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A note on one-dimensional varieties over the complex p-adic field

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Abstract. In this paper, we study the varieties $V \subseteq \mathbb{C}_p^4$ of dimension one that contain points of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$ by using tools from Non-Archimedean Analysis.

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1. Introduction

The algebraic (in)dependence between elements of the form $x, \exp(x)$ in the $p-$adic domain plays a fundamental role in the $p-$adic Transcendental Number Theory. Many results have been made towards this direction. For example, in 1932 K.Mahler, [N], proved that $\exp(\alpha)$ is transcendental over $\mathbb{Q}$ for any non-zero algebraic element $\alpha \in E$ (the domain of convergence of the exponential function). In 2008, Yu.V. Nesterenko proved that if $\alpha_1, ..., \alpha_n \in E$ are algebraic over $\mathbb{Q}$ and form a basis of a finite extension of degree n of $\mathbb{Q}$. Then, there exist at least $\lfloor \frac{n}{2} \rfloor$ among the elements $\exp(\alpha_1), ..., \exp(\alpha_n)$ which are $\mathbb{Q}-$algebraically independent. This result is usually called half of Lindemann-Weierstrass Conjecture in the $p-$adic domain, [N].

In this paper, we use Weierstrass Preparation Theorem to give necessary and sufficient conditions on a class of polynomials over $\mathbb{Z}$ so that each one of them has a root of the form $(x, \exp(x))$. Similarly, we use Hilbert Theorem on the ring of strictly convergent power series to give necessary and sufficient conditions on a class of polynomials over $\mathbb{Z}$ so that each one of them has a root of the form $(\exp(x_1), \exp(x_2))$. That enables us to put necessary conditions on certain varieties $V \subseteq \mathbb{C}_p^4$ of dimension one over $\mathbb{Q}$ in order to have points of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$. Also, we give a class of varieties $V \subseteq \mathbb{C}_p^4$ of dimension one over $\mathbb{Q}$ such that each variety contains a point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$. This point does not contradict Schanuel's conjecture for two elements. The conjecture asserts that for a given variety $V \subseteq \mathbb{C}_p^4$ over $\mathbb{Q}$ of dimension one and a tuple $(x_1, x_2, \exp(x_1), \exp(x_2)) \in V$, then x_1, x_2 are $\mathbb{Q}$-linearly dependent. Finally,

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we give some applications on Weierstrass Preparation Theorem and Hilbert Theorem concerning the algebraic dependence over $\mathbb{Q}_p$ and other related topics.
Many results concerning the existence of roots of $p-$adic exponential polynomials have been made by Poorten (see [P] and [PR]) and others. These results imply the existence of roots of polynomials $P[X,Y] \in \mathbb{Q}[X,Y]$ of the form $(x, \exp(x))$. In our work, we consider these polynomials directly where the coefficients and degrees of the variables play a role in the existence of such roots. We prove, as a Corollary, that there exist polynomials in $\mathbb{Q}[X,Y]$ which do not contain any root of the form $(x, \exp(x))$. This implies the existence of varieties $V \subseteq \mathbb{C}_p^4$ over $\mathbb{Q}$ of dimension one which do not contain points of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$. Using the same technic, we prove that there exist polynomials over $\mathbb{Z}$ with $2n$ variables which do not contain any root of the form $(x_1, .., x_n, \exp(x_1), .., \exp(x_n))$. Furthermore, we use Weierstrass Preparation Theorem to prove the existence of varieties $V \subseteq \mathbb{C}_p^4$ over $\mathbb{Q}_p$ of dimension one that contain points of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$.

2. Background

We recall some basic notations and results regarding the field of $p-$adic numbers and some elementary Non-Archimedian Analysis that will be needed later. For more details, see [BGR] and [G].
Let p be a prime number, $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the non-archimedian absolute value $|.|$ and $\mathbb{C}_p$ the completion of the algebraic closure of $\mathbb{Q}_p$. This field is Non-Archimedian (with respect to the extended p-adic absolute value $|.|$), complete and algebraically closed with the residue class field $\bar{\mathbb{F}}_p$ (the algebraic closure of the field $\mathbb{F}_p$) and the value group $p^{\mathbb{Q}} \cup \{0\}$. Moreover, The field $\mathbb{C}_p$ is endowed by the exponential map:

$$\exp : E \to 1 + E,$$

$$x \longmapsto \sum_{n \geq 0} \frac{x^n}{n!}$$

where $E = \{x \in \mathbb{C}_p; |x| < p^{\frac{-1}{p-1}}\}$.
It is well-known in the Non-Archimedian fields that a series $\sum_n a_n$ is convergent if and only if $\lim\limits_{n\to\infty} |a_n| = 0$. Therefore, Let $f(X) = \sum_{n=0}^{\infty} a_n X^n \in \mathbb{C}_p[[X]]$ be a power series. Then, $f(X)$ is convergent for each x in the closed ball $B(0,c)$ if and only if $\lim\limits_{n\to\infty} |a_n|c^n = 0$. Since $(|a_n|c^n)$ is convergent, it is bounded. i.e, it has a maximum. Therefore, the norm $\| \ . \ \|_c$ on $f(X)$ is defined as follows:

$$\| f(X) \|_c := \max\{|a_n|c^n\}.$$

We summaries the properties of $\| \ . \ \|_c$ as follows, [G]:
1) $\| f(X) \|_c = 0 \Leftrightarrow f(X) \equiv 0$,
2) $\| f(X) + g(X) \|_c \leq \max\{\| f(X) \|_c, \| g(X) \|_c\}$,
3) $\| \alpha \|_c = |\alpha|$, for any constant $\alpha \in \mathbb{C}_p$.

4) $|f(x)| \leq \| f(X) \|_c$, for any $x \in B(0,c)$,
where $f(X), g(X)$ are convergent power series on $B(0,c)$ and $|.|$ stands for the $p-$adic absolute value on $\mathbb{C}_p$.
Now, we are able to state Weierstrass Preparation Theorem:

Theorem 1. *(Weierstrass Preparation Theorem [G]) Let c be a positive real number of the form $p^\alpha, \alpha \in \mathbb{Q}$, and let*

$$f(X) = a_0 + a_1 X + .. + a_n X^n + .. \in \mathbb{C}_p[[X]]$$

be a power series convergent on the closed ball $B(0,c)$. Let $N \in \mathbb{N}$ be a number defined by the conditions:
$|a_N|c^N = \max_n \{|a_n|c^n\}$ *and* $|a_N|c^N > |a_n|c^n, \forall n > N$.
Then, there exist a polynomial $g(X) \in \mathbb{C}_p[X]$ of degree N, and a power series $h(X)$ convergent on the closed ball $B(0,c)$ such that
1) $f(X) = h(X)g(X)$. In addition, each root of $g(X)$, if exists, belongs to $B(0,c)$.
2) $\| h(X) - 1 \|_c < 1$. In particular, $h(X)$ has no roots in $B(0,c)$.

We need the following notions and results related to the ring of strictly convergent power series in order to study the polynomials in $\mathbb{Z}[X_1, X_2]$ that admit roots of the form $(\exp(x_1), \exp(x_2))$. See [S] and [BGR] for more details.
Let $(K, |.|)$ be a Non-Archimedian, complete and algebraically closed field. Then, a formal power series $f(X_1, .., X_n) = \sum_{I=(i_1,..,i_n)} a_I X_1^{i_1} ... X_n^{i_n} \in K[[X_1, .., X_n]]$ is convergent on a ball $B(0, \rho) := \{\bar{x} = (x_1, .., x_n) \in \mathbb{C}_p^n : \max |x_i| \leq \rho\}$ if and only if
$|a_I|\rho^{(i_1+...+i_n)} \to 0$ as $i_1 + ... + i_n \to \infty$. We define a norm $|.|_\rho$ on f as follows:

$$|f|_\rho := \max_{I=(i_1,..,i_n)} \{|a_I|\rho^{(i_1+...+i_n)}\}.$$

This norm is usually called *Gauss norm*, [S]. Let $T_n(\rho)$ be the set of all formal power series in $K[[X_1, .., X_n]]$ which are convergent on the ball $B(0, \rho)$. Then, $T_n(\rho)$ forms a complete normed $K-$algebra embeds $K[X_1, .., X_n]$ as a dense $K-$ subalgebra. In particular, for $\rho = 1$, $K\langle X_1, .., X_n \rangle$ denotes the ring of all power series which are convergent on the unit ball. Each element of this ring is usually called strictly convergent power series, [S]. Then, we have the following:

Lemma 1. *([S] Lemma 4.9, p.9) A strictly convergent power series*
$f = \sum_{I=(i_1,..,i_n)} a_I X_1^{i_1} ... X_n^{i_n} \in K\langle X_1, .., X_n \rangle$ is unit in $K\langle X_1, .., X_n \rangle$ if and only if $|a_{(0,..,0)}| = |f|$ and $|a_{(i_1,..,i_n)}| < |f|$ for all $i_1 + .. + i_n > 0$.

This lemma immediately implies that if $|a_{(0,0,..,0)}| < |f|$, then f is not unit in $K\langle X_1, .., X_n \rangle$.

Lemma 2. *(Hilbert Theorem [S], Corollary 5.10, p.14) There is a one to one correspondence between the maximal ideals of $K\langle X_1, .., X_n \rangle$ and the points in the unit ball $B(0,1) := \{\bar{x} = (x_1, .., x_n) \in \mathbb{C}_p^n : \max\{|x_i|\} \leq 1\}$. Under this correspondence, a point $\bar{x} = (x_1, .., x_n) \in B(0,1)$ determines the maximal ideal $\langle X_1 - x_1, .., X_n - x_n \rangle$.*

Throughout the paper, we use the standard notation $(\bar{x}, \exp(\bar{x}))$ for the $2n$- tuple $(x_1, ..., x_n, \exp(x_1), ..., \exp(x_n))$, [K].

3. The Main Results

It is clear that finding roots of a polynomial with rational coefficients can be reduced to the case of coefficients in $\mathbb{Z}$. Therefore, without loss of generality, we can take the polynomials over $\mathbb{Z}$. We only consider the class of polynomials $P[X,Y] \in \mathbb{Z}[X,Y]$ in which at least one of the degrees of the variable Y is relatively prime to p. Furthermore, we exclude the case of polynomials that contain the variable X in each term since they have the trivial root $(0, \exp(0))$.

Theorem 2. *The polynomial with rational integer coefficients*

$$P[X,Y] = c + \sum_{i=1}^{m} d_i Y^{\alpha_i} + e_1 X Y^{\beta_{1,2}} + \sum_{k=1}^{s} f_k X^{\gamma_{k,1}} Y^{\gamma_{k,2}}; \gamma_{k,1} \geq 2,$$

in which $(d_1\alpha_1 + .. + d_m\alpha_m + e_1, p) = 1$, *has a root of the form* $(x, \exp(x)) \in \mathbb{C}_p \times \mathbb{C}_p^*; p \geq 3$ *if and only if* $|c + d_1 + .. + d_m| \leq p^{-1}$.

Proof. (Proof of the necessary condition) If $(x, \exp(x))$ is a root of $P[X,Y]$, then x is a root of the power series $f(X) := P[X, \exp(X)]$ which is convergent on E (since at least one of the degrees of the variable Y is relatively prime to p, see [PR, Theorem 1]). Thus, $x \in E$. So,

$$c + \sum_{i=1}^{m} d_i \exp(\alpha_i x) = -\Big(e_1 x \exp(\beta_{1,2} x) + \sum_{k=1}^{s} f_k x^{\gamma_{k,1}} \exp(\gamma_{k,2} x)\Big).$$

We have $\mathbb{Z} \subseteq \mathbb{Z}_p$ and $|\exp(w)| = 1$ for every $w \in E$. Using the strong triangle inequality, it follows that

$$\begin{aligned} |c + \sum_{i=1}^{m} d_i \exp(\alpha_i x)| &\leq \max_k \{|e_1 x \exp(\beta_{1,2} x)|, |f_k x^{\gamma_{k,1}} \exp(\gamma_{k,2} x)|\} \\ &\leq \max_k \{|e_1||x||\exp(\beta_{1,2} x)|, |f_k||x^{\gamma_{k,1}}||\exp(\gamma_{k,2} x)|)\} \\ &\leq \max_k \{|x|, |x|^{\gamma_{k,1}}\} \\ &< p^{\frac{-1}{p-1}} < 1. \end{aligned}$$

We define $z_i = \alpha_i x; i = 1, 2, .., m$. Then,

$$|c + d_1 \exp(z_1) + ... + d_m \exp(z_m)| < 1. \tag{1}$$

Therefore, $|c + d_1 + ... + d_m| < 1$. This is because,

$$|c + d_1 \exp(z_1) + ... + d_m \exp(z_m)| = |c + d_1 + ... + d_m + d_1(\exp(z_1) - 1) + .. + d_m(\exp(z_m) - 1)|.$$

If $|c + d_1 + ... + d_m| = 1$, then we find that

$$|d_1(\exp(z_1) - 1) + .. + d_m(\exp(z_m) - 1)| \leq \max_{1 \leq i \leq m} \{|d_i(\exp(z_i) - 1)|\}$$

$$\begin{aligned} &\leq \max_{1\leq i\leq m}\{|(\exp(z_i)-1)|\} \\ (\text{ using the fact } |w| = |\exp(w)-1|, \forall w\in E) &\leq \max_{1\leq i\leq m}\{|(z_i|\} \\ &< p^{\frac{-1}{p-1}} \\ &< 1. \end{aligned}$$

So, by the isosceles triangle inequality, we find that

$$\begin{aligned} &|c+d_1\exp(z_1)+...+d_m\exp(z_m)| = \\ &= \max\{|c+d_1+...+d_m|, |d_1(\exp(z_1)-1)+..+d_m(\exp(z_m)-1)|\} = \\ &= |c+d_1+...+d_m| = 1. \end{aligned}$$

This contradicts (1). Therefore, $|c+d_1+...+d_m|<1$, so $|c+d_1+...+d_m|\leq p^{-1}$. This is because, $c+d_1+...+d_m\in\mathbb{Z}$, and the value group of $\mathbb{Z}$ is $p^{\mathbb{Z}}\cup\{0\}$.i.e, for each $q\in\mathbb{Z}^*$, $|q|=p^s$, for some $s\in\mathbb{Z}$.

Proof of the sufficient condition. Consider the polynomial

$$P[X,Y] = c+\sum_{i=1}^{m} d_iY^{\alpha_i} + e_1XY^{\beta_{1,2}} + \sum_{k=1}^{s} f_kX^{\gamma_{k,1}}Y^{\gamma_{k,2}} \in \mathbb{Z}[X,Y],$$

with the condition $|c+d_1+...+d_m|\leq p^{-1}$. We have to prove that $P[X,Y]$ has a root of the form $(x,\exp(x)), x\in E$. This is equivalent to prove that the power series $f(X) := P[X,\exp(X)]$ has a root $x\in E$. Suppose that the power series $f(X)$ takes the form $f(X) = a_0+a_1X+..+a_nX^n+..$

In our case, we have

$$a_0 = c+d_1+..+d_m,$$

$$a_1 = d_1\alpha_1+..+d_m\alpha_m+e_1,$$

$$a_n = \frac{d_1\alpha_1^n+..+d_m\alpha_m^n}{n!} + e_1\frac{\beta_{1,2}^{n-1}}{(n-1)!}; n<\min_{1\leq j\leq s}\{\gamma_{j,1}\},$$

$$a_n = \frac{d_1\alpha_1^n+..+d_m\alpha_m^n}{n!} + e_1\frac{\beta_{1,2}^{n-1}}{(n-1)!} + f_1\frac{\gamma_{1,2}^{n-\gamma_{1,1}}}{(n-\gamma_{1,1})!} + ..+ f_s\frac{\gamma_{s,2}^{n-\gamma_{s,1}}}{(n-\gamma_{s,1})!}; n\geq\min_{1\leq i\leq s}\{\gamma_{j,1}\}.$$

Let α be any rational number satisfying $-1<\alpha<\frac{-1}{p-1}$. In fact, we have chosen $\alpha\in\mathbb{Q}$ to guarantee that $p^\alpha\in|\mathbb{C}_p|$. Then $f(X)$ is convergent on the closed ball $B(0,p^\alpha)$. The general assumption of the theorem guarantees that $\text{ord}(a_1)=0$. Therefore, $|a_1|=1$. Also, by definition of α, we find that $p^{-1}<p^\alpha$. Thus,

$$|a_0|\leq p^{-1} < |a_1|p^{1.\alpha} \leq \max_{n\geq1}\{|a_n|p^{n\alpha}\}.$$

Therefore, the number N, defined in Weierstrass Preparation theorem, is strictly larger than zero.i.e, $N>0$. Weierstrass Preparation theorem guarantees that $f(X)$ can be written in the form $f(X)=h(X)g(X)$; $h(X)$ is a power series convergent and non-vanishing on $B(0, p^\alpha)$ and $g(X)$ is a polynomial with p-adic complex coefficients of degree $N > 0$. Since $\mathbb{C}_p$ is algebraically closes field, it follows that $g(X)$ has a root x. This root belongs to $B(0, p^\alpha)$.i.e., $x \in E$. Therefore, $f(x) = h(x).0 = 0$. Thus, $P(x, \exp(x)) = 0$.

Remark 1. *In the proof of the necessary condition, we did not use the assumption* $(d_1\alpha_1 + .. + d_m\alpha_m + e_1, p) = 1$. *This implies that any polynomial of the form*

$$P[X,Y] = c + \sum_{i=1}^{m} d_i Y^{\alpha_i} + \sum_{k=1}^{s} f_k X^{\xi_{k,1}} Y^{\xi_{k,2}}; \xi_{k,1} \geq 1,$$

with $(c + d_1 + .. + d_m, p) = 1$ *and at least one of the degrees of the variable* Y *is relatively prime to* p *does not have any root of the form* $(x, \exp(x))$.

Example 1. *We can use Remark 1 to prove that the polynomial* $P[X,Y] = X^2 + Y^2$ *has no roots of the form* $(x, \exp(x)) \in \mathbb{C}_p \times \mathbb{C}_p^*; p \geq 3$.

Example 2. *Consider the polynomial*

$$P[X,Y] = p - 1 + (p+1)Y^p + X + X^3Y^{p-1} + X^7Y^{15}.$$

Then, the domain of $f(X) = P[X, \exp(X)]$ *is* E, $(d_1\alpha_1 + .. + d_m\alpha_m + e_1, p) = (p(p+1) + 1, p) = 1$ *and* $|c + d_1 + .. + d_m| = |2p| = p^{-1}$. *According to Theorem 2, we find that* $P[X,Y]$ *has a root of the form* $(x, \exp(x)) \in \mathbb{C}_p \times \mathbb{C}_p^*; p \geq 3$. *This example shows that there exists a non trivial tuple of the form* $(x, \exp(x))$ *satisfies an algebraic dependence relation with rational integer coefficients relatively prime to* p.

Also we can use Hilbert Theorem to get a result concerning the roots of the form $(\exp(x_1), \exp(x_2))$ to the polynomials with rational integer coefficients and two variables .

Theorem 3. *The polynomial*

$$P[X_1, X_2] = a_{I_0} + a_{I_1} X_1^{i_{1,1}} X_2^{i_{1,2}} + ... + a_{I_m} X_1^{i_{m,1}} X_2^{i_{m,2}} \in \mathbb{Z}[X_1, X_2],$$

in which at least one of the elements $a_{I_1} i_{1,1} + ... + a_{I_m} i_{m,1}$, $a_{I_1} i_{1,2} + ... + a_{I_m} i_{m,2}$ *and all the degrees of* X_1 *and* X_2 *are relatively prime to* p *has a root of the form* $(\exp(x_1), \exp(x_2))$ *if and only if* $|a_{I_0} + ... + a_{I_m}| \leq p^{-1}$.

Proof. (*Proof of the necessary condition*). If P has a root of the form $(\exp(x_1), \exp(x_2))$ for some elements $x_1, x_2 \in E$, then

$$P(\exp(x_1), \exp(x_2)) = a_{I_0} + a_{I_1}(\exp(x_1))^{i_{1,1}}(\exp(x_2))^{i_{1,2}} + ... + a_{I_m}(\exp(x_1))^{i_{m,1}}(\exp(x_2))^{i_{m,2}} = 0.$$

Let $z_j := i_{j,1}x_1 + i_{j,2}x_2, \forall j = 1, 2, .., m$. Then $z_j \in E$. Using the universal property of the exponential function, we obtain

$$a_{I_0} + a_{I_1} \exp(z_1) + ... + a_{I_m} \exp(z_m) = 0.$$

Thus,

$$|a_{I_0} + a_{I_1}\exp(z_1) + ... + a_{I_m}\exp(z_m)| = 0 < 1.$$

By a similar argument to the necessary proof of Theorem 2, we find that
$|a_{I_0} + ... + a_{I_m}| \leq p^{-1}$.

Proof of the sufficient condition. Consider the polynomial

$$P[X_1, X_2] = a_{I_0} + a_{I_1}X_1^{i_{1,1}}X_2^{i_{1,2}} + ... + a_{I_m}X_1^{i_{m,1}}X_2^{i_{m,2}} \in \mathbb{Z}[X_1, X_2],$$

in which at least one of the elements $a_{I_1}i_{1,1} + ... + a_{I_m}i_{m,1}$, $a_{I_1}i_{1,2} + ... + a_{I_m}i_{m,2}$ and all the degrees of X_1 and X_2 are relatively prime to p.

Let $f \in \mathbb{C}_p[[X_1, X_2]]$ be an element defined by the relation

$$f(X_1, X_2) = P[\exp(X_1), \exp(X_2)].$$

Then, $P[X_1, X_2]$ has a root of the form $(\exp(x_1), \exp(x_2))$ if and only if (x_1, x_2) is a root of f. It is clear that f is convergent on the ball $B(0, \rho) := \{(x_1, x_2) : \max|x_i| \leq \rho, i = 1, 2\}$ for every $\rho < p^{\frac{-1}{p-1}}$ (since all the degrees of the variables X_1 and X_2 are relatively prime to p). Let α be a rational number satisfying the relation $-1 < \alpha < \frac{-1}{p-1}$. Then, $f(X_1, X_2)$ is convergent on the ball $B(0, p^\alpha)$. We define new variables: $Z_1 := p^\alpha X_1, Z_2 := p^\alpha X_2$. Also, we define a new power series $g(Z_1, Z_2)$ by the relation

$$g(Z_1, Z_2) := f(p^{-\alpha}Z_1, p^{-\alpha}Z_2).$$

It's clear that $g(Z_1, Z_2)$ is convergent on the unit ball $B(0, 1)$. Furthermore, $f(X_1, X_2)$ has a root in the ball $B(0, p^\alpha)$ if and only if $g(Z_1, Z_2)$ has a root in the unit ball. Since $g(Z_1, Z_2)$ is convergent on the unit ball, it follows that $g(Z_1, Z_2) \in \mathbb{C}_p\langle Z_1, Z_2\rangle$. Suppose that $g(Z_1, Z_2)$ takes the form $g = (g_0, g_1, ..., g_q, ...)$,where g_i is homogeneous polynomial of degree i. Then, in our case, we have

$$g_0 = g(0, 0) = a_{I_0} + ... + a_{I_m},$$

$$g_1 = (a_{I_1}i_{1,1} + ... + a_{I_m}i_{m,1})p^{-\alpha}Z_1 + (a_{I_1}i_{1,2} + ... + a_{I_m}i_{m,2})p^{-\alpha}Z_2.$$

Suppose that α takes the form $\alpha = \frac{-m}{n}$. Then, we have

$$|p^{-\alpha}|^n = |p^m| = p^{-m} \Rightarrow |p^{-\alpha}| = p^{\frac{-m}{n}} = p^\alpha.$$

We assume that $(a_{I_1}i_{1,1} + ... + a_{I_m}i_{m,1}, p) = 1$ (the other case can be done similarly). This implies that $|a_{I_1}i_{1,1} + ... + a_{I_m}i_{m,1}| = 1$.

Now, since $-1 < \alpha < \frac{-1}{p-1}$, it follows that $p^{-1} < p^\alpha$. Hence, we obtain the inequalities

$$|g_0| = |a_{I_0} + ... + a_{I_m}| \leq p^{-1} < p^\alpha = |(a_{I_1}i_{1,1} + ... + a_{I_m}i_{m,1})p^{-\alpha}| \leq$$

$$\leq \max_J\{|b_J|\} = |g|,$$

where $\{b_J\}$ are the coefficients of the power series g. Thus,

$$|g(0,0)| < |g|.$$

Using Lemma 1, it implies that g is not unit in the ring $\mathbb{C}_p\langle Z_1, Z_2\rangle$. Therefore, there exits a maximal ideal ϱ in $\mathbb{C}_p\langle Z_1, Z_2\rangle$ such that $g \in \varrho$. Using Lemma 2 and the fact that $\mathbb{C}_p$ is algebraically closed field, it follows that there exist the elements $z_1, z_2 \in B(0,1)$ such that

$$\varrho = \langle Z_1 - z_1, Z_2 - z_2\rangle.$$

Therefore, g can be written in the form $g = r_1(Z_1 - z_1) + r_2(Z_2 - z_2)$, for some $r_1, r_2 \in \mathbb{C}_p\langle Z_1, Z_2\rangle$. Thus, it is clear that

$$g(z_1, z_2) = 0.$$

Hence, g has a root in the unit ball. Therefore, f has a root in the ball $B(0, p^\alpha)$. Thus, the original polynomial $P[X_1, X_2]$ has a root of the form $(\exp(x_1), \exp(x_2))$.

Corollary 1. *Let $V \subseteq \mathbb{C}_p^4$ be a variety over $\mathbb{Q}$ of dimension one defined by a system of polynomials with rational integer coefficients of the form*

$$P_1[X_1, X_3] = c^{(1)} + \sum_{i=1}^{m} d_i^{(1)} X_3^{\alpha_i^{(1)}} + \sum_{l=1}^{r} f_k^{(1)} X_1^{\xi_{k,1}^{(1)}} X_3^{\xi_{k,2}^{(1)}}; \xi_{k,1}^{(1)} \geq 1$$
$$P_2[X_2, X_4] = c^{(2)} + \sum_{i=1}^{m} d_i^{(2)} X_4^{\alpha_i^{(2)}} + \sum_{l=1}^{r} f_k^{(2)} X_2^{\xi_{k,1}^{(2)}} X_4^{\xi_{k,2}^{(2)}}; \xi_{k,1}^{(2)} \geq 1$$
$$P_3[X_3, X_4] = a_{I_0} + a_{I_1} X_3^{i_{1,1}} X_4^{i_{1,2}} + ... + a_{I_m} X_3^{i_{m,1}} X_4^{i_{m,2}},$$

such that there exists a degree of each of the variables X_3 and X_4 in P_1 and P_2 respectively which is relatively prime to p and all the degrees of the variables X_3 and X_4 in P_3 are also relatively prime to p. If V contains a point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$, then the quantities $c^{(1)} + \sum_{i=1}^m d_i^{(1)}, c^{(2)} + \sum_{i=1}^m d_i^{(2)}$ and $a_{I_0} + ... + a_{I_m}$ are all divisible by p.

Proof. If V contains a point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$, then we have

$$P_1(x_1, \exp(x_1)) = P_2(x_2, \exp(x_2)) = P_3(\exp(x_1), \exp(x_2)) = 0.$$

Using Theorems 2 and 3, we find that the quantities $c^{(1)} + \sum_{i=1}^m d_i^{(1)}, c^{(2)} + \sum_{i=1}^m d_i^{(2)}$ and $a_{I_0} + ... + a_{I_m}$ are all divisible by p.

Remark 2. *From the previous corollary, we can deduce that if we have a variety $V \subseteq \mathbb{C}_p^4$ defined as in the previous Corollary in which one of the quantities $c^{(1)} + \sum_{i=1}^m d_i^{(1)}, c^{(2)} + \sum_{i=1}^m d_i^{(2)}$ or summation of coefficients of P_3 is relatively prime to p, then V has no point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$.*

We can also give sufficient conditions on a **class** of varieties such that each variety admits a point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$ as follows.

Corollary 2. *Let p be an odd prime and let $c, d, m \in \mathbb{Z}, m \geq 1$ with the conditions $(d+1, p) = (m, p) = 1, p|(c+d)$. Then the variety $V \subseteq \mathbb{C}_p^4$ of dimension one defined by the system of polynomials*

$$\begin{aligned} P_1[X_1, X_3] &= c + dX_3^m + mX_1 \\ P_2[X_2, X_4] &= c + dX_4 + X_2 \\ P_3[X_3, X_4] &= X_4 - X_3^m, \end{aligned}$$

has a point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$.

Proof. In fact, Theorem 2 guarantees that P_1 has a root of the form $(x, \exp(x))$. By a simple calculation, we find that $(mx, \exp(mx))$ is a root of P_2 which admits roots of the form $(x, \exp(x))$ according to Theorem 2.
It's clear that $(\exp(x), \exp(mx))$ is a root of P_3 which admits roots of the form $(\exp(x_1), \exp(x_2))$ according to Theorem 3. Hence $(x, mx, \exp(x), \exp(mx)) \in V$.

Remark 3. *Schanuel's conjecture in the case of two variables asserts that if $V \subseteq \mathbb{C}_p^4$ is a variety of dimension one over $\mathbb{Q}$ and has a point of the form $(x_1, x_2, \exp(x_1), \exp(x_2))$, then the point must take the form $(x, mx, \exp(x), \exp(mx))$, for some $m \in \mathbb{Q}$.*

4. Further Applications of Weierstrass Preparation Theorem and Hilbert Theorem

We can use Weierstrass Preparation Theorem to get a result concerning the algebraic dependence over $\mathbb{Q}_p$ as follows.

Theorem 4. *Let $P_1, P_2 \in \mathbb{Z}[X, Y]$ be polynomials defined as in the beginning of the previous section of the form*

$$\begin{aligned} P_1[X, Y] &= c^{(1)} + \sum_{i=1}^{m} d_i^{(1)} Y_i^{\alpha_i^{(1)}} + e^{(1)} X Y^{\beta_{1,2}^{(1)}} + \sum_{k=1}^{s} f_k^{(1)} X^{\gamma_{k,1}^{(1)}} Y^{\gamma_{k,2}^{(1)}}; \gamma_{k,1}^{(1)} \geq 2, \\ P_2[X, Y] &= c^{(2)} + \sum_{i=1}^{m} d_i^{(2)} Y_i^{\alpha_i^{(2)}} + e^{(2)} X Y^{\beta_{1,2}^{(2)}} + \sum_{k=1}^{s} f_k^{(2)} X^{\gamma_{k,1}^{(2)}} Y^{\gamma_{k,2}^{(2)}}; \gamma_{k,1}^{(2)} \geq 2, \end{aligned}$$

in which $(d_1^{(1)}\alpha_1^{(1)} + ... + d_m^{(1)}\alpha_m^{(1)}, p) = (d_1^{(2)}\alpha_1^{(2)} + ... + d_m^{(2)}\alpha_m^{(2)}, p) = 1$. If the quantities $c^{(1)} + \sum_{i=1}^{m} d_i^{(1)}, c^{(2)} + \sum_{i=1}^{m} d_i^{(2)}$ are divisible by p and $(x_1, \exp(x_1)), (x_2, \exp(x_2))$ are roots of P_1, P_2 receptively, then there exists a variety $V \subseteq \mathbb{C}_p^4$ over $\mathbb{Q}_p$ of dimension ≤ 1 containing the point $(x_1, x_2, \exp(x_1), \exp(x_2))$.

Proof. Since $\mathbb{Q} \subseteq \mathbb{Q}_p$ and $P_1(x_1, \exp(x_1)) = 0$, it follows that x_1 and $\exp(x_1)$ are $\mathbb{Q}_p$−algebraically dependent. The same holds true for x_2 and $\exp(x_2)$. It remains to show that x_1 and x_2 are $\mathbb{Q}_p$−algebraically dependent. For this, it suffices to show that x_1 and x_2 are algebraic over $\mathbb{Q}_p$. We briefly review the proof of Theorem 2. We have considered the power series $f[X] := P[X, \exp(X)] \in \mathbb{Q}[[X]]$ which is convergent on the closed ball $B(0, p^{\alpha}), \alpha \in (-1, \frac{-1}{p-1}) \cap \mathbb{Q}$. Weierstrass Preparation Theorem can be applied over any finite extension K of $\mathbb{Q}_p$ (For more details, see [G]). Also, the coefficients of $f(X)$ (which are rationals) can be considered as elements in any finite extension of $\mathbb{Q}_p$. Hence, we can take K to be $\mathbb{Q}_p$. Then, $f(X)$ can be factored in the form $f(X) = g(X)h(X)$, where $g(X) \in \mathbb{Q}_p[X]$ and $h(X) \in \mathbb{Q}_p[[X]]$ is non-vanishing and converging on the ball $B(0, p^{\alpha})$. The roots of $f(X)$ are exactly the roots of the polynomial g. That is, each root of $f(X)$ is algebraic over $\mathbb{Q}_p$. From this argument, we deduce that x_1 and x_2 are algebraic numbers over $\mathbb{Q}_p$. This clearly implies that x_1 and x_2 are $\mathbb{Q}_p-$algebraically dependent. Thus,

$$td_{\mathbb{Q}_p}\mathbb{Q}_p(x_1, x_2, \exp(x_1), \exp(x_2) \leq 1.$$

Hence, there exists a variety $V \subseteq \mathbb{C}_p^4$ over $\mathbb{Q}_p$ of dimension ≤ 1 containing the point $(x_1, x_2, \exp(x_1), \exp(x_2))$.

Finally, we generalize Theorem 2 to the case of polynomials $P[X_1, ..., X_n, Y_1, ..., Y_n] \in \mathbb{Q}[X_1, ..., X_n, Y_1, ..., Y_n]$.
As in the two variables case, we reduce the problem to find the roots of polynomials with rational integer coefficients and exclude the polynomials that have at least one of the variables $X_1, .., X_n$ in each term since it implies that the trivial point$(0, .., 0, \exp(0), .., \exp(0))$ is a root of these polynomials. Also, we only consider the polynomials in which all the degrees of the variables $Y_1, ..., Y_n$ are relatively prime to p. Then, we prove

Theorem 5. *The polynomial with rational integer coefficients*

$$\begin{aligned} P[X_1, ..., X_n, Y_1, ..., Y_n] &= c + \sum_{i=1}^{m} d_i Y_1^{\alpha_{i,1}} ... Y_n^{\alpha_{i,n}} + \\ & \sum_{j=1}^{n} e_j X_j Y_1^{\beta_{j,1}} .... Y_n^{\beta_{j,n}} + \sum_{k=1}^{s} f_k X_1^{\gamma_{k,1}} ... X_n^{\gamma_{k,n}} Y_1^{\gamma_{k,n+1}} ... Y_n^{\gamma_{k,2n}}; \\ & \gamma_{k,1} + ... + \gamma_{k,n} \geq 2, \end{aligned}$$

in which at least one of the elements
$(d_1\alpha_{1,1} + ... + d_m\alpha_{m,1} + e_1), ..., (d_1\alpha_{1,n} + ... + d_m\alpha_{m,n} + e_n)$ *is relatively prime to* $p; p \geq 3$, *has a root of the form* $(\bar{x}, \exp(\bar{x}))$ *if and only if*

$$|c + d_1 + ... + d_m| \leq p^{-1}.$$

Proof. Proof of the necessary condition. If $(\bar{x}, \exp(\bar{x}))$ is a root of the polynomial $P[X_1, ..., X_n, Y_1, ..., Y_n]$, then $\bar{x}$ is a root of the power series
$f(X_1, ..., X_n) := P[X_1, ..., X_n, \exp(X_1), .., \exp(X_n)]$ which is convergent on the disk
$\{\bar{x} : \max |x_i| < p^{\frac{-1}{p-1}}\}$. Thus, $\bar{x} \in E^n$. So,

$$c + \sum_{i=1}^{m} d_i \exp(\alpha_{i,1}x_1) ... \exp(\alpha_{i,n}x_n) =$$

$$= -\Big(\sum_{j=1}^{n} e_j x_j \exp(\beta_{j,1}x_1)....\exp(\beta_{j,n}x_n) + \sum_{k=1}^{s} f_k x_1^{\gamma_{k,1}}...x_n^{\gamma_{k,n}} \exp(\gamma_{k,n+1}x_1)...\exp(\gamma_{k,2n}x_n)\Big).$$

Using the fact $\mathbb{Z} \subseteq \mathbb{Z}_p, |\exp(w)| = 1, \forall w \in E$ and the strong triangle inequality, we find that

$$|c + \sum_{i=1}^{m} d_i \exp(\alpha_{i,1}x_1)...\exp(\alpha_{i,n}x_n)| < 1.$$

Let $z_i = \alpha_{i,1}x_1 + ... + \alpha_{i,n}x_n; i = 1, 2, .., m$. Using the universal property of the exponential function, we find that

$$|c + d_1 \exp(z_1) + ... + d_m \exp(z_m)| < 1.$$

By a similar fashion to the two variables case, we find that $|c + d_1 + ... + d_m| \leq p^{-1}$.
Proof of the sufficient condition. Consider the polynomial

$$\begin{aligned} P[X_1, ..., X_n, Y_1, ..., Y_n] =& c + \sum_{i=1}^{m} d_i Y_1^{\alpha_{i,1}}...Y_n^{\alpha_{i,n}} + \sum_{j=1}^{n} e_j X_j Y_1^{\beta_{j,1}}....Y_n^{\beta_{j,n}} + \\ & \sum_{k=1}^{s} f_k X_1^{\gamma_{k,1}}...X_n^{\gamma_{k,n}} Y_1^{\gamma_{k,n+1}}...Y_n^{\gamma_{k,2n}}; \gamma_{k,1} + ... + \gamma_{k,n} \geq 2, \end{aligned}$$

with the conditions:
1) At least one of the elements $(d_1\alpha_{1,1} + ... + d_m\alpha_{m,1} + e_1), ..., (d_1\alpha_{1,n} + ... + d_m\alpha_{m,n} + e_n)$ is relatively prime to p,
2) $|c + d_1 + ... + d_m| \leq p^{-1}$.
Consider the ring of the formal power series $\mathbb{C}_p[[X_1, ..., X_n]]$. Let $f \in \mathbb{C}_p[[X_1, ..., X_n]]$ be an element defined by the relation

$$f(X_1, ..., X_n) = P[X_1, ..., X_n, \exp(X_1), .., \exp(X_n)].$$

Then, $P[X_1, ..., X_n, Y_1, ..., Y_n]$ has a root of the form $(\bar{x}, \exp(\bar{x}))$ if and only if $(x_1, .., x_n)$ is a root of f. It is clear that f is convergent on the ball $B(0, \rho)$ for every $\rho < p^{\frac{-1}{p-1}}$. Applying the same argument in the proof of Theorem 3, we find that P has a root of the form $(\bar{x}, \exp(\bar{x}))$.

Remark 4. *As in the two variables case, the polynomial over* $\mathbb{Z}$

$$\begin{aligned} P[X_1, ..., X_n, Y_1, ..., Y_n] \;\; = \;\; & c + \sum_{i=1}^{m} d_i Y_1^{\alpha_{i,1}}...Y_n^{\alpha_{i,n}} \\ & + \sum_{k=1}^{s} f_k X_1^{\xi_{k,1}}...X_n^{\xi_{k,n}} Y_1^{\xi_{k,n+1}}...Y_n^{\xi_{k,2n}} \\ & ; \xi_{k,1} + ... + \xi_{k,n} \geq 1, \end{aligned}$$

with $(c + d_1 + ... + d_m, p) = 1$ *has no roots of the form* $(\bar{x}, \exp(\bar{x}))$.

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References

[1] [AD] S. Araci, U. Duran and M. Acikgoz, (ρ, q)-*Volkenborn Integration*, The Journal of Number Theory, V.171, 2017, pp.1830.

[2] [BGR] S. Bosch, U. Guntzer, and R. Remmert, *Non-Archimedian Analysis*, Springer-Verlag, Berlin, 1984.

[3] [DA] U. Duran, M. Acikgoz, *On* (ρ, q)*-Euler numbers and polynomials associated with* (ρ, q)*-Volkenborn integrals*, International Journal of Number Theory, 14 (1), 2018, 241-253. https://doi.org/10.1142/S179304211850015X.

[4] [G] F. Q. Gouvea, *p-Adic Numbers*, 2nd edition, Springer, New York, 2003.

[5] [K] J. Kirby, *Exponential algebraicity in exponential fields*, Bull. Lond. Math. Soc. 42:5(2010), 879890. MR 2011k:03070 Zbl 1203.03050.

[6] [N] Yu. V. Nesterenko, *Algebraic independence of p-adic numbers*, Izvestiya: Mathematics 72:3 565-579 (2008).

[7] [P] A.J.Van Der Poorten, *Zeros of p-adic Exponential Polynomials I*, School of Mathematics the university of NSW Kensington, NSW 2033, Australia, 1975.

[8] [PR] A.J.Van Der Poorten and Roberts Rumely, *Zeros of p-adic Exponential Polynomials II* .J.London Math.Soc.(2)36(1987)1-15.

[9] [R] A. Robert, *A Course in p-adic Analysis*, Graduate Texts in Mathematics 198, Springer-Verlag 2000.

[10] [S] H.Schoutens, *An Introduction to Rigid Analytic Geometry*, Ohio State University, June 2002, Lecture Notes, available at websupport1.citytech.cuny.edu/faculty/hschoutens/PDF/RAG Lecture Notes.pdf.