrix} \lambda_1 & (1+D^{-1}+D^{-2})\lambda_2 & (1+D^{-1}+D^{-2})\lambda_3 & \lambda_4 \\ 0 & \lambda_2 & (1+D^{-1})\lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$$\sigma_2 \longrightarrow \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ -\lambda_3 & \lambda_3 & 0 & 0 \\ D\lambda_2 & -(D+1)\lambda_2 & \lambda_2 & 0 \\ -D^3\lambda_1 & (D^3+D^2+D)\lambda_1 & -(D^2+D+1)\lambda_1 & \lambda_1 \end{pmatrix}, \text{ where}$$

$\lambda_1, \lambda_2, \lambda_3,$ *and* λ_4 *are indeterminates and* $D = \sqrt{\frac{\lambda_2\lambda_3}{\lambda_1\lambda_4}}$.

Proposition 2. [7, p.503] *Tuba's representation of B_3 of dimension four is irreducible if and only if*
$-\gamma^{-2}\left(\lambda_r^2+\gamma^2\right)\left(\lambda_s^2+\gamma^2\right)\left(\gamma^2+\lambda_r\lambda_k+\lambda_s\lambda_l\right)\left(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k\right) \neq 0,$
where $\{r, s, k, l\} = \{1, 2, 3, 4\}$, *and γ^2 is a square root of the* $det(\sigma_1)$.

A similar result is obtained for the pure braid group P_3, the normal subgroup of the braid group B_3.

Proposition 3. *[6] Tuba's representation of P_3 is irreducible if and only if*
i) $\lambda_1 \neq -\lambda_2$ *and* $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 \neq 0$ *for dimension* $d = 2$,
ii) $\lambda_i \neq -\lambda_j$ *and* $(\lambda_m^2 + \lambda_k\lambda_n)(\lambda_n^2 + \lambda_k\lambda_m) \neq 0$, *for dimension* $d = 3$.
Here $i \neq j$, $m \neq n \neq k$, *and* $i, j, m, n, k \in \{1, 2, 3\}$.

4. Tuba's Representation of P_3

Let P_3 be the pure braid group on three strings. Applying Tuba's representation on the normal subgroup of the braid group , namely the pure braid group, we get the following representation of dimension $d = 4$.

Definition 5. *Tuba's representation of the pure braid group P_3 of dimension $d = 4$ is defined as follows:*

$$A_{12} = \begin{pmatrix} \lambda_1^2 & J\lambda_2(\lambda_1+\lambda_2) & J\lambda_3[\lambda_1+\lambda_3+I\lambda_2] & \lambda_4[\lambda_1+\lambda_4+J(\lambda_2+\lambda_3)] \\ 0 & \lambda_2^2 & I\lambda_3(\lambda_2+\lambda_3) & \lambda_4[\lambda_2+\lambda_4+I\lambda_3] \\ 0 & 0 & \lambda_3^2 & \lambda_4(\lambda_3+\lambda_4) \\ 0 & 0 & 0 & \lambda_4^2 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} \lambda_4^2 & 0 & 0 & 0 \\ -\lambda_3(\lambda_3+\lambda_4) & \lambda_3^2 & 0 & 0 \\ \lambda_2[D\lambda_4+(D+1)\lambda_3+D\lambda_2] & -(D+1)\lambda_2(\lambda_2+\lambda_3) & \lambda_2^2 & 0 \\ L & K & M & \lambda_1^2 \end{pmatrix},$$

where

$$\begin{aligned} I &= 1+D^{-1}, \\ J &= 1+D^{-1}+D^{-2}, \\ K &= \lambda_1(D^2+D+1)[D(\lambda_1+\lambda_2+\lambda_3)+\lambda_2], \\ L &= \lambda_1[-D^3(\lambda_4+\lambda_1)-(D^3+D^2+D)(\lambda_3+\lambda_2)], \\ M &= -(D^2+D+1)\lambda_1(\lambda_1+\lambda_2). \end{aligned}$$

As for $A_{13}=\sigma_2\sigma_1^2\sigma_2^{-1}$, *we will not need it in the proof of Theorem* 11.

5. Irreducibility of Tuba's Representation of the Pure Braid Group P_3 with Dimension d=4

We specialize the indeterminates $\lambda_1, \lambda_2, \lambda_3$, and λ_4 to non zero complex numbers. Then we find sufficient conditions for the irreducibility of the complex specialization of Tuba's representation of the pure braid group P_3 with dimension $d=4$.

Definition 6. *Principal square root function is defined as follows:*

For $z=(1,\theta)$, $\sqrt{z}=e^{\frac{\theta}{2}i}$, *where* $-\pi \prec \theta \leq \pi$. *Since* $\theta \in (-\pi,\pi]$, *it follows that* $\sqrt{z^2}=z$ *for any complex number* z.

In what follows, we take $\sqrt{z^2}=z$ for any complex number z.

Lemma 1. *Let* $\varphi : P_3 \longrightarrow GL_4(\mathbb{C})$ *be the complex specialization of Tuba's representation of the pure braid group* P_3. *Hence, the following are true:*

i) $D+1=0$ *if and only if* $\gamma^2+\lambda_1\lambda_4=0$.

ii) $D\lambda_n+\lambda_m=0$ *if and only if* $\gamma^2+\lambda_n^2=0$, *where* $\{m,n\}=\{2,3\}$.

iii) $1+D+D^2=0$ *implies that* $\gamma^2+\lambda_1\lambda_4+\lambda_2\lambda_3=0$.

Proof. The proof of (i) follows from the fact that $\det(\sigma_1)=\lambda_1\lambda_2\lambda_3\lambda_4$ and $\gamma^2=\lambda_1\lambda_4 D$.

The proof of (ii) follows from the fact that $\gamma^2=\lambda_2\lambda_3 D^{-1}$.

To prove (iii) : If $1+D+D^2=0$ then $D^3-1=0$. This implies that

$(\lambda_1\lambda_4 D)^3-(\lambda_1\lambda_4)^3=0$, which is equivalent to $\gamma^6-(\lambda_1\lambda_4)^3=0$. Hence

$(\gamma^2-\lambda_1\lambda_4)(\gamma^4+\lambda_1\lambda_4\gamma^2+(\lambda_1\lambda_4)^2)=0$

In the case $\gamma^2-\lambda_1\lambda_4=0$, we get $\lambda_1\lambda_4 D-\lambda_1\lambda_4=0$. This implies that $D=1$,

a contradiction. Thus $\gamma^4+\lambda_1\lambda_4\gamma^2+(\lambda_1\lambda_4)^2=0$. It follows that

$\lambda_2\lambda_3\gamma^4+\lambda_1\lambda_2\lambda_3\lambda_4\gamma^2+\lambda_1^2\lambda_2\lambda_3\lambda_4^2=0$. This implies that $\lambda_2\lambda_3\gamma^4+\gamma^6+\lambda_1\lambda_4\gamma^4=0$.

Thus $\lambda_2\lambda_3+\gamma^2+\lambda_1\lambda_4=0$.

Theorem 1. *Tuba's representation $\varphi : P_3 \longrightarrow GL_4(\mathbb{C})$ is irreducible if the following hold true:*

(1) $\lambda_i \neq -\lambda_j$, where $i,j\in\{1,2,3,4\}$
(2) $\gamma^2+\lambda_l\lambda_4\neq 0$, where $l\in\{1,2,3\}$
(3) $\gamma^2(\lambda_i+\lambda_3+\lambda_4)+\lambda_i\lambda_j\lambda_3+\lambda_i\lambda_j\lambda_4+\lambda_j\lambda_3\lambda_4\neq 0$, where $\{i,j\}\in\{1,2\}$
(4) $(\gamma^2+\lambda_r^2)(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k)\neq 0$, where $\{r,s,l,k\}=\{1,2,3,4\}$

Proof. To get contradiction, suppose that this representation $\varphi : P_3 \longrightarrow GL_4(C)$ is reducible .That is, there exists a proper non-zero invariant subspace S, of dimension 1 , 2 or 3.We consider 15 cases. We use e_1, e_2, e_3, and e_4 as the canonical basis of $\mathbb{C}^4$. Let α, β and δ be non-zero complex numbers.

Case 1: Let $e_1\in S$, it follows that

$A_{23}e_1-\lambda_4^2 e_1\in S$, then $\begin{pmatrix}0\\R_2\\R_3\\R_4\end{pmatrix}\in S.$

Here, the constants are given by

$$\begin{aligned}R_2 &= -\lambda_3(\lambda_3+\lambda_4),\\ R_3 &= \lambda_2[D\lambda_4+(D+1)\lambda_3+D\lambda_2],\\ R_4 &= \lambda_1[-D^3(\lambda_4+\lambda_1)-(D^3+D^2+D)(\lambda_3+\lambda_2)].\end{aligned}$$

We have $A_{23}(R_2e_2 + R_3e_3 + R_4e_4) - \lambda_3^2(R_2e_2 + R_3e_3 + R_4e_4) \in S$.

Then $\begin{pmatrix} 0 \\ 0 \\ P_3 \\ P_4 \end{pmatrix} \in S$, where

$$P_3 = -R_2(D+1)\lambda_2(\lambda_2 + \lambda_3) + R_3(\lambda_2^2 - \lambda_3^2),$$

$$P_4 = KR_2 - R_3(D^2 + D + 1)\lambda_1(\lambda_1 + \lambda_2) + R_4(\lambda_1^2 - \lambda_3^2).$$

Also, we have $A_{23}(P_3e_3 + P_4e_4) - \lambda_2^2(P_3e_3 + P_4e_4) \in S$. Then $\begin{pmatrix} 0 \\ 0 \\ 0 \\ T \end{pmatrix} \in S$.

Here,

$$T = -P_3(D^2 + D + 1)\lambda_1(\lambda_1 + \lambda_2) + P_4(\lambda_1^2 - \lambda_2^2).$$

If $T \neq 0$, then $e_4 \in S$. This implies that $P_3e_3 \in S$.

In the case $P_3 = 0$, We get $-R_2(D+1)\lambda_2 + R_3(\lambda_2 - \lambda_3) = 0$,

and so $(D+1)\lambda_2\lambda_3(\lambda_3 + \lambda_4) + (\lambda_2 - \lambda_3)\lambda_2[D\lambda_4 + (D+1)\lambda_3 + D\lambda_2] = 0$.

Thus $\lambda_2(\lambda_3 + D\lambda_2)(\lambda_4 + \lambda_2) = 0$, which implies that $\lambda_3 + D\lambda_2 = 0$. This is equivalent to $\gamma^2 + \lambda_2^2 = 0$ (Lemma 10), a contradiction.

In the case $P_3 \neq 0$, we get $e_3 \in S$. But we have $R_2e_2 + R_3e_3 + R_4e_4 \in S$,
and $R_2 = -\lambda_3(\lambda_3 + \lambda_4) \neq 0$. So $e_2 \in S$, and also $e_1 \in S$.
Thus $S = \mathbb{C}^4$, a contradiction.

Therefore, we have $T = 0$.

This implies that $\lambda_2\lambda_3 + D^3\lambda_1^2 + D\lambda_1\lambda_2 + D\lambda_1\lambda_3 + D^2\lambda_1\lambda_2 + D^2\lambda_1\lambda_3 = 0$.
We get $\lambda_1\lambda_4^2(\lambda_2\lambda_3 + D^3\lambda_1^2 + D\lambda_1\lambda_2 + D\lambda_1\lambda_3 + D^2\lambda_1\lambda_2 + D^2\lambda_1\lambda_3) = 0$.
Thus $\gamma^2[\gamma^2(\lambda_2 + \lambda_3 + \lambda_4) + \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4] = 0$, a contradiction.

Case 2: Let $e_2 \in S$, it follows that

$A_{12}e_2 - \lambda_2^2 e_2 \in S$. Then $\begin{pmatrix} (1+D^{-1}+D^{-2})\lambda_2\left(\lambda_1+\lambda_2\right) \\ 0 \\ 0 \\ 0 \end{pmatrix} \in S.$

But $(1+D^{-1}+D^{-2})\lambda_2\left(\lambda_1+\lambda_2\right) \neq 0$, a contradiction (Case 1).

Case 3: Let $e_4 \in S$, it follows that

$\frac{1}{\lambda_4}[A_{12}e_4 - \lambda_4^2 e_4] \in S$. Then $\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ 0 \end{pmatrix} \in S.$

Here, the constants are given by

$N_1 = \lambda_1 + \lambda_4 + (1+D^{-1}+D^{-2})\left(\lambda_2+\lambda_3\right),$
$N_2 = \lambda_2 + \lambda_4 + (1+D^{-1})\lambda_3,$
$N_3 = \lambda_3 + \lambda_4.$

We have $A_{12}(N_1e_1 + N_2e_2 + N_3e_3) - \lambda_3^2(N_1e_1 + N_2e_2 + N_3e_3) \in S$.

This implies that $\begin{pmatrix} M_1 \\ M_2 \\ 0 \\ 0 \end{pmatrix} \in S,$

where

$M_1 = N_1(\lambda_1^2 - \lambda_3^2) + N_2J\lambda_2\left(\lambda_1+\lambda_2\right) + N_3J\lambda_3[\lambda_1+\lambda_3+(1+D^{-1})\lambda_2],$
$M_2 = N_2(\lambda_2^2 - \lambda_3^2) + N_3I\lambda_3\left(\lambda_2+\lambda_3\right).$

Also, we have $A_{12}(M_1e_1 + M_2e_2) - \lambda_2^2(M_1e_1 + M_2e_2) \in S$.

This implies that $\begin{pmatrix} T \\ 0 \\ 0 \\ 0 \end{pmatrix} \in S$, where

$T = M_1(\lambda_1^2 - \lambda_2^2) + M_2J\lambda_2\left(\lambda_1+\lambda_2\right).$

If $T \neq 0$, then we get a contradiction (Case 1).

If $T = 0$. Then $\lambda_2\lambda_3 + D^3\lambda_1^2 + D\lambda_1\lambda_2 + D\lambda_1\lambda_3 + D^2\lambda_1\lambda_2 + D^2\lambda_1\lambda_3 = 0$.
Thus $\gamma^2[\gamma^2(\lambda_2+\lambda_3+\lambda_4) + \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4] = 0$, a contradiction.

Case 4: Let $e_3 \in S$, it follows that $A_{23}e_3 - \lambda_2^2 e_3 \in S$.

So $\begin{pmatrix} 0 \\ 0 \\ 0 \\ -(D^2+D+1)\lambda_1(\lambda_1+\lambda_2) \end{pmatrix} \in S.$

But $-(D^2+D+1)\lambda_1(\lambda_1+\lambda_2) \neq 0$, a contradiction (Case 3).

Case 5: Let $e_1 + \alpha e_2 \in S$, it follows that

$A_{12}(e_1+\alpha e_2) - \lambda_2^2(e_1+\alpha e_2) \in S$. This implies that $\begin{pmatrix} T \\ 0 \\ 0 \\ 0 \end{pmatrix} \in S$, where

$T = \lambda_1^2 - \lambda_2^2 + \alpha J \lambda_2(\lambda_1+\lambda_2)$.

If $T \neq 0$, then $e_1 \in S$, a contradiction (Case 1). If $T = 0$ then $\lambda_1 - \lambda_2 + \alpha J \lambda_2 = 0$. It follows that $J\lambda_2 e_1 + (\lambda_2 - \lambda_1)e_2 \in S$. Thus

$A_{23}[J\lambda_2 e_1 + (\lambda_2-\lambda_1)e_2] - \lambda_4^2[J\lambda_2 e_1 + (\lambda_2-\lambda_1)e_2] \in S$. Then $\begin{pmatrix} 0 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \in S.$

Here, the constants are given by

$R_2 = -\lambda_2 J\lambda_3(\lambda_4+\lambda_3) + (\lambda_2-\lambda_1)(\lambda_3^2-\lambda_4^2),$

$R_3 = \lambda_2^2 J[D\lambda_4 + (D+1)\lambda_3 + D\lambda_2] - (\lambda_2-\lambda_1)(D+1)\lambda_2(\lambda_3+\lambda_2),$

$R_4 = \lambda_2 JL + (\lambda_2-\lambda_1)K.$

Since $R_2e_2 + R_3e_3 + R_4e_4 \in S$, it follows that

$A_{23}(R_2e_2+R_3e_3+R_4e_4) - \lambda_3^2(R_2e_2+R_3e_3+R_4e_4) \in S$. Thus $\begin{pmatrix} 0 \\ 0 \\ P_3 \\ P_4 \end{pmatrix} \in S,$

where

$P_3 = -R_2(D+1)\lambda_2(\lambda_2+\lambda_3) + R_3(\lambda_2^2-\lambda_3^2),$

$P_4 = R_2K + R_3M + R_4(\lambda_1^2-\lambda_3^2).$

On the other hand, $A_{23}(P_3e_3 + P_4e_4) - \lambda_2^2(P_3e_3 + P_4e_4) \in S$. Then $\begin{pmatrix} 0 \\ 0 \\ 0 \\ T_1 \end{pmatrix} \in S$.

Here, the constant T_1 is given by

$T_1 = -P_3(D^2 + D + 1)\lambda_1(\lambda_1 + \lambda_2) + P_4(\lambda_1^2 - \lambda_2^2)$.

If $T_1 \neq 0$, then $e_4 \in S$, a contradiction (by Case 3).

If $T_1 = 0$ then $D^2\lambda_1^3 + D^2\lambda_1\lambda_2\lambda_4 + D\lambda_1^2\lambda_2 + D\lambda_1\lambda_2\lambda_3 + D\lambda_2^2\lambda_4 + \lambda_1\lambda_2^2 = 0$.

Hence, $\frac{\lambda_1\lambda_4}{\lambda_2}(D^2\lambda_1^3 + D^2\lambda_1\lambda_2\lambda_4 + D\lambda_1^2\lambda_2 + D\lambda_1\lambda_2\lambda_3 + D\lambda_2^2\lambda_4 + \lambda_1\lambda_2^2) = 0$.

This implies that $(\gamma^2 + \lambda_1^2)(\gamma^2 + \lambda_1\lambda_3 + \lambda_2\lambda_4) = 0$, a contradiction.

Case 6: Let $e_1 + \alpha e_3 \in S$, it follows that

$A_{12}(e_1 + \alpha e_3) - \lambda_3^2(e_1 + \alpha e_3) \in S$. Then $\begin{pmatrix} N_1 \\ N_2 \\ 0 \\ 0 \end{pmatrix} \in S$.

Here, the constants are given by

$N_1 = \lambda_1^2 - \lambda_3^2 + \alpha J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2]$,
$N_2 = \alpha I\lambda_3\left(\lambda_2 + \lambda_3\right)$.

We have $N_2 \neq 0$ ($I \neq 0$ by Lemma 10). If $N_1 = 0$, then $e_2 \in S$, a contradiction (Case 2). If $N_1 \neq 0$, we get a contradiction (Case 5).

Case 7: Let $e_3 + \alpha e_4 \in S$, it follows that

$A_{23}(e_3 + \alpha e_4) - \lambda_2^2(e_3 + \alpha e_4) \in S$. So $\begin{pmatrix} 0 \\ 0 \\ 0 \\ T \end{pmatrix} \in S$, where

$T = -D^2J\lambda_1(\lambda_1 + \lambda_2) + \alpha(\lambda_1^2 - \lambda_2^2)$.

If $T \neq 0$, then $e_4 \in S$, a contradiction.

If $T = 0$ then $-D^2J\lambda_1 + \alpha(\lambda_1 - \lambda_2) = 0$.

On the other hand, we have $(e_3 + \alpha e_4)(\lambda_1 - \lambda_2) \in S$.
It follows that $(\lambda_1 - \lambda_2)e_3 + D^2J\lambda_1 e_4 \in S$.

Hence $A_{12}[(\lambda_1 - \lambda_2)e_3 + D^2J\lambda_1 e_4] - \lambda_4^2[(\lambda_1 - \lambda_2)e_3 + D^2J\lambda_1 e_4] \in S$.

$$\text{Thus } \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ 0 \end{pmatrix} \in S.$$

Here, the constants are given by

$N_1 = (\lambda_1 - \lambda_2)J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2] + D^2J\lambda_1\lambda_4[\lambda_1 + \lambda_4 + J(\lambda_2 + \lambda_3)]$,
$N_2 = (\lambda_1 - \lambda_2)I\lambda_3(\lambda_2 + \lambda_3) + D^2J\lambda_1\lambda_4[\lambda_2 + \lambda_4 + I\lambda_3]$,
$N_3 = (\lambda_1 - \lambda_2)(\lambda_3^2 - \lambda_4^2) + D^2J\lambda_1\lambda_4(\lambda_3 + \lambda_4)$.

Also, we have $A_{12}(N_1e_1 + N_2e_2 + N_3e_3) - \lambda_3^2(N_1e_1 + N_2e_2 + N_3e_3) \in S$.
Then $\begin{pmatrix} M_1 \\ M_2 \\ 0 \\ 0 \end{pmatrix} \in S$, where

$M_1 = N_1(\lambda_1^2 - \lambda_3^2) + N_2J\lambda_2(\lambda_1 + \lambda_2) + N_3J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2]$,

$M_2 = N_2(\lambda_2^2 - \lambda_3^2) + N_3I\lambda_3(\lambda_2 + \lambda_3)$.

If $M_1 \neq 0$ or $M_2 \neq 0$. Then we get a contradiction (Case 1, Case 2, and Case 5).

If $M_1 = 0$ and $M_2 = 0$. Then

$\lambda_2\lambda_3(\lambda_1 + \lambda_4) + D^2\lambda_1\lambda_3\lambda_4 + D\lambda_1\lambda_2\lambda_3 + D\lambda_1\lambda_2\lambda_4 + D\lambda_2\lambda_3\lambda_4 = 0$.

So $D^{-1}(\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + D^2\lambda_1\lambda_3\lambda_4 + D\lambda_1\lambda_2\lambda_3 + D\lambda_1\lambda_2\lambda_4 + D\lambda_2\lambda_3\lambda_4) = 0$.

This implies that $\gamma^2(\lambda_1 + \lambda_3 + \lambda_4) + \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 = 0$,
a contradiction.

Case 8: Let $e_1 + \alpha e_4 \in S$, it follows that $A_{23}(e_1 + \alpha e_4) - \lambda_4^2(e_1 + \alpha e_4) \in S$.

$$\text{Then} \begin{pmatrix} 0 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \in S.$$

Here, the constants are given by

$R_2 = -\lambda_3 (\lambda_3 + \lambda_4)$,
$R_3 = \lambda_2[D\lambda_4 + (D+1)\lambda_3 + D\lambda_2]$,
$R_4 = L + \alpha(\lambda_1^2 - \lambda_4^2)$.

Also, we have $A_{23}(R_2e_2 + R_3e_3 + R_4e_4) - \lambda_3^2(R_2e_2 + R_3e_3 + R_4e_4) \in S$,
then $\begin{pmatrix} 0 \\ 0 \\ P_3 \\ P_4 \end{pmatrix} \in S$.

Here, $P_3 = -R_2(D+1)\lambda_2(\lambda_2 + \lambda_3) + R_3(\lambda_2^2 - \lambda_3^2)$,
$P_4 = R_2K + R_3M + R_4(\lambda_1^2 - \lambda_3^2)$.

If $P_3 \neq 0$ or $P_4 \neq 0$, then we get a contradiction (Case 3, Case 4, and Case 7).

Otherwise, if $P_3 = 0$ then $\lambda_2 (\lambda_2 + \lambda_4) (\lambda_2 + \lambda_3) (D\lambda_2 + \lambda_3) = 0$.

Hence $D\lambda_2 + \lambda_3 = 0$, which is equivalent to $\gamma^2 + \lambda_2^2 = 0$ (Lemma 10), a contradiction.

Case 9: Let $e_2 + \alpha e_3 \in S$, it follows that

$A_{12}(e_2 + \alpha e_3) - \lambda_3^2(e_2 + \alpha e_3) \in S$. Then $\begin{pmatrix} N_1 \\ N_2 \\ 0 \\ 0 \end{pmatrix} \in S$.

Here, the constants are given by

$N_1 = J\lambda_2 (\lambda_1 + \lambda_2) + \alpha J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2]$,
$N_2 = \lambda_2^2 - \lambda_3^2 + \alpha I\lambda_3 (\lambda_2 + \lambda_3)$.

If $N_1 \neq$ or $N_2 \neq 0$, then we get a contradiction (Case 1, Case 2, and Case 5).

Otherwise, $N_2 = 0$ and so

$\frac{1}{D} (\lambda_2 + \lambda_3) (D\lambda_2 - D\lambda_3 + \alpha\lambda_3 + \alpha D\lambda_3) = 0$.

This implies that $D\lambda_2 - D\lambda_3 + \alpha(\lambda_3 + D\lambda_3) = 0$. (1)

Also, we have $N_1 = 0$. It follows that

$\frac{1}{D^3} D^2 J(D\lambda_2^2 + D\lambda_1\lambda_2 + \alpha\lambda_2\lambda_3 + \alpha D\lambda_3^2 + \alpha D\lambda_1\lambda_3 + \alpha D\lambda_2\lambda_3) = 0.$

Hence $D\lambda_2^2 + D\lambda_1\lambda_2 + \alpha\lambda_2\lambda_3 + \alpha D\lambda_3^2 + \alpha D\lambda_1\lambda_3 + \alpha D\lambda_2\lambda_3 = 0.$ (2)

Now, after subtracting equation (2) from equation $\lambda_2(1)$, we get
$D\lambda_2\lambda_3 + D\lambda_1\lambda_2 + \alpha D\lambda_3^2 + \alpha D\lambda_1\lambda_3 = 0$. This implies that
$D\left(\lambda_1 + \lambda_3\right)\left(\lambda_2 + \alpha\lambda_3\right) = 0$. Thus $\alpha\lambda_3 = -\lambda_2$. Substituting $\alpha\lambda_3 = -\lambda_2$ in (1).
We get $-D\lambda_3 - \lambda_2 = 0$, which is equivalent to $\gamma^2 + \lambda_3^2 = 0$ (Lemma 10), a contradiction.

Case 10: Let $e_2 + \alpha e_4 \in S$, it follows that

$A_{23}(e_2 + \alpha e_4) - \lambda_3^2(e_2 + \alpha e_4) \in S$. Then $\begin{pmatrix} 0 \\ 0 \\ P_3 \\ P_4 \end{pmatrix} \in S.$

Here, the constants are given by

$P_3 = -(D+1)\lambda_2\left(\lambda_2 + \lambda_3\right),$
$P_4 = \lambda_1(D^2 + D + 1)[D(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_2] + \alpha(\lambda_1^2 - \lambda_3^2).$

If $P_3 \neq 0$ or $P_4 \neq 0$, then we get a contradiction (Case 3, Case 4, and Case 7).

Otherwise, if $P_3 = 0$ then $-(D+1)\lambda_2\left(\lambda_2 + \lambda_3\right) = 0$, a contradiction (Lemma 10).

Case 11: Let $\alpha e_1 + \beta e_2 + e_3 \in S$, it follows that

$A_{12}(\alpha e_1 + \beta e_2 + e_3) - \lambda_3^2(\alpha e_1 + \beta e_2 + e_3) \in S$.Then $\begin{pmatrix} N_1 \\ N_2 \\ 0 \\ 0 \end{pmatrix} \in S.$

Here, the constants are given by

$N_1 = \alpha(\lambda_1^2 - \lambda_3^2) + \beta J\lambda_2\left(\lambda_1 + \lambda_2\right) + J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2],$
$N_2 = \beta(\lambda_2^2 - \lambda_3^2) + I\lambda_3\left(\lambda_2 + \lambda_3\right).$

If $N_1 \neq 0$ or $N_2 \neq 0$, then we get a contradiction (Case 1, Case 2, and Case 5).

Otherwise, $N_2 = 0$ and so $\beta(\lambda_2 - \lambda_3) + I\lambda_3 = 0$.

In the case $\lambda_2 - \lambda_3 = 0$, we get $I\lambda_3 = 0$, a contradiction (Lemma 10). Hence $\beta = \frac{I\lambda_3}{\lambda_3 - \lambda_2}$.

Substituting, $\beta = \frac{I\lambda_3}{\lambda_3 - \lambda_2}$ in the equation $(\lambda_3 - \lambda_2)N_1 = 0$, we get

$\alpha\left(\lambda_1^2 - \lambda_3^2\right)(\lambda_3 - \lambda_2) + IJ\lambda_3\lambda_2(\lambda_1 + \lambda_2) + J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2](\lambda_3 - \lambda_2) = 0.$

Then $\alpha\left(\lambda_1^2 - \lambda_3^2\right)(\lambda_3 - \lambda_2) + IJ\lambda_2\lambda_3(\lambda_1 + \lambda_3) + \lambda_3 J(\lambda_3 - \lambda_2)(\lambda_1 + \lambda_3) = 0.$

This implies that $\alpha(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2) + IJ\lambda_2\lambda_3 + \lambda_3 J(\lambda_3 - \lambda_2) = 0.$

Hence $\alpha(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2) + \lambda_3(1 + D^{-1} + D^{-2})(\lambda_3 + D^{-1}\lambda_2) = 0.$

If $(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) = 0$, then $\lambda_3(1 + D^{-1} + D^{-2})(\lambda_3 + D^{-1}\lambda_2) = 0.$

So $\lambda_3 + D^{-1}\lambda_2 = 0$, which is equivalent to $\gamma^2 + \lambda_3^2 = 0$ (Lemma 10), a contradiction.

That is $(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \neq 0$. Thus $\alpha = \frac{\lambda_3(1+D^{-1}+D^{-2})(\lambda_3+D^{-1}\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$.

On the other hand, $A_{23}(\alpha e_1 + \beta e_2 + e_3) - \lambda_4^2(\alpha e_1 + \beta e_2 + e_3) \in S$. It follows that

$$\begin{pmatrix} 0 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \in S.$$

Here, the constants are given by

$$\begin{aligned} R_2 &= -\alpha\lambda_3(\lambda_4 + \lambda_3) + \beta(\lambda_3^2 - \lambda_4^2), \\ R_3 &= \alpha\lambda_2[D\lambda_4 + (D+1)\lambda_3 + D\lambda_2] - \beta(D+1)\lambda_2(\lambda_2 + \lambda_3) + \lambda_2^2 - \lambda_4^2, \\ R_4 &= \alpha L + \beta K + M. \end{aligned}$$

We have $A_{23}(R_2e_2 + R_3e_3 + R_4e_4) - \lambda_3^2(R_2e_2 + R_3e_3 + R_4e_4) \in S$.

It follows that $\begin{pmatrix} 0 \\ 0 \\ P_3 \\ P_4 \end{pmatrix} \in S.$

Here,

$$\begin{aligned} P_3 &= -R_2(D+1)\lambda_2(\lambda_2 + \lambda_3) + R_3(\lambda_2^2 - \lambda_3^2), \\ P_4 &= R_2K + R_3M + R_4(\lambda_1^2 - \lambda_3^2). \end{aligned}$$

If $P_3 \neq 0$ or $P_4 \neq 0$, then we get a contradiction (Case 3, Case 4, and Case 7).

Thus $P_3 = 0$ and $P_4 = 0$.

But $\alpha = \frac{\lambda_3(1+D^{-1}+D^{-2})(\lambda_3+D^{-1}\lambda_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$, and $\beta = \frac{(1+D^{-1})\lambda_3}{(\lambda_3-\lambda_2)}$.

After substituting the obtaining values of α and β in the equation $P_3 = 0$, we get

$\frac{(\lambda_2+\lambda_3)(\lambda_2+D\lambda_3)(\lambda_3+D\lambda_2)(\lambda_2+\lambda_4)}{D^3(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}(D^2(\lambda_1\lambda_2 + \lambda_3\lambda_4 - \lambda_1\lambda_4) + D\lambda_2\lambda_3 + \lambda_2\lambda_3) = 0.$

It follows that, either

$(\lambda_2 + D\lambda_3)(\lambda_3 + D\lambda_2) = 0$ or $D^2(\lambda_1\lambda_2 + \lambda_3\lambda_4 - \lambda_1\lambda_4) + D\lambda_2\lambda_3 + \lambda_2\lambda_3 = 0.$

If $(\lambda_2 + D\lambda_3)(\lambda_3 + D\lambda_2) = 0$, which is equivalent to $(\gamma^2 + \lambda_3^2)(\gamma^2 + \lambda_2^2) = 0$ (Lemma 10), we get a contradiction,

If $D^2(\lambda_1\lambda_2 + \lambda_3\lambda_4 - \lambda_1\lambda_4) + D\lambda_2\lambda_3 + \lambda_2\lambda_3 = 0$, we get that
$D(\lambda_1\lambda_4\lambda_2\lambda_3 + D\lambda_1\lambda_4\lambda_1\lambda_2 + D\lambda_1\lambda_4\lambda_3\lambda_4) = 0$, which is equivalent to
$D\gamma^2(\gamma^2 + \lambda_1\lambda_2 + \lambda_3\lambda_4)$. This give a contradiction.

Case 12: Let $e_2 + \alpha e_3 + \beta e_4 \in S$, it follows that

$A_{23}(e_2 + \alpha e_3 + \beta e_4) - \lambda_3^2(e_2 + \alpha e_3 + \beta e_4) \in S.$
.

$$\text{Then} \begin{pmatrix} 0 \\ 0 \\ P_3 \\ P_4 \end{pmatrix} \in S.$$

Here, the constants are given by

$P_3 = -(D+1)\lambda_2\,(\lambda_2 + \lambda_3) + \alpha(\lambda_2^2 - \lambda_3^2),$
$P_4 = K + \alpha M + \beta(\lambda_1^2 - \lambda_3^2).$

If $P_3 \neq 0$ or $P_4 \neq 0$, then we get a contradiction (Case 3, Case 4, and Case 7).

Otherwise, if $P_3 = 0$ then $-(D+1)\lambda_2 + \alpha(\lambda_2 - \lambda_3) = 0$.

In the case $\lambda_2 - \lambda_3 = 0$, we get $-(D+1)\lambda_2 = 0$, a contradiction (Lemma 10). Hence $\alpha = \frac{(D+1)\lambda_2}{\lambda_2-\lambda_3}$.

Substituting $\alpha = \frac{(D+1)\lambda_2}{\lambda_2-\lambda_3}$ in the equation $P_4(\lambda_2-\lambda_3)=0$, we get
$K(\lambda_2-\lambda_3)+(D+1)\lambda_2 M+\beta(\lambda_1^2-\lambda_3^2)=0$.
Then $-\lambda_1(D^2+D+1)(\lambda_1+\lambda_3)(D\lambda_3+\lambda_2)+\beta(\lambda_1^2-\lambda_3^2)=0$ and so
$-\lambda_1(D^2+D+1)(D\lambda_3+\lambda_2)+\beta(\lambda_1-\lambda_3)=0$.

If $\lambda_1-\lambda_3=0$, then $D\lambda_3+\lambda_2=0$, which is equivalent to $\gamma^2+\lambda_3^2=0$ (Lemma 10), a contradiction.

Hence $\beta = \frac{\lambda_1(D^2+D+1)(D\lambda_3+\lambda_2)}{\lambda_1-\lambda_3}$.

On the other hand, $A_{12}(e_2+\alpha e_3+\beta e_4)-\lambda_4^2(e_2+\alpha e_3+\beta e_4)\in S$.

$$\text{Then} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ 0 \end{pmatrix} \in S.$$

Here, the constants are given by

$N_1 = J\lambda_2(\lambda_1+\lambda_2)+\alpha J\lambda_3[\lambda_1+\lambda_3+I\lambda_2]+\beta\lambda_4[\lambda_1+\lambda_4+J(\lambda_2+\lambda_3)]$,
$N_2 = \lambda_2^2-\lambda_4^2+\alpha I\lambda_3(\lambda_2+\lambda_3)+\beta\lambda_4[\lambda_2+\lambda_4+I\lambda_3]$,
$N_3 = \alpha(\lambda_3^2-\lambda_4^2)+\beta\lambda_4(\lambda_3+\lambda_4)$.

If $N_1\neq 0$ or $N_2\neq 0$ or $N_3\neq 0$,
then we get a contradiction (Case 1,Case 2, Case 4, Case 5, Case 6, Case 9, and Case 11).

Otherwise, if $N_3=0$ then $\beta\lambda_4=\alpha(\lambda_4-\lambda_3)$.
Substituting $\beta\lambda_4=\alpha(\lambda_4-\lambda_3)$ in the equation $N_2=0$, we get

$\lambda_2^2-\lambda_4^2+\alpha I\lambda_3(\lambda_2+\lambda_3)+\alpha(\lambda_4-\lambda_3)[\lambda_2+\lambda_4+I\lambda_3]=0 \quad (3)$.
Substituting $\alpha=\frac{(D+1)\lambda_2}{\lambda_2-\lambda_3}$ in (1), we get $\frac{(\lambda_2+\lambda_4)(\lambda_2+D\lambda_4)(\lambda_3+D\lambda_2)}{D(\lambda_2-\lambda_3)}=0$.
This implies that $\lambda_2+D\lambda_4=0$ or $\lambda_3+D\lambda_2=0$.

If $\lambda_2+D\lambda_4=0$ then $D^{-1}\lambda_2\lambda_3+\lambda_3\lambda_4=0$, which is equivalent to $\gamma^2+\lambda_3\lambda_4=0$, a contradiction,

If $\lambda_3+D\lambda_2=0$ then $\gamma^2+\lambda_2^2=0$ (Lemma 10), a contradiction.

Case 13 : Let $\alpha e_1+\beta e_2+e_4\in S$, it follows that

$A_{12}(\alpha e_1+\beta e_2+e_4)-\lambda_4^2(\alpha e_1+\beta e_2+e_4)\in S$.

Then $\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ 0 \end{pmatrix} \in S$.

Here, the constants are given by

$N_1 = \alpha(\lambda_1^2 - \lambda_4^2) + \beta J \lambda_2 (\lambda_1 + \lambda_2) + \lambda_4[\lambda_1 + \lambda_4 + J(\lambda_2 + \lambda_3)]$,
$N_2 = \beta(\lambda_2^2 - \lambda_4^2) + \lambda_4[\lambda_2 + \lambda_4 + I\lambda_3]$,
$N_3 = \lambda_4 (\lambda_3 + \lambda_4)$.

If $N_1 \neq 0$ or $N_2 \neq 0$ or $N_3 \neq 0$,
then we get a contradiction (Case 1, Case 2, Case 4, Case 5, Case 6, Case 9, and Case 11).

Otherwise, if $N_3 = 0$ then $\lambda_4 (\lambda_3 + \lambda_4) = 0$, a contradiction.

Case 14: Let $e_1 + \alpha e_3 + \beta e_4 \in S$, it follows that

$A_{23}(e_1 + \alpha e_3 + \beta e_4) - \lambda_4^2(e_1 + \alpha e_3 + \beta e_4) \in S$.

$$\text{Then } \begin{pmatrix} 0 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \in S.$$

Here, the constants are given by

$R_2 = -\lambda_3 (\lambda_3 + \lambda_4)$,
$R_3 = \lambda_2[D\lambda_4 + (D+1)\lambda_3 + D\lambda_2] + \alpha(\lambda_2^2 - \lambda_4^2)$,
$R_4 = L + \alpha M + \beta(\lambda_1^2 - \lambda_4^2)$.

If $R_2 \neq 0$ or $R_3 \neq 0$ or $R_4 \neq 0$ then we get a contradiction (Case 2, Case 3, Case 4, Case 7,Case 9, Case 10, and Case 12).

Otherwise, if $R_2 = 0$ then $-\lambda_3 (\lambda_3 + \lambda_4) = 0$, a contradiction.

Case 15: Let $\alpha e_1 + \beta e_2 + \delta e_3 + e_4 \in S$, it follows that

$A_{12}(\alpha e_1 + \beta e_2 + \delta e_3 + e_4) - \lambda_4^2(\alpha e_1 + \beta e_2 + \delta e_3 + e_4) \in S$.

.

$$\text{Then} \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ 0 \end{pmatrix} \in S.$$

Here, the constants are given by

$N_1 = \alpha(\lambda_1^2 - \lambda_4^2) + \beta J\lambda_2 (\lambda_1 + \lambda_2) + \delta J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2] + \lambda_4[\lambda_1 + \lambda_4 + J (\lambda_2 + \lambda_3)]$,
$N_2 = \beta(\lambda_2^2 - \lambda_4^2) + \delta I\lambda_3 (\lambda_2 + \lambda_3) + \lambda_4[\lambda_2 + \lambda_4 + I\lambda_3]$,
$N_3 = \delta(\lambda_3^2 - \lambda_4^2) + \lambda_4 (\lambda_3 + \lambda_4)$.

If $N_1 \neq 0$ or $N_2 \neq 0$ or $N_3 \neq 0$ then we get a contradiction (Case 1, Case 2, Case 4, Case 5, Case 6, Case 9, and Case 11).

Otherwise, if $N_3 = 0$ then $\delta(\lambda_3 - \lambda_4) + \lambda_4 = 0$. This implies that $\delta = \frac{\lambda_4}{\lambda_4-\lambda_3}$.

Substituting $\delta = \frac{\lambda_4}{\lambda_4-\lambda_3}$ in the equation $N_2(\lambda_4 - \lambda_3) = 0$, we get
$\beta(\lambda_2^2 - \lambda_4^2)(\lambda_4 - \lambda_3) + \lambda_4 I\lambda_3 (\lambda_2 + \lambda_3) + \lambda_4(\lambda_4 - \lambda_3)[\lambda_2 + \lambda_4 + I\lambda_3] = 0$.

So $\beta(\lambda_2^2 - \lambda_4^2)(\lambda_4 - \lambda_3) + \lambda_4 I\lambda_3(\lambda_2 + \lambda_4) + \lambda_4(\lambda_4 - \lambda_3)(\lambda_2 + \lambda_4) = 0$.

This implies that $(\lambda_2 + \lambda_4)[\beta(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_3) + \lambda_4(\lambda_4 + \lambda_3 D^{-1})] = 0$.

Hence $\beta(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_3) + \lambda_4(\lambda_4 + \lambda_3 D^{-1}) = 0$.

In the case $(\lambda_2 - \lambda_4)(\lambda_4 - \lambda_3) = 0$, we get $\lambda_4 + \lambda_3 D^{-1} = 0$.
This implies that $\lambda_2\lambda_4 + \lambda_2\lambda_3 D^{-1} = 0$, which is equivalent to $\gamma^2 + \lambda_2\lambda_4 = 0$,
a contradiction. Hence $\beta = \frac{\lambda_4(\lambda_4+\lambda_3 D^{-1})}{(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}$.

Let us substitute $\delta = \frac{\lambda_4}{\lambda_4-\lambda_3}$, and $\beta = \frac{\lambda_4(\lambda_4+\lambda_3 D^{-1})}{(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}$ in the equation

$N_1(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4) = 0$. It follows that

$\alpha f + \lambda_4(\lambda_4 + \lambda_3 D^{-1})J\lambda_2 (\lambda_1 + \lambda_2) - \lambda_4(\lambda_2 - \lambda_4)J\lambda_3[\lambda_1 + \lambda_3 + I\lambda_2] + g = 0$.

Then

$\alpha f + D^{-3}\lambda_4 (\lambda_1 + \lambda_4) (\lambda_2\lambda_3 + D^3\lambda_4^2 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 + D^2\lambda_2\lambda_4 + D^2\lambda_3\lambda_4) = 0.(4)$

Here, the constants are f and g are given by

$f = (\lambda_1^2 - \lambda_4^2)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)$,

$g = \lambda_4[\lambda_1 + \lambda_4 + J(\lambda_2 + \lambda_3)](\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)$.

In the case $f = 0$, we get $\lambda_1 = \lambda_4$. Let us substitute $\lambda_1 = \lambda_4$ in (4), we get $D^{-3}\lambda_4(\lambda_1 + \lambda_4)(\lambda_2\lambda_3 + D^3\lambda_4^2 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 + D^2\lambda_2\lambda_4 + D^2\lambda_3\lambda_4) = 0$.

This implies that $\lambda_4(\lambda_2\lambda_3 + D^3\lambda_4^2 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 + D^2\lambda_2\lambda_4 + D^2\lambda_3\lambda_4) = 0$.

But $D^3 = \frac{D\lambda_2\lambda_3}{\lambda_4^2}$, for $\lambda_1 = \lambda_4$.
So $\lambda_2\lambda_3 + D\lambda_2\lambda_3 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 + D^2\lambda_2\lambda_4 + D^2\lambda_3\lambda_4 = 0$.

Then $(D+1)(\lambda_2\lambda_3 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4) = 0$. Hence $\lambda_2\lambda_3 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 = 0$.

This implies that $D^{-1}\lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 0$.

It follows that $\gamma^2 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = 0$, a contradiction. Thus $f \neq 0$.

Now, (4) implies that $\alpha = \frac{\lambda_4(\lambda_2\lambda_3 + D^3\lambda_4^2 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 + D^2\lambda_2\lambda_4 + D^2\lambda_3\lambda_4)}{D^3(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_4-\lambda_3)}$.

On the other hand, $A_{23}(\alpha e_1 + \beta e_2 + \delta e_3 + e_4) - \lambda_4^2(\alpha e_1 + \beta e_2 + \delta e_3 + e_4) \in S$.

$$\text{Then} \begin{pmatrix} 0 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} \in S.$$

Here, the constants are given by

$R_2 = -\alpha\lambda_3(\lambda_3 + \lambda_4) + \beta(\lambda_3^2 - \lambda_4^2)$,
$R_3 = \alpha\lambda_2[D\lambda_4 + (D+1)\lambda_3 + D\lambda_2] - \beta(D+1)\lambda_2(\lambda_2 + \lambda_3) + \delta(\lambda_2^2 - \lambda_4^2)$,
$R_4 = \alpha L + \beta K + \delta M + \lambda_1^2 - \lambda_4^2$.

If $R_2 \neq 0$ or $R_3 \neq 0$ or $R_4 \neq 0$ then we get a contradiction (Case 2, Case 3, Case 4, Case 7, Case 9, Case 10, and Case 12).

Otherwise, if $R_2 = 0$ then $-\alpha\lambda_3 + \beta(\lambda_3 - \lambda_4) = 0$.

Hence $\alpha = \frac{\beta(\lambda_3-\lambda_4)}{\lambda_3}$, and we have $\beta = \frac{\lambda_4(\lambda_4+\lambda_3 D^{-1})}{(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}$.

Thus $\alpha = \frac{\lambda_4(\lambda_4+\lambda_3 D^{-1})}{\lambda_3(\lambda_2-\lambda_4)}$. Also, $\alpha = \frac{\lambda_4(\lambda_2\lambda_3 + D^3\lambda_4^2 + D\lambda_2\lambda_4 + D\lambda_3\lambda_4 + D^2\lambda_2\lambda_4 + D^2\lambda_3\lambda_4)}{D^3(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_4-\lambda_3)}$.

So $-\frac{\lambda_4(\lambda_2\lambda_3+D^3\lambda_4^2+D\lambda_2\lambda_4+D\lambda_3\lambda_4+D^2\lambda_2\lambda_4+D^2\lambda_3\lambda_4)}{D^3(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_4-\lambda_3)}+\frac{\lambda_4(\lambda_4+\lambda_3 D^{-1})}{\lambda_3(\lambda_2-\lambda_4)}=0.$

This implies that

$$\frac{\lambda_2\lambda_3^2+D^3\lambda_4^3+D^2\lambda_1\lambda_3^2-D^3\lambda_1\lambda_4^2+D^2\lambda_3\lambda_4^2+D\lambda_3^2\lambda_4-D^2\lambda_1\lambda_3\lambda_4+D^3\lambda_1\lambda_3\lambda_4+(D+1)D\lambda_2\lambda_3\lambda_4}{D^3\lambda_3(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}=0.$$

It's easy to see that, $\lambda_2\lambda_3^2-D^2\lambda_1\lambda_3\lambda_4$, so we get $h=0$, where

$$h=D^3\lambda_4^3+D^2\lambda_1\lambda_3^2-D^3\lambda_1\lambda_4^2+D^2\lambda_3\lambda_4^2+D\lambda_3^2\lambda_4+D^3\lambda_1\lambda_3\lambda_4+(D+1)D\lambda_2\lambda_3\lambda_4.$$

Now, we multiply the equation $h=0$ by $\frac{\lambda_1\lambda_4}{D}$, we get

$$\lambda_2\lambda_3\lambda_4^3+\lambda_1\lambda_3^2\gamma^2-\lambda_4\gamma^4+\lambda_3\lambda_4^2\gamma^2+\lambda_1\lambda_3^2\lambda_4^2+\lambda_3\gamma^4+\lambda_2\lambda_3\lambda_4\gamma^2+\lambda_4\gamma^4=0.$$

Hence $\lambda_3(\gamma^2+\lambda_4^2)(\gamma^2+\lambda_1\lambda_3+\lambda_2\lambda_4)=0$, a contradiction.

Therefore, there is no non-zero invariant proper subspace. This implies that we have determined sufficient condition under which the representation $\varphi: P_3 \longrightarrow GL_4(C)$ is irreducible.

If we require further the conditions $\lambda_1=\lambda_3$ and $\lambda_2=\lambda_4$, we get a necessary and sufficient condition for the irreducibility of $\varphi: P_3 \longrightarrow GL_4(\mathbb{C})$.

Corollary 1. *Let $\varphi: P_3 \longrightarrow GL_4(\mathbb{C})$ be the complex specialization of Tuba's representation of the pure braid group P_3. Assume that $\lambda_1=\lambda_3$, $\lambda_2=\lambda_4$ and $\lambda_1\neq-\lambda_2$.*
Then φ is irreducible if and only if $(\gamma^2+\lambda_r^2)(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k)\neq 0$, where
$\{r,s,l,k\}=\{1,2,3,4\}$.

Proof. Let us show that if $(\gamma^2+\lambda_r^2)(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k)=0$, where $\{r,s,l,k\}=\{1,2,3,4\}$, then the representation φ is reducible.

Assume that $(\gamma^2+\lambda_r^2)(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k)=0$, where $\{r,s,l,k\}=\{1,2,3,4\}$, then the reducibility on P_3 follows from reducibility on B_3 (see Proposition 6).

Now, let us show that if $(\gamma^2+\lambda_r^2)(\gamma^2+\lambda_r\lambda_l+\lambda_s\lambda_k)\neq 0$, where $\{r,s,l,k\}=\{1,2,3,4\}$, then φ is irreducible.

Given that $\lambda_1=\lambda_3$, $\lambda_2=\lambda_4$, and $\lambda_1\neq-\lambda_2$. In this case, $\gamma^2=\lambda_1\lambda_2$. Hence, it's easy to verify that all the conditions of Theorem 11 are satisfied: For instance, the second condition of Theorem 11 is equivalent to $\lambda_1\neq-\lambda_2$. Also, the third condition is equivalent

to $\lambda_1 \neq -\lambda_2$.
Therefore, by Theorem 11, φ is irreducible.

Note that, provided that $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$, we have to require $\lambda_1 \neq -\lambda_2$ in order for the matrices of the generators of B_3 not to be constant matrices.

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