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On α-prime and weakly α-prime submodules

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Abstract. We have introduced the notion of α-prime and weakly α-prime submodules as a generalization of prime submodules. Some basic properties of α-prime and weakly α-prime submodules are the extension of prime submodules. Finally, after introducing the notion of α-prime submodules, we also define and study the concept of α-prime ideals in a ring.

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1. Introduction

All rings are assumed to be commutative with nonzero identity and all modules are left unital. Let $(G, +)$ be a group. For a subset H of G, denote $\alpha(H) = \{h \in G \mid h + h \in H\}$ and $\beta(H) = \{h + h \mid h \in H\}$. It is clear that $\beta(H) \subseteq H \subseteq \alpha(H)$. If I is an ideal of a ring R, then $\alpha(I)$ and $\beta(I)$ are ideals of R. If N is a submodule of a module M, then $\alpha(N)$ and $\beta(N)$ are submodules of M. We recall the definition of prime submodules from [1]. A proper submodule P of a left R-module M is called prime if $rm \in P$ for some $r \in R$ and $m \in M$, then $r \in (P : M)$ or $m \in P$ where $(N : M) = \{r \in R \mid rM \subseteq N\}$.

Let M be a left R-module, $m \in M$ and N be a submodule of M. For convenience, we denote $(0 : m) = \{r \in R \mid rm = 0\}$ and $(N : m) = \{r \in R \mid rm \in N\}$. With these notations, we have both of $(N : m)$ and $(0 : m)$ are ideals of R.

It is well known that there are several authors have extended the notion of prime submodules. All of those definitions focus on multiplication between element of rings and of modules. This motivates us to study α-prime submodules by taking care on all operations of a left module structure. Our extension obtains a generalization of prime submodules which call α-prime submodules. Its definition and results appear in section 1.

In section 2, we introduce α-prime submodules and also give some examples of an α-prime submodule which is not a prime submodule. Characterization of α-prime submodules of $\mathbb{Z}$-module $\mathbb{Z}$ is completely given.

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In section 3, we extend the notion of α-prime submodules to weakly α-prime submodules. We study properties the product of submodules in the Cartesian product of modules.

In section 4, we move the investigation of α-prime submodules to α-prime ideals.

2. α-prime submodules

First, we present fundamental definitions of α-prime submodules which will be studied in this paper.

Definition 1. *Let P be a proper submodule of M. We call P is* ***α-prime*** *if for any element $r \in R$ and $m \in M$ such that $r(m+m) \in P$, we have $r+r \in (P:M)$ or $m+m \in P$.*

By this definition, every prime submodule is an α-prime submodule, but the converse is not true in general.

Example 1. *Let $\mathbb{Z}$ be an $\mathbb{Z}$-module and $p \in \mathbb{Z}$. Then $p\mathbb{Z}$ is an α-prime submodule of $\mathbb{Z}$ if and only if $p = 0$ or p is a prime number or $p = 2q$ where q is a prime number.*

Proof. ($\rightarrow$) Assume that $p\mathbb{Z}$ is an α-prime submodule of $\mathbb{Z}$. Suppose that $p \neq 0$ and p is not prime number. Then $p = ab$ for some integers a and b with $1 < a, b < p$. We see that $p \mid a(b+b)$. This implies that $p \mid a+a$ or $p \mid b+b$. Now, we assume that $p \mid a+a$. This means $p \leq 2a$. Hence $ab \leq 2a$. Therefore $b \leq 2$. That is $b = 2$. Next, suppose that a is not a prime number. Then $a = cd$ for some integers c and d with $1 < c, d < a$. We have $p = 2a = 2cd = c(d+d)$. Since $p\mathbb{Z}$ is an α-prime submodule of $\mathbb{Z}$, $p \mid c+c$ or $p \mid d+d$. Hence $a \mid c$ or $a \mid d$. This implies that $a \leq c$ or $a \leq d$ which is a contradiction. This prove that $p = 2q$ for some prime numbers q.

($\leftarrow$) It is clear that $p\mathbb{Z}$ is an α-prime submodule of $\mathbb{Z}$ where $p = 0$ or p is a prime number or $p = 2q$ for some prime numbers q.

Example 1 obtains that $4\mathbb{Z}$ is α-prime but is not prime submodule of $\mathbb{Z}$. The following first result gives the characterization of α-prime submodules.

Theorem 1. *Let P be a proper submodule of an R-module M. The following statements are equivalent.*

(i) P is an α-prime submodule of M.

(ii) For all ideals I of R and for all submodules N of M,

$$\text{if } I\beta(N) \subseteq P, \text{ then } I \subseteq \alpha((P:M)) \text{ or } N \subseteq \alpha(P).$$

(iii) For all $a \in R$ and for all submodules N of M,

$$\text{if } a\beta(N) \subseteq P, \text{ then } a \in \alpha((P:M)) \text{ or } N \subseteq \alpha(P).$$

(iv) *For all ideals I of R and for all $m \in M$,*

$$\text{if } I(m+m) \subseteq P, \text{ then } I \subseteq \alpha((P:M)) \text{ or } m \in \alpha(P).$$

(v) *For all $a \in R$ and for all $m \in M$,*

$$\text{if } aR(m+m) \subseteq P, \text{ then } a \in \alpha((P:M)) \text{ or } m \in \alpha(P).$$

(vi) *For all $m \in M$, if $m+m \notin P$, then $\alpha((P:M)) = \alpha((P:m))$.*

Proof. $(i) \to (ii)$ Assume that P is an α-prime submodule of M. Let I be an ideal of R and N be a submodule of M such that $I\beta(N) \subseteq P$ and $N \nsubseteq \alpha(P)$. To show that $I \subseteq \alpha((P:M))$, let $r \in I$ and $n \in N$ be such that $n \notin \alpha(P)$. Then $n+n \notin P$ and $n+n \in \beta(N)$. This implies that $r(n+n) \in P$. Since P is an α-prime submodule of M and $n+n \notin P$, $r+r \in (P:M)$. Hence $I \subseteq \alpha((P:M))$.

$(ii) \to (iii)$ Assume that (ii) holds. Let $a \in R$ and N be a submodule of M such that $a\beta(N) \subseteq P$. Then $(Ra)\beta(N) = R(a\beta(N)) \subseteq RP \subseteq P$. By (ii), we have $Ra \subseteq \alpha((P:M))$ or $N \subseteq \alpha(P)$. Therefore $a \in \alpha((P:M))$ or $N \subseteq \alpha(P)$.

$(iii) \to (iv)$ Assume that (iii) holds. To prove that (iv) holds, let I be an ideal of R and $m \in M$ such that $I(m+m) \subseteq P$ and $m \notin \alpha(P)$. Let $a \in I$. Then $a\beta(Rm) \subseteq P$. By (iii) and $m \notin \alpha(P)$, $a \in \alpha((P:M))$. Hence $I \subseteq \alpha((P:M))$.

$(iv) \to (v)$, $(v) \to (i)$ and $(vi) \to (i)$ are obvious.

$(i) \to (vi)$ Assume that P is an α-prime submodule of M. Let $m \in M$ be such that $m+m \notin P$. It is clear that $\alpha((P:M)) \subseteq \alpha((P:m))$. Let $r \in \alpha((P:m))$. Then $r+r \in (P:m)$. Hence $r(m+m) = (r+r)m \in P$. Since P is α-prime and $m+m \notin P$, $r+r \in (P:M)$. That is $r \in \alpha((P:M))$. Therefore $\alpha((P:M)) = \alpha((P:m))$.

Lemma 1. *Let $\phi: M_1 \to M_2$ be an R-module homomorphism, P be a submodule of M_1 and K be a submodule of M_2. Then*

(i) *If ϕ is an epimorphism and $r+r \in (P:M_1)$, then $r+r \in (\phi(P):M_2)$.*

(ii) *If $r+r \in (K:M_2)$, then $r+r \in (\phi^{-1}(K):M_1)$.*

Proof. (i) Assume that ϕ is an epimorphism and $(r+r)M_1 \subseteq P$. Let $m_2 \in M_2$. Then $\phi(m_1) = m_2$ for some $m_1 \in M_1$. Thus $(r+r)m_1 \in P$. This implies that $(r+r)m_2 = (r+r)\phi(m_1) \in \phi(P)$. That is $r+r \in (\phi(P):M_2)$.

(ii) Assume that $(r+r)M_2 \subseteq K$. Let $m_1 \in M_1$. Then $\phi((r+r)m_1) = (r+r)\phi(m_1) \in K$. Hence $(r+r)m_1 \in \phi^{-1}(K)$. Therefore $r+r \in (\phi^{-1}(K):M_1)$.

Proposition 1. *Let $\phi: M_1 \to M_2$ be an R-module homomorphism. Then*

(i) *If ϕ is an epimorphism and P is an α-prime submodule of M_1 containing $\ker\phi$, then $\phi(P)$ is an α-prime submodule of M_2.*

(ii) *If K is an α-prime submodule of M_2, then $\phi^{-1}(K)$ is an α-prime submodule of M_1.*

Proof. (i) Assume that ϕ is an epimorphism and P is an α-prime submodule of M_1 containing $\ker\phi$. Let $r \in R$ and $m \in M_2$ be such that $r(m+m) \in \phi(P)$. There exist elements $n \in M_1$ and $p \in P$ such that $r(m+m) = \phi(p)$ and $\phi(n) = m$. Then $\phi(p) = r(m+m) = r(\phi(n)+\phi(n)) = r(\phi(n+n)) = \phi(r(n+n))$. This implies that $r(n+n) - p \in \ker\phi$. Since $\ker\phi \subseteq P$, $r(n+n) \in P$. Since P is an α-prime submodule of M_1, $r+r \in (P : M_1)$ or $n+n \in P$. Since ϕ is onto, $r+r \in (\phi(P) : M_2)$ or $m+m \in \phi(P)$. Hence $\phi(P)$ is an α-prime submodule of M_2.

(ii) Assume that K is an α-prime submodule of M_2. Let $r \in R$ and $m \in M$ be such that $r(m+m) \in \phi^{-1}(K)$. Then $r(\phi(m)+\phi(m)) \in K$. Since K is an α-prime submodule of M_2, $r+r \in (K : M_2)$ or $\phi(m)+\phi(m) \in K$. This implies that $r+r \in (\phi^{-1}(K) : M_1)$ or $m+m \in \phi^{-1}(K)$. Hence $\phi^{-1}(K)$ is an α-prime submodule of M_1.

Corollary 1. *Let N be a submodule of M. Then*

(i) If P is an α-prime submodule of M and K is a submodule of M contained in P, then ${}^{P}/_{K}$ is an α-prime submodule of ${}^{M}/_{K}$.

(ii) If K' is an α-prime submodule of ${}^{M}/_{N}$, then $K' = {}^{K}/_{N}$. for some α-prime submodule K of M.

Proof. (i) Assume that P is an α-prime submodule of M and K is a submodule of M contained in P. Define a homomorphism $\varphi : M \to {}^{M}/_{K}$ by $\varphi(m) = m+K$ for all $m \in M$. Then φ is an epimorphism and $\ker\varphi = K$. By Proposition 1 (i), $\varphi(P) = {}^{P}/_{K}$ is an α-prime submodule of ${}^{M}/_{K}$.

(ii) Assume that K' is an α-prime submodule of ${}^{M}/_{N}$. Then the set $K = \{x \in M \mid x+N \in K'\}$ is an α-prime submodule of M. Clearly, $K' = {}^{K}/_{N}$.

For subgroups A and B of a group $(G,+)$, we have $A \subseteq \alpha(B)$ if and only if $\beta(A) \subseteq B$.

Definition 2. *Let R be a ring and M be an R-module. A nonempty set $S \subseteq M\backslash\{0\}$ is called an α-**multiplicative system** if for all ideal I of R and for all submodules K and N of M, if $\Big(K+\beta(I)M\Big)\cap S \neq \emptyset$ and $\Big(K+\beta(N)\Big)\cap S \neq \emptyset$, then $\Big(K+I\beta(N)\Big)\cap S \neq \emptyset$.*

Proposition 2. *Let P be a submodule of an R-module M. Then P is an α-prime submodule of M if and only if $M\backslash P$ is an α-multiplicative system.*

Proof. $(\to)$ Assume that P is an α-prime submodule of M. Let I be an ideal of R and let K and N be submodules of M such that $\Big(K+I\beta(N)\Big)\cap M\backslash P = \emptyset$. Then $K+I\beta(N) \subseteq P$. It follows that $K \subseteq P$ and $I\beta(N) \subseteq P$. Since P is an α-prime submodule of M, $I \subseteq \alpha((P:M))$ or $N \subseteq \alpha(P)$. This implies that $\beta(I) \subseteq (P:M)$ or $\beta(N) \subseteq P$. Hence $K+\beta(I)M \subseteq P$ or $K+\beta(N) \subseteq P$. Hence $\Big(K+\beta(I)M\Big)\cap M\backslash P = \emptyset$ or $\Big(K+\beta(N)\Big)\cap M\backslash P = \emptyset$. This shows that $M\backslash P$ is an α-multiplicative system.

($\leftarrow$) Assume that $M\backslash P$ is an α-multiplicative system. Let I be an ideals of R and N be a submodule of M such that $I\beta(N) \subseteq P$. Hence $\Big(I\beta(N)\Big) \cap M\backslash P = \emptyset$. Since $M\backslash P$ is an α-multiplicative system, $\Big(\beta(I)M\Big) \cap M\backslash P = \emptyset$ or $\Big(\beta(N)\Big) \cap M\backslash P = \emptyset$. That is, $\beta(I)M \subseteq P$ or $\beta(N) \subseteq P$. We already show that $\beta(I) \subseteq (P : M)$ or $\beta(N) \subseteq P$. This means $I \subseteq \alpha((P : M))$ or $N \subseteq \alpha(P)$. Therefore P is an α-prime submodule of M.

Proposition 3. *Let M be an R-module and X be an α-multiplicative system. If P is a submodule of M maximal with respect to the property that $P \cap X = \emptyset$, then P is an α-prime submodule of M.*

Proof. Assume that P is a submodule of M maximal with respect to the property that $P \cap X = \emptyset$. Let I be an ideal of R and N be a submodule of M. Now, assume that $I \nsubseteq \alpha((P : M))$ and $N \nsubseteq \alpha(P)$. Hence $\beta(I)M \nsubseteq P$ and $\beta(N) \nsubseteq P$. Then $\Big(P + \beta(I)M\Big) \cap X \neq \emptyset$ and $\Big(P + \beta(N)\Big) \cap X \neq \emptyset$. Since X is an α-multiplicative system, $\Big(P + I\beta(N)\Big) \cap X \neq \emptyset$. Since $P \cap X = \emptyset$, $I\beta(N) \nsubseteq P$. This implies that P is an α-prime submodule of M.

Definition 3. *Let M be an R-module and N be a submodule of M. If there is an α-prime submodule of M containing N, then we define*

$$\sqrt[\alpha]{N} = \{x \in M \mid \text{ every } \alpha\text{-multiplicative system containing } x \text{ meets } N\}.$$

If there is no a α-prime submodule of M containing N, then we define $\sqrt[\alpha]{N} = M$.

Theorem 2. *Let M be an R-module and N be a submodule of M. Then either $\sqrt[\alpha]{N} = M$ or $\sqrt[\alpha]{N}$ is the intersection of all α-prime submodule of M containing N.*

Proof. Assume that $\sqrt[\alpha]{N} \neq M$. Let $x \in \sqrt[\alpha]{N}$ and P be an α-prime submodule of M containing N. By Proposition 2, $M\backslash P$ is an α-multiplicative system and $N \cap (M\backslash P) = \emptyset$. Hence $x \in P$. Conversely, let $x \in M$ be such that $x \notin \sqrt[\alpha]{N}$. Let S be an α-multiplicative system such that $x \in S$ and $S \cap N = \emptyset$. By Zorn's Lemma on the set of submodule J of M containing N and $S \cap J = \emptyset$, there exists a maximal submodule K of M such that $S \cap K = \emptyset$. By Proposition 3, K is a α-prime submodule of M. Hence $x \notin K$.

3. Weakly α-prime submodules

In this section we begin with the definition of weakly α-prime submodules which is a generalization of α-prime submodules. In [2], S.E. Atani and F. Farzalipour gave the notion of weakly prime submodules stated that a proper submodule P of a left R-module M is called weakly prime if $0 \neq rm \in P$ for some $r \in R$ and $m \in M$, then $r \in (P : M)$ or $m \in P$ where $(N : M) = \{r \in R \mid rM \subseteq N\}$.

Definition 4. *Let P be a proper submodule of M. We call P is* ***weakly α-prime*** *if for any elements $r \in R$ and $m \in M$ such that $r(m+m) \in P \backslash \{0\}$, we have $r+r \in (P:M)$ or $m+m \in P$.*

Every α-prime submodule is weakly α-prime submodule. But the converse need not be true. For example, $\{\bar{0}\}$ is weakly α-prime but is not α-prime submodule of $\mathbb{Z}$-module $\mathbb{Z}_8$ because $2 \cdot (\bar{2}+\bar{2}) = 2 \cdot \bar{4} = \bar{8} = \bar{0}$ and $(2+2)\mathbb{Z}_8 \nsubseteq \{\bar{0}\}$ and $\bar{2}+\bar{2} \neq \bar{0}$.

Next we give several characterizations of weakly α-prime submodules.

Theorem 3. *Let M be an R-module and P be a submodule of M. The following statements are equivalent.*

(i) P is a weakly α-prime submodule of M.

(ii) For any $m \in M$, if $m+m \notin P$, then $(P : m+m) = \alpha((P : M)) \cup \alpha((0 : m))$.

(iii) For any $m \in M$, if $m+m \notin P$, then $(P : m+m) = \alpha((P : M))$ or $(P : m+m) = \alpha((0 : m))$.

Proof. $(i) \to (ii)$ Assume that P is a weakly α-prime submodule of M. Let $m \in M$ be such that $m+m \notin P$. Let $r \in (P : m+m)$. Then $r(m+m) \in P$. If $r(m+m) = 0$, then $r \in \alpha((0 : m))$. Suppose that $r(m+m) \neq 0$. Since P is weakly α-prime and $m+m \notin P$, $r+r \in (P : M)$. That is $r \in \alpha((P : M))$. Conversely, let $r \in \alpha((P : M)) \cup \alpha((0 : m))$. Then $r+r \in (P : M)$ or $rm+rm = 0$. These implie that $r \in (P : m+m)$.

$(ii) \to (iii)$ Obvious.

$(iii) \to (i)$ Assume that (iii) holds. Let $r \in R$ and $m \in M$ be such that $r(m+m) \in P \backslash \{0\}$ and $m+m \notin P$. Then $r \in (P : m+m)$. Since $r(m+m) \neq 0$, $r \notin \alpha((0 : m))$. By (iii), $(P : m+m) = \alpha((P : M))$. Hence $r \in \alpha((P : M))$. Therefore $r+r \in (P : M)$. This proves that P is a weakly α-prime submodule of M.

Let M_1 and M_2 be R-modules. Then $M_1 \times M_2$ is an R-module under the operation $(a,b)+(c,d) = (a+c, b+d)$ and $r(a,b) = (ra, rb)$ for all $a, c \in M_1$, $b, d \in M_2$ and $r \in R$. We denote this module by $M_1 \oplus M_2$.

Proposition 4. *Let N_1 be a submodule of M_1 and N_2 be a submodule of M_2. If $N_1 \times N_2$ is a weakly α-prime submodule of $M_1 \oplus M_2$, then N_1 is a weakly α-prime submodule of M_1 and N_2 is a weakly α-prime submodule of M_2.*

Proof. It is straightforward.

Let R_1 and R_2 be commutative rings with identity, M_i be a unital R_i-module where $i = 1, 2$. Then $M_1 \times M_2$ is an $(R_1 \times R_2)$-module under the operation $(r_1, r_2)(m_1, m_1) = (r_1 m_1, r_2 m_2)$ for all $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$. We set up these notation for the next two results.

Proposition 5. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ and let N_1 be an R_1-submodule of M_1. Consider the following statements.*

(i) N_1 *is an* α*-prime submodule of* M_1.

(ii) $N_1 \times M_2$ *is an* α*-prime submodule of* $M_1 \times M_2$.

(iii) $N_1 \times M_2$ *is a weakly* α*-prime submodule of* $M_1 \times M_2$.

Then $(i) \to (ii) \to (iii)$. *Moreover, if* $\beta(M_2) \neq \{0\}$, *then* $(i), (ii)$ *and* (iii) *are equivalent.*

Proof. $(i) \to (ii)$ Assume that N_1 is an α-prime submodule of M_1. Let $(a, b) \in R_1 \times R_2$ and $(x, y) \in M_1 \times M_2$ be such that $(a, b)[(x, y) + (x, y)] \in N_1 \times M_2$. Then $[a(x+x), b(y+y)] \in N_1 \times M_2$. Thus $a(x+x) \in N_1$. Since N_1 is an α-prime submodule of M_1, $a+a \in (N_1 : M_1)$ or $x+x \in N_1$. This leads to $(a+a, b+b) \in (N_1 \times M_2 : M_1 \times M_2)$ or $(x, y)+(x, y) \in N_1 \times M_2$. Therefore $N_1 \times M_2$ is an α-prime prime submodule of $M_1 \times M_2$. $(ii) \to (iii)$ It is obvious.

Next, let $w \in M_2$ be such that $w + w \neq 0$ and assume that $N_1 \times M_2$ is a weakly α-prime submodule of $M_1 \times M_2$. Let $r \in R_1$ and $m \in M_1$ such that $r(m + m) \in N_1$. Then $(r, 1)[(m, w) + (m, w)] = (r(m+m), w+w) \in N_1 \times M_2 \backslash \{(0, 0)\}$. Since $N_1 \times M_2$ is a weakly α-prime submodule of $M_1 \times M_2$, we have $(r+r, 1+1) \in (N_1 \times M_1 : M_1 \times M_2)$ or $(m, w) + (m, w) \in N_1 \times M_2$. This implies that $r + r \in (N_1 : M_1)$ or $m + m \in N_1$. Hence N_1 is an α-prime submodule of M_1.

The following example shows that, in general, the condition $\beta(M_2) \neq \{0\}$ in Proposition 5 can not be omitted.

Example 2. *Let* $M_1 = \mathbb{Z}_8$, $M_2 = \{0\}$, $R_1 = R_2 = \mathbb{Z}$. *It is clear that* $\{\bar{0}\} \times \{0\}$ *is a weakly* α*-prime submodule of* $M_1 \times M_2$. *However,* $\{\bar{0}\}$ *is not an* α*-prime submodule of* $\mathbb{Z}$*-module* $\mathbb{Z}_8$.

Proposition 6. *Let* M_1, M_2 *be* R_1, R_2*-modules respectively and* $N_1 \times N_2$ *be a submodule of* $M_1 \times M_2$. *Then* $\beta(N_1 \times N_2) = \{(0, 0)\}$ *if and only if* $\beta(N_1) = \{0\}$ *and* $\beta(N_2) = \{0\}$.

Proof. It is evident.

Proposition 7. *Let* M_1, M_2 *be* R_1, R_2*-modules respectively. Then*

(i) If $N_1 \times N_2$ *is a weakly* α*-prime submodule of* $M_1 \times M_2$, *then either* $\beta(N_1) = \{0\}$ *or* $\alpha(N_1) = M_1$ *or* $\alpha(N_2) = M_2$.

(ii) If $N_1 \times N_2$ *is a weakly* α*-prime submodule of* $M_1 \times M_2$, *then either* $\beta(N_2) = \{0\}$ *or* $\alpha(N_1) = M_1$ *or* $\alpha(N_2) = M_2$.

(iii) If $N_1 \times N_2$ *is a weakly* α*-prime submodule of* $M_1 \times M_2$, *then* $\beta(N_1) = \{0\}$ *or* $\alpha(N_2) = M_2$ *or* $N_1 \times N_2$ *is an* α*-prime submodule of* $M_1 \times M_2$.

(iv) If $N_1 \times N_2$ *is a weakly* α*-prime submodule of* $M_1 \times M_2$, *then* $\beta(N_2) = \{0\}$ *or* $\alpha(N_1) = M_1$ *or* $N_1 \times N_2$ *is an* α*-prime submodule of* $M_1 \times M_2$.

Proof. (i) Assume that $N_1 \times N_2$ is a weakly α-prime submodule of $M_1 \times M_2$ and $\beta(N_1) \neq \{0\}$ and $\alpha(N_1) \neq M_1$. Let $a \in N_1$ be such that $a + a \neq 0$. Let $r \in (N_2 : M_2)$ and $y \in M_2$. Then $(0,0) \neq (a+a, r(y+y)) = (1,r)[(a,y)+(a,y)] \in N_1 \times N_2$. Since $N_1 \times N_2$ is a weakly α-prime submodule of $M_1 \times M_2$, we have $(1+1, r+r)\Big(M_1 \times M_2\Big) \subseteq N_1 \times N_2$ or $(a,y)+(a,y) \in N_1 \times N_2$. This implies that $(1+1)M_1 \subseteq N_1$ or $y + y \in N_2$. Since $\alpha(N_1) \neq M_1$, there is $m \in M_1$ such that $m + m \notin N_1$. This means $(1+1)M_1 \nsubseteq N_1$. Therefore $y \in \alpha(N_2)$.

(ii) The proof is similar to (i).

(iii) Assume that $N_1 \times N_2$ is a weakly α-prime submodule of $M_1 \times M_2$ and $\beta(N_1) \neq \{0\}$ and $\alpha(N_2) \neq M_2$. By (i), $\alpha(N_1) = M_1$. Let $(r_1, r_2) \in R_1 \times R_2$ and $(m_1, m_2) \in M_1 \times M_2$ be such that $(r_1, r_2)[(m_1, m_2) + (m_1, m_2)] \in N_1 \times N_2$. Then $r_1(m_1 + m_1) \in N_1$ and $r_2(m_2+m_2) \in N_2$. Let $a \in N_1$ be such that $a+a \neq 0$. Then $(0,0) \neq (a+a, r_2(m_2+m_2)) = (1, r_2)[(a, m_2) + (a, m_2)] \in N_1 \times N_2$. Since $N_1 \times N_2$ is a weakly α-prime submodule of $M_1 \times M_2$, we have $(1+1, r_2+r_2)\Big(M_1 \times M_2\Big) \subseteq N_1 \times N_2$ or $(a, m_2)+(a, m_2) \in N_1 \times N_2$. Since $N_1 \times N_2$ is a submodule of $M_1 \times M_2$ and $\alpha(N_1) = M_1$, $(r_1+r_1, r_2+r_2)\Big(M_1 \times M_2\Big) \subseteq N_1 \times N_2$ or $(m_1, m_2) + (m_1, m_2) \in N_1 \times N_2$. This implies that $N_1 \times N_2$ is an α-prime submodule of $M_1 \times M_2$.

(iv) The proof is similar to (iii).

The following example obtains that the assumption $\alpha(N_2) \neq M_2$ in the proof of Proposition 7 (iii) is necessary.

Example 3. *Consider a submodule $4\mathbb{Z} \times 3\mathbb{Z}$ of a $\mathbb{Z} \times \mathbb{Z}$-module $\mathbb{Z} \times 3\mathbb{Z}$, by Proposition 5, $4\mathbb{Z} \times 3\mathbb{Z}$ is a weakly α-prime submodule of $\mathbb{Z} \times 3\mathbb{Z}$. However, $4\mathbb{Z} \times 3\mathbb{Z}$ is not an α-prime submodule of $\mathbb{Z} \times 3\mathbb{Z}$ because $(1,3)[(2,1)+(2,1)] = (4,6) \in 4\mathbb{Z} \times 3\mathbb{Z}$ and $(2,6)\Big(\mathbb{Z} \times 3\mathbb{Z}\Big) \nsubseteq 4\mathbb{Z} \times 3\mathbb{Z}$ and $(4,2) \notin 4\mathbb{Z} \times 3\mathbb{Z}$. In particular, $\alpha(3\mathbb{Z}) = 3\mathbb{Z}$.*

4. The traveling of α-prime from modules to rings

In this section we apply the notion of (weakly) α-prime submodules to (weakly) α-prime ideals.

Definition 5. *A proper ideal P of a ring R is called an **α-prime ideal** of R if P is an α-prime submodule of an R-modules R.*

*Similarly, a proper ideal P of a ring R is called a **weakly α-prime ideal** of R if P is an weakly α-prime submodule of an R-modules R.*

It is easy to show that for an ideal P of R, P is an α-prime ideal of R if and only if for all $a, b \in R$, if $a(b+b) \in P$, then $a + a \in P$ or $b + b \in P$. Similarly, P is a weakly α-prime ideal of R if and only if for all $a, b \in R$, if $a(b+b) \in P \backslash \{0\}$, then $a + a \in P$ or $b + b \in P$.

Proposition 8. *If P is an α-prime submodule of an R-module M, then $(P : M)$ is an α-prime ideal of R.*

Proof. Assume that P is an α-prime submodule of an R-module M. Let $a, b \in R$ be such that $a(b+b) \in (P : M)$ and $b+b \notin (P : M)$. Then there exists an element $m \in M$ such that $(b+b)m \notin P$ and $a(b+b)m \in P$. Since P is α-prime and $(b+b)m \notin P$, $a+a \in (P : M)$. Therefore $(P : M)$ is an α-prime ideal of R.

Let R be a ring. The Cartesian product $R \times R$ is a ring under componentwise addition and the multiplication $(a, b) * (c, d) = (ac, ad + bc)$. We use the notation $R(+)R$ for this ring.

Proposition 9. *If I is an α-prime ideal of a ring R, then $I \times R$ is an α-prime ideal of $R(+)R$.*

Proof. It is straightforward.

Example 4. *We know that $4\mathbb{Z}$ and $6\mathbb{Z}$ are α-prime ideal of $\mathbb{Z}$. In $\mathbb{Z}(+)\mathbb{Z}$, we have $(2, 1)[(1, 1) + (1, 1)] = (2, 1)(2, 2) = (4, 6) \in 4\mathbb{Z} \times 6\mathbb{Z}$. However, $(4, 2) \notin 4\mathbb{Z} \times 6\mathbb{Z}$ and $(2, 2) \notin 4\mathbb{Z} \times 6\mathbb{Z}$. This is an example shows that $I \times J$ may be not an α-prime ideal of $R(+)R$ even if I and J are α-prime ideals of R.*

Proposition 10. *If P is a weakly α-prime submodule of M and $(P : M)\beta(P) \neq 0$, then P is an α-prime submodule of M*

Proof. Assume that P is a weakly α-prime submodule of M and $(P : M)\beta(P) \neq 0$. Let $r \in R$ and $m \in M$ be such that $r(m + m) \in P$. If $r(m + m) \neq 0$, $r + r \in (P : M)$ or $m + m \in P$. Assume that $r(m + m) = 0$. We consider the following two cases.
Case 1. $r\beta(P) \neq 0$.
Then $r(n_0 + n_0) \neq 0$ for some $n_0 \in P$. Hence $r(m + m + n_0 + n_0) = r(n_0 + n_0) \in P$. Since P is a weakly α-prime submodule of M, $r + r \in (P : M)$ or $m + m + n_0 + n_0 \in P$. Since $n_o \in P$, $r + r \in (P : M)$ or $m + m \in P$. Hence P is an α-prime submodule of M.
Case 2. $r\beta(P) = 0$.

Subcase 2.1. $(P : M)(m + m) \neq 0$.

Let $k \in (P : M)$ be such that $k(m + m) \neq 0$. Then $(r + k)(m + m) = k(m + m) \in P$. Since P is a weakly α-prime submodule of M, $r + k + r + k \in (P : M)$ or $m + m \in P$. Since $k \in (P : M)$, $r + r \in (P : M)$ or $m + m \in P$. Hence P is an α-prime submodule of M.

Subcase 2.2. $(P : M)(m + m) = 0$.
Since $(P : M)\beta(P) \neq 0$, we have $k(n + n) \neq 0$ for some $k \in (P : M)$ and $n \in P$. Then $(r+k)(m+m+n+n) = r(m+m)+r(n+n)+k(m+m)+k(n+n) = k(n+n) \in P$. Since P is a weakly α-prime submodule of M, $r + k + r + k \in (P : M)$ or $m + m + n + n \in P$. Since $k \in (P : M)$ and $n \in P$, $r + r \in (P : M)$ or $m + m \in P$. Hence P is an α-prime submodule of M.

The following result directly implies from Proposition 8 and 10.

Corollary 2. *If P is a weakly α-prime submodule of M and $(P : M)\beta(P) \neq 0$, then $(P : M)$ is a weakly α-prime ideal of R.*

We prove in Proposition 8 that if P is an α-prime submodule of an R-module M, then $(P : M)$ is an α-prime ideal of R. However, this situation is false for weakly α-prime submodules.

Example 5. *In $\mathbb{Z}/_{8\mathbb{Z}}$ as a $\mathbb{Z}$-module, we have $\{\bar{0}\}$ is a weakly α-prime submodule of $\mathbb{Z}/_{8\mathbb{Z}}$. However, $(\{\bar{0}\} : \mathbb{Z}/_{8\mathbb{Z}}) = 8\mathbb{Z}$ is not a α-prime ideal of $\mathbb{Z}$.*

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