The strong semilattice of π-groups

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Abstract. A semigroup is called a GV-inverse semigroup if and only if it is isomorphic to a semilattice of π-groups. In this paper, we give the sufficient and necessary conditions for a GV-inverse semigroup to be a strong semilattice of π-groups. Some conclusions about Clifford semigroups are generalized.

Key Words and Phrases: GV-inverse semigroup, π-group, strong semilattice, homomorphism

1. Introduction

First of all, we give the basic definition for this paper. Let Y be a semilattice. For each $\alpha \in Y$, let S_α be a semigroup and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, there exists a homomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ such that:

(C1) $\phi_{\alpha,\alpha} = 1_{S_\alpha}$ for any $\alpha \in Y$.

(C2) For any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

Define a multiplication on $S = \cup_{\alpha \in Y} S_\alpha$, in terms of the multiplications in the components S_α and the homomorphisms $\phi_{\alpha,\beta}$, for each x in S_α and y in S_β,

$$xy = x\phi_{\alpha,\alpha\beta}y\phi_{\beta,\alpha\beta}.$$

Then S with the multiplication defined above is a strong semilattice Y of semigroup S_α, to be denoted by $S[Y; S_\alpha, \phi_{\alpha,\beta}]$. The homomorphisms $\phi_{\alpha,\beta}$ are called the structure homomorphisms of S. And if $S_\alpha \in \mathfrak{H}$ for all $\alpha \in Y$ and some class of semigroups $\mathfrak{H}$, then S is a strong semilattice of type $\mathfrak{H}$.

As is well known, a semigroup is a Clifford semigroup if and only if it is isomorphic to a strong semilattice of groups. A semigroup S is a π-group if it is a nil-extension of a group, which means that there exists a subgroup G of S and G is an ideal, and for any $a \in S$, there exists a number $n \in N$ such that $a^n \in G$, where N is the natural number set. A semigroup is called a GV-inverse semigroup if and only if it is isomorphic to a semilattice of π-groups. It is natural to ask how about the strong semilattice of π-groups. In this paper, we give the sufficient and necessary conditions for a GV-inverse semigroup to be a strong semilattice of π-groups.

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2. Preliminaries

The class of π-regular semigroups is one of the important classes of non-regular semigroups. Recall that a semigroup S is said to be a π-regular semigroup if for any $a \in S$, there exists a positive integer m such that $a^m \in a^m S a^m$. Denote $r(a) = \min\{m \in N : a^m \in a^m S a^m\}$ and call it the least regular index of a. A π-regular semigroup is called a GV-semigroup if every regular element is completely regular. Furthermore, a GV-semigroup S is called a GV-inverse semigroup if every regular element of S possesses a unique inverse. GV-semigroups and GV-inverse semigroups are the generalizations of completely regular semigroups and Clifford semigroups in the range of π-regular semigroups respectively.

Throughout this paper, we denote the set of all regular elements of a π-regular semigroup S by $RegS$. We will write maps on the right of the objects on which they act.

Let S be a π-regular semigroup. Generalized Green's equivalences were defined by:

$$a\mathcal{L}^*b \Leftrightarrow S^1 a^{r(a)} = S^1 b^{r(b)}, \ a\mathcal{R}^*b \Leftrightarrow a^{r(a)}S^1 = b^{r(b)}S^1, \ a\mathcal{J}^*b \Leftrightarrow S^1 a^{r(a)} S^1 = S^1 b^{r(b)} S^1$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*, \ \mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.$$

If S is a regular semigroup, then $\mathcal{K} = \mathcal{K}^*$ on S for any $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}$. The class of π-regular semigroups and some of its subclasses have been studied in in [1], [2], [4], [5], [6].

For notations and terminologies not mentioned here, the reader is referred to [1] and [3].

3. Strong Semilattice of π-groups

In this section, we characterize the strong semilattice of π-groups. At first, we give some characterizations of GV-inverse semigroups.

Lemma 1. ([1]) *Let S be a semigroup and x be an element of S such that x^n lies in a subgroup G of S for some positive integer n. If e is the identity of G, then*
   (i) *$ex = xe \in G$.*
   (ii) *$x^m \in G$ for every $m \geq n$ and $m \in N$.*

Lemma 2. ([1]) *Let S be a semigroup. Then the following conditions are equivalent:*
   (i) *S is a GV-inverse semigroup.*
   (ii) *S is π-regular, and $a = axa$ implies that $ax = xa$.*
   (iii) *S is a semilattice of π-groups.*

For convenience, we always denote a GV-inverse semigroup by $S = \cup_{\alpha \in Y} S_\alpha$ in this section, where Y is a semilattice, S_α is a π-group for each $\alpha \in Y$ by Lemma 2. Further, we write $S_\alpha = G_\alpha \cup Q_\alpha$, where G_α is the group kernel of S_α, and the identity of G_α is denoted by e_α for any $\alpha \in Y$. And $Q_\alpha = S_\alpha \backslash G_\alpha$ is the set of non-regular elements of S_α and it is a partial semigroup by the definition of π-group. Certainly, if S_α is just a group, then Q_α is an empty set.

According to the results in [1], we know that $\mathcal{H}^* = \mathcal{L}^* = \mathcal{R}^* = \mathcal{D}^* = \mathcal{J}^*$ on a GV-inverse semigroup. On the other hand, we have the following results.

Lemma 3. [5] *Let S be a GV-inverse semigroup. Then for any $\mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}$, $\mathcal{K} \subseteq \mathcal{K}^*$ on S.*

Lemma 4. *Let S be a GV-inverse semigroup and $RegS$ be an ideal of S. For any $\alpha \in Y$, if $a \in G_\alpha$, then $H_a = L_a = R_a = J_a = G_\alpha$; if $a \in Q_\alpha$, then $J_a = \{a\}$.*

Proof. For any $\alpha \in Y$, let $a, b \in Q_\alpha$. Suppose that $a \mathcal{R} b$. Then there exist $s \in S_\beta$ and $t \in S_\gamma$ such that $a = bs$, $b = at$ for some $\beta, \gamma \in Y$ with $\beta, \gamma \geq \alpha$. Further,

$$a = bs = ats = a(ts)^2 = \cdots = a(ts)^m,$$
$$b = at = bst = b(st)^2 = \cdots = b(st)^m$$

where $m \geq max\{r(st), r(ts)\}$. By Lemma 1, $(ts)^m, (st)^m \in G_{\beta\gamma} \subseteq RegS$. Since $RegS$ is an ideal of S, we get that $s = t = 1$ and $a = b$, so $R_a = \{a\}$. Similarly,

$$K_a = \{a\} \text{ for any } \mathcal{K} \in \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}\}, a \in Q_\alpha. \tag{1}$$

On the other hand, by Lemma 2, it is easy to see that if $a \in G_\alpha$,

$$H_a = L_a = R_a = J_a = G_\alpha. \tag{2}$$

Let S be a GV-inverse semigroup. Define a mapping ψ from S into $RegS$ as follows: for any $a \in S$,

$$\psi : S \to RegS; a \mapsto ae_\alpha, \text{ if } a \in S_\alpha$$

where e_α is the unique idempotent of π-group S_α. Then it is obvious that $\psi|_{G_\alpha} = 1_{G_\alpha}$. For any $a, b \in S$, define the following relation: $a \widetilde{\psi} b$ if and only if $a\psi = b\psi$. Then it is clear that $\widetilde{\psi}$ is an equivalence on S and $\widetilde{\psi}|_{RegS} = \varepsilon$, where ε is the equality relation. Since $a \widetilde{\psi} ae_\alpha$ for any $a \in S_\alpha$, $\widetilde{\psi}|_S = \varepsilon$ if and only if S is a Clifford semigroup. In general, $\widetilde{\psi}|_{S \backslash RegS}$ is not necessary the equality relation. Next, we give an example.

Example 1. *Let $S = \{e, a, b\}$ with the following Cayley table*

	a	b	e
a	b	e	e
b	e	e	e
e	e	e	e

It is clear that S is a π-group and $RegS = \{e\}$ and $a\psi = ae = e = be = b\psi$, but $a \neq b$.

Lemma 5. *Let S be a GV-inverse semigroup. For any $a, b \in S_\alpha$, if $a\psi = b\psi$, then there exists $M \in N$ such that $a^m = b^m$ for any $m \in N$ with $m \geq M$. In particular, $(a\psi)^{r(a)} = a^{r(a)}$. And if $a\psi = a^{r(a)}$, then $a^{r(a)^2 - r(a)} = e_\alpha$.*

Proof. Let $a, b \in S_\alpha$. If $a\psi = b\psi$, then $ae_\alpha = be_\alpha$, and $(ae_\alpha)^n = (be_\alpha)^n$ for any $n \in N$. By Lemma 1, $a^n e_\alpha = b^n e_\alpha$. Take $M = max\{r(a), r(b)\}$, then for any $m \in N$ with $m \geq M$, $a^m, b^m \in RegS$ and $a^m = b^m$. In particular, $(a\psi)^{r(a)} = a^{r(a)} e_\alpha = a^{r(a)}$. If $a\psi = a^{r(a)}$, then $a\psi = (a\psi)^{r(a)}$ and $(a\psi)^{r(a)-1} = e_\alpha$, and hence $a^{r(a)^2 - r(a)} = e_\alpha$.

Lemma 6. *Let S be a GV-inverse semigroup and $RegS$ be an ideal of S. Then for any $\alpha, \beta \in Y$ and $a \in S_\alpha$, $e_\beta \in S_\beta$, $ae_\beta = e_\beta a$, which means that $E(S) \subseteq C(S)$, where $C(S)$ is the center of S.*

Proof. By Lemma 2, $RegS$ is a Clifford subsemigroup of S, then $ae_\beta, e_\beta a \in G_{\alpha\beta}$. And hence $ae_\beta = e_{\alpha\beta}(ae_\beta)e_{\alpha\beta} = (e_{\alpha\beta}a)e_{\alpha\beta} = e_{\alpha\beta}(e_\beta a)e_{\alpha\beta} = e_\beta a$.

Lemma 7. *Let S be a GV-inverse semigroup and $RegS$ be an ideal of S. Then ψ is a homomorphism and $\widetilde{\psi}$ is the least Clifford congruence on S.*

Proof. Let $a, b \in S$ and $a \in S_\alpha$, $b \in S_\beta$. Then $a\psi = ae_\alpha$ and $b\psi = be_\beta$. By Lemma 6, we can get that

$$a\psi b\psi = ae_\alpha be_\beta = ae_\alpha e_\beta b = ae_{\alpha\beta}b = abe_{\alpha\beta} = (ab)\psi.$$

So ψ is a homomorphism and it is easy to understand that $\widetilde{\psi}$ is a congruence on S. For any $a \in S$, it is easy to see that $a\widetilde{\psi}a\psi(\in RegS)$ and so $\widetilde{\psi}$ is a regular congruence. Further, since $a\widetilde{\psi}b$ if and only if $a = b$ for any $a, b \in RegS$, $\widetilde{\psi}$ is the least Clifford congruence on S.

Definition 1. *Let S be a partial semigroup and T be a semigroup. A mapping $f : S \to T$ is called a partial semigroup homomorphism if $(ab)f = (af)(bf)$ for any $a, b \in S$ and $ab \in S$.*

Now we give the main result of this paper.

Theorem 1. *Let $S = \cup_{\alpha \in Y} S_\alpha$ be a GV-inverse semigroup. If the following conditions are satisfied:*

(i) $RegS = \cup_{\alpha \in Y} G_\alpha$ is a Clifford subsemigroup of S, denoted by $G[Y; G_\alpha, \theta_{\alpha,\beta}]$, and it is an ideal of S, which means that S is a nil-extension of a Clifford semigroup.

(ii) For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, if $Q_\alpha \neq \varnothing$, there is a partial semigroup homomorphism $\varphi_{\alpha,\beta} : Q_\alpha \to S_\beta$ such that

(1) $\varphi_{\alpha,\alpha} = 1_{Q_\alpha}$ for any $\alpha \in Y$.

(2) For any $\alpha, \beta, \gamma \in Y$ with $\gamma \leq \alpha\beta$ and $a \in Q_\alpha, b \in Q_\beta$, if $ab \notin Q_{\alpha\beta}$, then $(a\varphi_{\alpha,\gamma})(b\varphi_{\beta,\gamma}) \notin Q_\gamma$.

(3) For any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$ and $a \in Q_\alpha$, if $a\varphi_{\alpha,\beta} \in Q_\beta$, then $a\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = a\varphi_{\alpha,\gamma}$. If $a\varphi_{\alpha,\beta} \notin Q_\beta$, then $a\varphi_{\alpha,\gamma} \notin Q_\gamma$.

(4) For any $\alpha, \beta \in Y$ and $a \in Q_\alpha$, $b \in Q_\beta$, if $ab \in Q_{\alpha\beta}$, then $ab = (a\varphi_{\alpha,\alpha\beta})(b\varphi_{\beta,\alpha\beta})$.

(iii) For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, $\varphi_{\alpha,\beta}\psi = \psi\theta_{\alpha,\beta}$, where ψ is the homomorphism in Lemma 7.

Define a mapping $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $a \in S_\alpha$,

$$a\phi_{\alpha,\beta} = \begin{cases} a\theta_{\alpha,\beta}, & a \in G_\alpha, \\ a\varphi_{\alpha,\beta}, & a \in Q_\alpha. \end{cases}$$

Then S is a strong semilattice of π-groups, denoted by $S[Y; S_\alpha, \phi_{\alpha,\beta}]$. Conversely, every strong semilattice of π-groups can be so obtained.

Proof. Let S be a GV-inverse semigroup and the given conditions are satisfied. In order to prove that S is a strong semilattice of π-groups, we firstly show that the mapping $\phi_{\alpha,\beta}$ defined is a homomorphism from S_α to S_β. For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, suppose that $a, b \in S_\alpha$.

Case 1: if $a \in Q_\alpha$, $b \in Q_\alpha$ and $ab \in Q_\alpha$, since $\varphi_{\alpha,\beta}$ is a partial semigroup homomorphism,

$$(ab)\phi_{\alpha,\beta} = (ab)\varphi_{\alpha,\beta} = (a\varphi_{\alpha,\beta})(b\varphi_{\alpha,\beta}) = (a\phi_{\alpha,\beta})(b\phi_{\alpha,\beta}).$$

Case 2: if $a \in Q_\alpha$, $b \in Q_\alpha$ and $ab \in G_\alpha$, since G_α is the group kernel of S_α,

$$
\begin{aligned}
(ab)\phi_{\alpha,\beta} &= (abe_\alpha)\phi_{\alpha,\beta} = ((ae_\alpha)(be_\alpha))\theta_{\alpha,\beta} = (ae_\alpha)\theta_{\alpha,\beta}(be_\alpha)\theta_{\alpha,\beta} \\
&= (a\psi\theta_{\alpha,\beta})(b\psi\theta_{\alpha,\beta}) = (a\varphi_{\alpha,\beta}\psi)(b\varphi_{\alpha,\beta}\psi)(\text{by Condition (iii)}) \\
&= (a\varphi_{\alpha,\beta}e_\beta)(b\varphi_{\alpha,\beta}e_\beta) = a\varphi_{\alpha,\beta}b\varphi_{\alpha,\beta}e_\beta \\
&= a\varphi_{\alpha,\beta}b\varphi_{\alpha,\beta} = a\phi_{\alpha,\beta}b\phi_{\alpha,\beta} \ (\text{by Condition (ii)(2)}).
\end{aligned}
$$

Case 3: if $a \in Q_\alpha$, $b \in G_\alpha$, then $ab \in G_\alpha$ since G_α is the group kernel of S_α,

$$
\begin{aligned}
(ab)\phi_{\alpha,\beta} &= (ab)\theta_{\alpha,\beta} = (a(e_\alpha b))\theta_{\alpha,\beta} = ((ae_\alpha)b)\theta_{\alpha,\beta} = (ae_\alpha)\theta_{\alpha,\beta}b\theta_{\alpha,\beta} \\
&= (a\psi\theta_{\alpha,\beta})b\theta_{\alpha,\beta} = (a\varphi_{\alpha,\beta}e_\beta)(b\theta_{\alpha,\beta}) \ (\text{by Condition (iii)}) \\
&= (a\varphi_{\alpha,\beta})(e_\beta b\theta_{\alpha,\beta}) = a\varphi_{\alpha,\beta}b\theta_{\alpha,\beta} = a\phi_{\alpha,\beta}b\phi_{\alpha,\beta}.
\end{aligned}
$$

Case 4: if $a \in G_\alpha$, $b \in Q_\alpha$, then $ab \in G_\alpha$, similar to the above case,

$$
\begin{aligned}
(ab)\phi_{\alpha,\beta} &= (ab)\theta_{\alpha,\beta} = ((ae_\alpha)b)\theta_{\alpha,\beta} = (a(e_\alpha b))\theta_{\alpha,\beta} \\
&= a\theta_{\alpha,\beta}(e_\alpha b)\theta_{\alpha,\beta} = a\theta_{\alpha,\beta}(be_\alpha)\theta_{\alpha,\beta} \ (\text{by Lemma 1}) \\
&= a\theta_{\alpha,\beta}b\varphi_{\alpha,\beta}e_\beta = a\theta_{\alpha,\beta}b\varphi_{\alpha,\beta} \ (\text{by Condition (iii)}) \\
&= a\phi_{\alpha,\beta}b\phi_{\alpha,\beta}.
\end{aligned}
$$

Case 5: if $a \in G_\alpha$, $b \in G_\alpha$, then $ab \in G_\alpha$, by Condition (i),

$$(ab)\phi_{\alpha,\beta} = (ab)\theta_{\alpha,\beta} = a\theta_{\alpha,\beta}b\theta_{\alpha,\beta} = a\phi_{\alpha,\beta}b\phi_{\alpha,\beta}.$$

And so the mapping $\phi_{\alpha,\beta}$ defined is a homomorphism from S_α to S_β. By Condition (i), (ii)(1) and the definition of $\phi_{\alpha,\beta}$, it is clear that $\phi_{\alpha,\alpha} = 1_{S_\alpha}$.

For any $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$, let $a \in Q_\alpha$.

If $a\varphi_{\alpha,\beta} \in Q_\beta$, by Condition (ii)(3),

$$a\phi_{\alpha,\beta}\phi_{\beta,\gamma} = a\varphi_{\alpha,\beta}\phi_{\beta,\gamma} = a\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = a\varphi_{\alpha,\gamma} = a\phi_{\alpha,\gamma}.$$

If $a \in Q_\alpha$ and $a\varphi_{\alpha,\beta} \notin Q_\beta$, then $a\varphi_{\alpha,\gamma} \notin Q_\gamma$ by Condition (ii)(3),

$$
\begin{aligned}
a\phi_{\alpha,\beta}\phi_{\beta,\gamma} &= a\varphi_{\alpha,\beta}\theta_{\beta,\gamma} = (a\varphi_{\alpha,\beta}e_\beta)\theta_{\beta,\gamma} \\
&= (a\psi\theta_{\alpha,\beta})\theta_{\beta,\gamma} = (ae_\alpha)\theta_{\alpha,\beta}\theta_{\beta,\gamma} \ (\text{by Condition (iii)}) \\
&= (ae_\alpha)\theta_{\alpha,\gamma} = (a\psi)\theta_{\alpha,\gamma} = a\varphi_{\alpha,\gamma}\psi \ (\text{by Condition (i), (iii)}) \\
&= a\varphi_{\alpha,\gamma}e_\gamma = a\varphi_{\alpha,\gamma} = a\phi_{\alpha,\gamma}.
\end{aligned}
$$

Let $a \in G_\alpha$.

$$a\phi_{\alpha,\beta}\phi_{\beta,\gamma} = a\theta_{\alpha,\beta}\phi_{\beta,\gamma} = a\theta_{\alpha,\beta}\theta_{\beta,\gamma} = a\theta_{\alpha,\gamma} = a\phi_{\alpha,\gamma}.$$

Now we consider the multiplication on S. For any $\alpha, \beta \in Y$, suppose that $a \in S_\alpha$ and $b \in S_\beta$.

Case i: if $a \in Q_\alpha$, $b \in Q_\beta$, and $ab \in Q_{\alpha\beta}$, then by Condition (ii)(4),

$$ab = a\varphi_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta} = a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta}.$$

Case ii: if $a \in Q_\alpha$, $b \in Q_\beta$, and $ab \in G_{\alpha\beta}$, then

$$
\begin{aligned}
ab &= (ab)e_{\alpha\beta} = a(be_{\alpha\beta}) = ae_{\alpha\beta}(be_{\alpha\beta}) \text{ (since RegS is an ideal of S)} \\
&= ae_\alpha e_{\alpha\beta}be_\beta e_{\alpha\beta} = (ae_\alpha)e_{\alpha\beta}(be_\beta)e_{\alpha\beta} \\
&= (ae_\alpha)\theta_{\alpha,\alpha\beta}(be_\beta)\theta_{\beta,\alpha\beta} \text{ (since RegS is a Clifford subsemigroup)} \\
&= (a\psi\theta_{\alpha,\alpha\beta})(b\psi\theta_{\beta,\alpha\beta}) = (a\varphi_{\alpha,\alpha\beta}e_{\alpha\beta})(b\varphi_{\beta,\alpha\beta}e_{\alpha\beta}) \text{ (by Condition (iii))} \\
&= a\varphi_{\alpha,\alpha\beta}(e_{\alpha\beta}b\varphi_{\beta,\alpha\beta}e_{\alpha\beta}) = a\varphi_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta}e_{\alpha\beta} \\
&= a\varphi_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta} \text{ (by Condition (ii)(2))} \\
&= a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta}.
\end{aligned}
$$

Case iii: if $a \in Q_\alpha$, $b \in G_\beta$, then $ab \in G_{\alpha\beta}$ by Condition (i), and

$$
\begin{aligned}
ab &= (ab)e_{\alpha\beta} = ae_{\alpha\beta}(be_{\alpha\beta}) = (ae_\alpha e_{\alpha\beta})be_{\alpha\beta} \\
&= (ae_\alpha)\theta_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta} = (a\psi\theta_{\alpha,\alpha\beta})b\theta_{\beta,\alpha\beta} \\
&= (a\varphi_{\alpha,\alpha\beta}\psi)b\theta_{\beta,\alpha\beta} = a\varphi_{\alpha,\alpha\beta}e_{\alpha\beta}b\theta_{\beta,\alpha\beta} \text{ (by Condition (iii))} \\
&= a\varphi_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta} = a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta}.
\end{aligned}
$$

Case iv: if $a \in G_\alpha$, $b \in Q_\beta$, $ab \in G_{\alpha\beta}$, then similar to the above case

$$
\begin{aligned}
ab &= (ab)e_{\alpha\beta} = ae_{\alpha\beta}(be_{\alpha\beta}) = ae_{\alpha\beta}be_\beta e_{\alpha\beta} \\
&= a\theta_{\alpha,\alpha\beta}(be_\beta)\theta_{\beta,\alpha\beta} = a\theta_{\alpha,\alpha\beta}(b\psi\theta_{\beta,\alpha\beta}) \\
&= a\theta_{\alpha,\alpha\beta}(b\varphi_{\beta,\alpha\beta}\psi) = a\theta_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta}e_{\alpha\beta} \text{ (by Condition (iii))} \\
&= a\theta_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta} = a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta}.
\end{aligned}
$$

Case v: if $a \in G_\alpha$, $b \in G_\beta$, then $ab \in G_{\alpha\beta}$, since $RegS$ is a Clifford subsemigroup of S,

$$ab = a\theta_{\alpha,\alpha\beta}b\theta_{\beta,\alpha\beta} = a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta}.$$

By now we have proved that GV-inverse semigroup S is a strong semilattice of π-groups if it satisfies the conditions.

Conversely, suppose that GV-inverse semigroup S is a strong semilattice of π-groups, denoted by $S[Y; S_\alpha, \phi_{\alpha,\beta}]$. For any $\alpha, \beta \in Y$ and $a \in S_\alpha$, $b \in G_\beta \subseteq RegS$, $ab = a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta}$. Because the structure homomorphisms $\phi_{\alpha,\beta}$ preserve the Green's relations on S, $b\phi_{\beta,\alpha\beta} \in G_{\alpha\beta}$ by Lemma 4. On the other hand, $G_{\alpha\beta}$ is an ideal of $S_{\alpha\beta}$, and so $ab \in G_{\alpha\beta} \subseteq RegS$. Similarly, we can prove that $ab \in RegS$ for any $a \in RegS$, $b \in S$. And so $RegS$ is an ideal of S. Since $RegS = \cup_{\alpha \in Y}G_\alpha$, it is clear that $RegS$ is a Clifford subsemigroup of S. Denote $\phi_{\alpha,\beta}|_{G_\alpha}$ by $\theta_{\alpha,\beta}$, then $RegS = G[Y; G_\alpha, \theta_{\alpha,\beta}]$.

By Lemma 2, Q_α is a partial semigroup for any $\alpha \in Y$. Denote $\phi_{\alpha,\beta}|_{Q_\alpha}$ by $\varphi_{\alpha,\beta}$, then it is easy to understand that $\varphi_{\alpha,\beta}$ is a partial semigroup homomorphism from Q_α to S_β such that Condition (ii) and (iii) are satisfied. In fact, for any $\alpha, \beta, \gamma \in Y$ with $\gamma \le \alpha\beta$ and $a \in Q_\alpha, b \in Q_\beta$, since

$$\begin{aligned}(a\varphi_{\alpha,\gamma})(b\varphi_{\beta,\gamma}) &= (a\phi_{\alpha,\gamma})(b\phi_{\beta,\gamma}) = (a\phi_{\alpha,\alpha\beta}\phi_{\alpha\beta,\gamma})(b\phi_{\beta,\alpha\beta}\phi_{\alpha\beta,\gamma}) \\ &= (a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta})\phi_{\alpha\beta,\gamma} = (ab)\phi_{\alpha\beta,\gamma}.\end{aligned}$$

If $ab \notin Q_{\alpha\beta}$, then $(a\varphi_{\alpha,\gamma})(b\varphi_{\beta,\gamma}) = (ab)\phi_{\alpha\beta,\gamma} \notin Q_\gamma$. So Condition (ii)(2) holds. And it is easy to see that Condition (ii)(1),(3),(4) hold by the definition of strong semilattice.

Further, for any $\alpha, \beta \in Y$ with $\alpha \ge \beta$ and $a \in Q_\alpha$, since ae_α is regular,

$$a\psi\theta_{\alpha,\beta} = (ae_\alpha)\theta_{\alpha,\beta} = (ae_\alpha)\phi_{\alpha,\beta} = (a\phi_{\alpha,\beta})(e_\alpha\phi_{\alpha,\beta}) = a\varphi_{\alpha,\beta}e_\beta = a\varphi_{\alpha,\beta}\psi.$$

So each strong semilattice of π-groups satisfies the conditions. The proof is completed.

At last, we consider the homomorphisms between two strong semilattices of π-groups. Let $S[Y; S_\alpha, \phi_{\alpha,\beta}]$ and $T[M; T_\alpha, \omega_{\alpha,\beta}]$ be two strong semilattices of π-groups. Denote the set of non-regular elements of S and T by Q^S and Q^T respectively. Every homomorphism f from S to T induces a homomorphism from Y to M, we denoted this semilattice homomorphism by f_L, and it is obvious that $e_\alpha f = e_{\alpha f_L}$ for any $\alpha \in Y$. On the other hand, since $f|_{S_\alpha} \in Hom(S_\alpha, T_{\alpha f_L})$, we can get a family of π-group homomorphisms denoted by $\{f_\alpha : \alpha \in Y\}$.

Theorem 2. *Let S and T be be two strong semilattices of π-groups as defined above. Given a semilattice homomorphism $f_L : Y \to M$ and a family of π-group homomorphisms $\{f_\alpha : \alpha \in Y\}$, where $f_\alpha \in Hom(S_\alpha, T_{\alpha f_L})$. Define $f : S \to T$ by $sf = sf_\alpha$ for any $\alpha \in Y$, $s \in S_\alpha$. And the follwing conditions are satisfied.*

(1) $f|_{Q^S}$ is a partial semigroup homomorphism from Q^S to T. And for any $a, b \in Q^S$, if $ab \notin Q^S$, then $(af)(bf) \notin Q^T$.

(2) $\phi_{\alpha,\beta}f_\beta\psi = f_\alpha\omega_{\alpha f_L, \beta f_L}\psi$ for any $\alpha, \beta \in Y$ with $\alpha \ge \beta$.

Then f is a homomorphism. Conversely, every homomorphism between two strong semilattices of π-groups satisfies the conditions.

Proof. It is obvious that f is a map from S to T. We only need to prove that f is a homomorphism. Let $a \in S_\alpha$ and $b \in S_\beta$.

Case 1: if $ab \in Q_{\alpha\beta}$, then $a \in Q_\alpha$, $b \in Q_\beta$, then by Condition (1), $afbf = (ab)f$.

Case 2: if $ab \notin Q_{\alpha\beta}$, then $(ab)f \notin Q^T$,

$$\begin{aligned}(ab)f &= (ab)f_{\alpha\beta} = (a\phi_{\alpha,\alpha\beta}b\phi_{\beta,\alpha\beta})f_{\alpha\beta} = (a\phi_{\alpha,\alpha\beta}f_{\alpha\beta})(b\phi_{\beta,\alpha\beta}f_{\alpha\beta})e_{(\alpha\beta)f_L} \\ &= (a\phi_{\alpha,\alpha\beta}f_{\alpha\beta}e_{(\alpha\beta)f_L})(b\phi_{\beta,\alpha\beta}f_{\alpha\beta}e_{(\alpha\beta)f_L}) = (a\phi_{\alpha,\alpha\beta}f_{\alpha\beta}\psi)(b\phi_{\beta,\alpha\beta}f_{\alpha\beta}\psi).\end{aligned}$$

On the other hand,

$$\begin{aligned}afbf &= (af_\alpha)(bf_\beta) = (af_\alpha\omega_{\alpha f_L,(\alpha\beta)f_L})(bf_\beta\omega_{\beta f_L,(\alpha\beta)f_L}) \\ &= (af_\alpha\omega_{\alpha f_L,(\alpha\beta)f_L}e_{(\alpha\beta)f_L})(bf_\beta\omega_{\beta f_L,(\alpha\beta)f_L}e_{(\alpha\beta)f_L}) \text{ (by Condition (1))} \\ &= (af_\alpha\omega_{\alpha f_L,(\alpha\beta)f_L}\psi)(bf_\beta\omega_{\beta f_L,(\alpha\beta)f_L}\psi).\end{aligned}$$

By Condition (2), we get $afbf = (ab)f$.

Conversely, let f be a homomorphism from S to T, f_L and $\{f_\alpha : \alpha \in Y\}$ be the corresponding semilattice homomorphism and the family of π-group homomorphisms as considered above this theorem. That Condition (1) holds is clear. We only need to prove the equation holds in Condition (2).

At first, we give the following fact. Let $R[Z; R_\alpha, \chi_{\alpha,\beta}]$ be a strong semilattices of π-groups. For any $\alpha, \beta \in Z$ with $\alpha \geq \beta$ and for any $a \in R_\alpha$,

$$a(\psi \chi_{\alpha,\beta}) = (ae_\alpha)\chi_{\alpha,\beta} = a\chi_{\alpha,\beta}e_\alpha\chi_{\alpha,\beta} = a\chi_{\alpha,\beta}e_\beta = a(\chi_{\alpha,\beta}\psi).$$

And so

$$\psi \chi_{\alpha,\beta} = \chi_{\alpha,\beta}\psi. \tag{3}$$

Now we return to the proof. For any $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and for any $a \in S_\alpha$,

$$(ae_\beta)f = (ae_\alpha e_\beta)f = a(\psi \phi_{\alpha,\beta} f_\beta),$$

$$(af)(e_\beta f) = (af_\alpha)(e_{\beta f_L}) = (af_\alpha)(e_{\alpha f_L})(e_{\beta f_L}) = a(f_\alpha \psi \omega_{\alpha f_L, \beta f_L}).$$

And hence,

$$\psi \phi_{\alpha,\beta} f_\beta = f_\alpha \psi \omega_{\alpha f_L, \beta f_L}. \tag{4}$$

Specially, for any $\alpha \in Y$ and $a \in S_\alpha$,

$$a(\psi f_\alpha) = (ae_\alpha)f_\alpha = af_\alpha e_{\alpha f_L} = a(f_\alpha \psi).$$

Which means that

$$\psi f_\alpha = f_\alpha \psi. \tag{5}$$

By the equations (3), (4) and (5), it is easy to understand that

$$\phi_{\alpha,\beta} f_\beta \psi = \psi \phi_{\alpha,\beta} f_\beta = f_\alpha \psi \omega_{\alpha f_L, \beta f_L} = f_\alpha \omega_{\alpha f_L, \beta f_L} \psi. \tag{6}$$

The proof is completed.

Corollary 1. *Let $G_1[Y_1; G_\alpha, \phi_{\alpha,\beta}]$ and $G_2[Y_2; G_\alpha, \omega_{\alpha,\beta}]$ be two Clifford semigroups. Given a semilattice homomorphism $f_L : Y_1 \to Y_2$ and a family of group homomorphisms $\{f_\alpha : \alpha \in Y_1\}$, where $f_\alpha \in Hom(G_\alpha, G_{\alpha f_L})$. Define $f : G_1 \to G_2$ by $sf = sf_\alpha$ for any $s \in G_\alpha$. And for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, the follwing equation is satisfied.*

$$\phi_{\alpha,\beta} f_\beta = f_\alpha \omega_{\alpha f_L, \beta f_L}.$$

Then f is a homomorphism. Conversely, every homomorphism between this two Clifford semigroups satisfies the conditions.

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