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The Minimality and Maximality of n-ideals in n-ary Semigroups

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Abstract. One knows that the concept of minimality and maximality of left ideals and right ideals play an important role in semigroups. In this paper, we extend this concept to consider in n-ary semigroups. A number of results concerning relationships between minimality and maximality of n-ideals of n-ary semigroups and n-simple (0-n-simple) n-ary semigroups as well as some characterizations of minimality and maximality of n-ideals of n-ary semigroups are given.

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1. Introduction

The generalization of classical algebraic structures to n-ary structures was first introduced by Kasner [10] in 1904. In [12], Sioson introduced regular n-ary semigroups and verified their properties. In [3], Dudek and Grozdinska investigated the nature of regular n-ary semigroups in detail; moreover, Dudek proved several results and gave many examples of n-ary groups in [4], [5] and [6]. Furthermore, Dudek also investigated the properties of ideals of some elements of n-ary ($n \geq 3$) semigroups containing an idempotent in [7]. In [15], the relation between soft regular n-ary semigroups and regular n-ary semigroups was discussed by Wang, Zhou and Zhan. Nowadays, the theory of n-ary systems has many applications, for instance, application in physics ([11] and [14]) and application in automata theory [8]. Recently, Solano, Suebsung and Chinram studied ideals of fuzzy points n-ary semigroups in [13].

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In 2000, Cao and Xu studied about the minimal and maximal left ideals in ordered semigroups and gave some characterizations of them in [2]. After that, in [1], Arslanov and Kehayopulu characterized the minimal and maximal ideals in ordered semigroups. In 2010, Iampan gave some characterization of minimality and maximality of left ideals and right ideals in ternary semigroups in [9] and this is an our motivation to do this paper. In this paper, we extend those results in [9] to n-ary semigroups. We investigate the minimality and maximality of n-ideals in n-ary semigroups and give some characterizations of minimality and maximality of n-ideals in n-ary semigroups.

2. Preliminaries

For the sake of completeness, we state some definitions in the same fashion as found in [15] and [9] which are used throughout this paper. First, we would like to introduce the definition of n-ary semigroup which was stated in [15], a nonempty set S together with an n-ary operation given by $f : S^n \to S$, where $n \geq 2$, is called an *n-ary groupoid* and is denoted by (S, f). According to the general convention used in the theory of n-ary groupoids, the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $j < i$, it is the empty symbol. If $x_{i+1} = x_{i+2} = \cdots = x_{i+t} = x$, then we write x^t instead of x_{i+1}^{i+t}. In this convention,

$$f(x_1, x_2, \ldots, x_n) = f(x_1^n)$$

and

$$f(x_1, \ldots, x_i, \underbrace{x \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n) = f(x_1^i, x^t, x_{i+t+1}^n).$$

An n-ary groupoid (S, f) is called *(i, j)-associative* if

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

hold for all $x_1, x_2, \ldots, x_{2n-1} \in S$. The operation f is *associative* if the above identity holds for every $1 \leq i \leq j \leq n$, and (S, f) is called an *n-ary semigroup.*

A nonempty subset H of an n-ary semigroup (S, f) is called an *n-ary subsemigroup* of S if $f(a_1^n) \in H$ for all $a_1, a_2, \ldots, a_n \in H$.

A nonempty subset I of S is called an *i-ideal* of S if for every $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n \in S$ with $a \in I$, then $f(x_1^{i-1}, a, x_{i+1}^n) \in I$. A nonempty subset I of S is called an *ideal* of S if I is an i-ideal for every $1 \leq i \leq n$.

For nonempty subset $A_1, A_2, \ldots, A_n$ of S, let

$$f(A_1^n) := \{f(a_1^n) \mid a_i \in A_i \text{ for all } i \in \{1, 2, \ldots, n\}\}.$$

If $A_1 = \{a_1\}$, then we write $f(\{a_1\}, A_2^n)$ as $f(a_1, A_2^n)$, and similarly in another case such as we write $f(\{a_1\}, A_2^{n-1}, \{a_n\})$ as $f(a_1, A_2^n, a_n)$ and so on.

The intersection of all n-ideals of an n-ary subsemigroup H of an n-ary semigroup S containing a nonempty subset A of H is the *n-ideal of H generated by A*. For $A = \{a\}$, we donote $I_{n,H}(a)$ to be the n-ideal of H generate by $\{a\}$. If $H = S$, then we write $I_{n,S}(a)$ as $I_n(a)$.

An element a of an n-ary semigroup S with at least two elements is called *zero element* of S if $f(x_1^{i-1}, a, x_{i+1}^n) = a$ for all $x_1, x_2, \ldots, x_{n-1}, x_{n+1}, \ldots, x_n \in S$ and denote it by 0. If an n-ary semigroup S contains a zero element, then every n-ideal of S also contains a zero element.

An n-ary semigroup S without zero is called *n-simple* if it has no proper n-ideals. An n-ary semigroup S with zero is called *0-n-simple* if it has no nonzero proper n-ideals and $f(S^n) \neq \{0\}$.

An n-ideal I of an n-ary semigroup S without zero is called a *minimal n-ideal* of S if there is no n-ideal J of S such that $J \subsetneq I$. This implies that if there is an n-ideal J of S such that $J \subseteq I$, we obtain that $J = I$. A nonzero n-ideal I of an n-ary semigroup S with zero is called a *0-minimal n-ideal* of S if there is no nonzero n-ideal J of S such that $J \subsetneq I$. Equivalently, if S has an n-ideal J such that $J \subsetneq I$, we acquire that $J = \{0\}$. A proper n-ideal I of an n-ary semigroup S is called a *maximal n-ideal* of S if for any n-ideal J of S such that $I \subsetneq J$, we have $J = S$. Equivalently, if J is a proper n-ideal of S such that $I \subseteq J$, we gain that $J = I$.

3. Main Results

Throughout this paper, S is assumed to be an n-ary semigroup. In this section, we provide some idea, elementary properties and some our fundamental results which relate to n-ideals, n-simples, and 0-n-simples.

Lemma 1. *Let A be any nonempty subset of S. Then $f(S^{n-1}, A) \cup A$ is the smallest n-ideal of S containing A.*

Proof. First, we show that $f(S^{n-1}, A) \cup A$ is an n-ideal of S. Let $x_1, x_2, \ldots, x_{n-1} \in S$ and $y \in f(S^{n-1}, A) \cup A$. We divide into two cases.

Case 1: If $y \in f(S^{n-1}, A)$, then $y = f(s_1^{n-1}, a)$ for some $s_1, s_2, \ldots, s_{n-1} \in S$ and for some $a \in A$. Then $f(x_1^{n-1}, y) = f(x_1^{n-1}, f(s_1^{n-1}, a)) = f(f(x_1^{n-1}, s_1), s_2^{n-1}, a) \in f(S^{n-1}, A) \subseteq f(S^{n-1}, A) \cup A$.

Case 2: If $y \in A$, then $f(x_1^{n-1}, y) \in f(S^{n-1}, A) \subseteq f(S^{n-1}, A) \cup A$.

From Case 1 and Case 2, we can conclude that $f(S^{n-1}, A) \cup A$ is a n-ideal of S.

Next, we show that $f(S^{n-1}, A) \cup A$ is a smallest n-ideal of S containing A. Let I be any n-ideal of S containing A. Let $y \in f(S^{n-1}, A) \cup A$. If $y \in A$, then $y \in I$ because $A \subseteq I$. If $y \in f(S^{n-1}, A)$, then $y = f(s_1^{n-1}, a)$ for some $s_1, s_2, \ldots, s_{n-1} \in S$ and for some $a \in A$. Thus $a \in I$ because $A \subseteq I$. Hence $y = f(s_1^{n-1}, a) \in I$ since I is an n-ideal of S. Therefore, we obtain $f(S^{n-1}, A) \cup A \subseteq I$. Since I is an arbitrary n-ideal of S containing A, we obtain that $f(S^{n-1}, A) \cup A$ is a smallest n-ideal of S containing A.

Corollary 1. *For any an element a of S, $I_n(a) = f(S^{n-1}, a) \cup \{a\}$.*

Proof. This follows from Lemma 1.

Lemma 2. *Let A be any nonempty subset of S. Then $f(S^{n-1}, A)$ is an n-ideal of S.*

Proof. This follows from one of the proof of Lemma 1.

Lemma 3. *If S has no zero element, then the following statements are equivalent:*

(1) *S is n-simple.*

(2) *$f(S^{n-1}, a) = S$ for all $a \in S$.*

(3) *$I_n(a) = S$ for all $a \in S$.*

Proof. First, we show (1) $\Rightarrow$ (2). Assume that S is n-simple. By Lemma 2, $f(S^{n-1}, a)$ is an n-ideal of S for all $a \in S$. Hence $f(S^{n-1}, a) = S$ for all $a \in S$ because S is n-simple. Next, we show (2) $\Rightarrow$ (3). Suppose that $f(S^{n-1}, a) = S$ for all $a \in S$. By Corollary 1, we gain $I_n(a) = f(S^{n-1}, a) \cup \{a\} = S \cup \{a\} = S$. Therefore, $I_n(a) = S$ for all $a \in S$. Finally, we show (3) $\Rightarrow$ (1). Assume the statement (3) holds. Let I be any n-ideal of S. Since I is a nonempty set, there exists $x \in I$. Then $S = I_n(x) \subseteq I \subseteq S$. This implies that $I = S$. Therefore, S is n-simple.

Example 1. Consider $\mathbb{Z}_{30}$, let $S = \{\overline{5}, \overline{25}\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{30}$. It is easy to see that S is n-simple.

Lemma 4. *If S has a zero element, then the following statements hold:*

(1) *If S is 0-n-simple, then $I_n(a) = S$ for all $a \in S \setminus \{0\}$.*

(2) *If $I_n(a) = S$ for all $a \in S \setminus \{0\}$, then either $f(S^n) = \{0\}$ or S is 0-n-simple.*

Proof. (1) Suppose that S is 0-n-simple. Since $I_n(a)$ is a nonzero n-ideal of S for all $a \in S \setminus \{0\}$, we obtain that $I_n(a) = S$ for all $a \in S \setminus \{0\}$.

(2) Assume that $I_n(a) = S$ for all $a \in S \setminus \{0\}$ and suppose that $f(S^n) \neq \{0\}$. Let I be a nonzero n-ideal of S. Then there exists $x \in I \setminus \{0\}$. Hence $S = I_n(x) \subseteq I \subseteq S$, and so $I = S$. Therefore, S is 0-n-simple.

Example 2. Consider $\mathbb{Z}_{30}$, let $S = \{\overline{0}, \overline{5}, \overline{25}\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{30}$. It is easy to see that S is 0-n-simple.

Lemma 5. *let $\{I_\gamma \mid \gamma \in \Gamma\}$ be a family of n-ideals of S. Then $\bigcup_{\gamma\in\Gamma} I_\gamma$ is an n-ideal of S and $\bigcap_{\gamma\in\gamma} I_\gamma$ is also an n-ideal of S if it's not empty.*

Proof. The proof is straightforward.

Lemma 6. *Let I be an n-ideal of S and H be an n-ary subsemigroup of S, then the following statements hold:*

(1) *If H is n-simple such that $H \cap I \neq \emptyset$, then $H \subseteq I$.*

(2) *If H is 0-n-simple such that $(H \setminus \{0\}) \cap I \neq \emptyset$, then $H \subseteq I$.*

Proof. (1) Assume that H is n-simple such that $H \cap I \neq \emptyset$. Then there exists $a \in H \cap I$. By Lemma 2, we obtain that $f(H^{n-1}, a) \cap H$ is an n-ideal of H. Since H is n-simple, we gain $f(H^{n-1}, a) \cap H = H$. This implies $H \subseteq f(H^{n-1}, a) \subseteq f(S^{n-1}, I) \subseteq I$. Therefore, $H \subseteq I$.

(2) Suppose that H is 0-n-simple such that $(H \setminus \{0\}) \cap I \neq \emptyset$. Then there exists $a \in H \setminus \{0\} \cap I$. By Lemma 4(1) and Corollary 1, we obtain $H = I_{n,H}(a) = (f(H^{n-1}, a) \cup \{a\}) \cap H \subseteq f(S^{n-1}, a) \cup \{a\} = I_n(a) \subseteq I$. Therefore, $H \subseteq I$ as desire.

Lemma 7. *Let A be a nonempty subset of an n-ideal I of S. Then $f(I^{n-1}, A)$ is an n-ideal of S.*

Proof. Let $s_1, s_2, \ldots, s_{n-1} \in S$ and let $y \in f(I^{n-1}, A)$. Then $y = f(x_1^{n-1}, a)$ for some $x_1, x_2, \ldots, x_{n-1} \in I$ and for some $a \in A$. Then $f(s_1^{n-1}, y) = f(s_1^{n-1}, f(x^{n-1}, a)) = f(f(s_1^{n-1}, x_1), x_2^{n-1}, a) \in f(I^{n-1}, A)$ because I is an n-ideal of S and $x_i \in I$ for all $i \in \{1, 2, \ldots, n-1\}$. This implies that $f(I^{n-1}, A)$ is an n-ideal of S.

4. Minimality of n-ideals

In this section, we investigate the relationship between the minimality of n-ideals and n-simple (0-n-simple) n-ary semigroups.

Theorem 1. *Let S be an n-ary semigroup without zero and I be an n-ideal of S. Then I is a minimal n-ideal of S if and only if I is n-simple.*

Proof. (1) Assume that I is a minimal n-ideal of S. Let J be any n-ideal of I. Thus $f(I^{n-1}, J) \subseteq J \subseteq I$. By Lemma 7, $f(I^{n-1}, J)$ is an n-ideal of S. Since I is a minimal n-ideal, $I \subseteq f(I^{n-1}, J)$ and then $f(I^{n-1}, J) = I$. Therefore, I is n-simple. Conversely, suppose that I is n-simple. Let J be an n-ideal of S such that $J \subseteq I$. So $I \cap I \neq \emptyset$, and hence $I \subseteq J$ by Lemma 6(1). This implies that $J = I$. Therefore, I is a minimal n-ideal of S.

Example 3. Consider $\mathbb{Z}_{30}$, let $S = \{\overline{1}, \overline{5}, \overline{25}\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{30}$. It is easy to see that $I = \{\overline{5}, \overline{25}\}$ is a minimal n-ideal of S.

Theorem 2. *If S has a zero element and I is a nonzero n-ideal of S, then the following statement hold:*

(1) *If I is a 0-minimal n-ideal of S, then either $f(I^{n-1}, J) = \{0\}$ for some nonzero n-ideal J of I or I is 0-n-simple.*

(2) *If I is 0-n-simple, then I is a 0-minimal n-ideal of S.*

Proof. (1) Assume that I is a 0-minimal n-ideal of S and assume that $f(I^{n-1}, J) \neq \{0\}$ for any nonzero n-ideal J of I. Let J be a nonzero n-ideal of I. Then $\{0\} \neq f(I^{n-1}, J) \subseteq J \subseteq I$. Moreover, we obtain that $f(I^{n-1}, J)$ is an n-ideal of S by Lemma 7. Since I is a 0-minimal n-ideal of S, $I \subseteq f(I^{n-1}, J)$. This implies that $f(I^{n-1}, J) = J = I$. Therefore, I is an 0-n-simple.

(2) Assume that I is 0-n-simple. Let J be a nonzero n-ideal of S such that $J \subseteq I$. This implies that $I \setminus \{0\} \cap J \neq \emptyset$ and so $I \subseteq J$ by Lemma 6(2). Hence $J = I$. Therefore, I is a 0-minimal n-ideal of S.

Example 4. Consider $\mathbb{Z}_{30}$, let $S = \{\overline{0}, \overline{1}, \overline{5}, \overline{25}\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the multiplication of $\mathbb{Z}_{30}$. It is easy to see that $I = \{\overline{0}, \overline{5}, \overline{25}\}$ is a 0-minimal n-ideal of S.

Theorem 3. *If S has no zero element but it has proper n-ideals, then every proper n-ideal of S is minimal if and only if S contains exactly one proper n-ideal or S contains exactly two proper n-ideals I_1 and I_2 such that $I_1 \cup I_2 = S$ and $I_1 \cap I_2 = \emptyset$.*

Proof. Assume that every proper n-ideal of S is minimal. Let I be a proper n-ideal of S. Then I is a minimal n-ideal of S. We divide into two cases:

Case 1: Suppose that $S = I_n(a)$ for all $a \in S \setminus I$. Let J be a proper n-ideal of S and $J \neq I$, then $J \setminus I \neq \emptyset$ because I is a minimal n-ideals of S. Thus there exists $a \in J \setminus I \subseteq S \setminus I$. Hence $S = I_n(a) \subseteq J \subseteq S$. So $J = S$, which is a contradiction. This implies that $J = I$. In this case, we can conclude that S contains exactly one proper n-ideal of S.

Case 2: Suppose that there exists $a \in S \setminus I$ such that $S \neq I_n(a)$. This implies that $I_n(a) \neq I$ and $I_n(a)$ is a minimal n-ideal of S by the fact that $I_n(a)$ is a proper n-ideal of S. By Lemma 5, we gain that $I_n(a) \cup I$ is an n-ideal of S. Since I is a minimal n-ideal of S and $I \subsetneq I_n(a) \cup I$, we acquire that $I_n(a) \cup I = S$ otherwise $I_n(a) \cup I$ must be a minimal n-ideal of S, it is impossible. By the minimality of an n-ideal $I_n(a)$ and by the fact that $I_n(a) \cap I \subsetneq I_n(a)$, we have that $I_n(a) \cap I = \emptyset$. Next, we show that S has exactly two proper n-ideals I and $I_n(a)$. Suppose that M is a proper n-ideal of S. Then M is a minimal n-ideal of S by the hypothesis. Thus $M = M \cap S = M \cap (I_n(a) \cup I) = (M \cap I_n(a)) \cup (M \cap I)$. If $M \cap I \neq \emptyset$, then $M = I$ because M and I are both minimal n-ideals of S. If $M \cap I = \emptyset$, then $M = M \cap I_n(a)$. Hence $M \cap I_n(a) \neq \emptyset$. This implies that $M = I_n(a)$ because M and $I_n(a)$ are both minimal n-ideals of S. In this case, we can conclude that S contains exactly two proper n-ideals I and $I_n(a)$, moreover, we obtain $I_n(a) \cup I = S$ and $I_n(a) \cap I = \emptyset$.

Conversely, if S has exactly one proper n-ideal, then it is clearly that it is just a minimal n-ideal. Next, suppose that S has exactly two proper n-ideals I_1 and I_2 such that $I_1 \cup I_2 = S$ and $I_1 \cap I_2 = \emptyset$. Since $I_1 \cap I_2 = \emptyset$, we have that $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. Hence I_1 and I_2 are both minimal n-ideals of S. So we can conclude that every proper n-ideal of S is minimal.

Therefore, the proof is completed.

Theorem 4. *If S has a zero element and nonzero proper n-ideals, then every nonzero proper n-ideal of S is 0-minimal if and only if S contains exactly one nonzero proper n-ideal or S contains exactly two nonzero proper n-ideals I_1 and I_2 such that $I_1 \cup I_2 = S$ and $I_1 \cap I_2 = \{0\}$.*

Proof. It follows from the proof of Theorem 3 and use the fact that every n-ideal of S contains a zero element.

5. Maximality of n-ideals

As a result of this section, we give some characterization of the minimality of n-ideals of n-ary semigroups as well as the relationship between maximality of n-ideals and the union $\mathcal{U}$ of all (nonzero) proper n-ideals of n-ary semigroups are characterized.

Theorem 5. *If S has no zero element but it has proper n-ideals, then every proper n-ideal of S is maximal if and only if S contains exactly one proper n-ideal or S contains exactly two proper n-ideals I_1 and I_2 such that $I_1 \cup I_2 = S$ and $I_1 \cap I_2 = \emptyset$.*

Proof. Assume that every proper n-ideal of S is maximal. Let I be a proper n-ideal of S. Then I is maximal n-ideal of S. We divide into two cases:

Case 1: Suppose that $S = I_n(a)$ for all $a \in S \setminus I$. Let J be also a proper n-ideal of S and $J \neq I$. Then J is a maximal n-ideal of S, and so $J \setminus I \neq \emptyset$. Then there exists $a \in J \setminus I \subseteq S \setminus I$. Hence $S = I_n(a) \subseteq J \subseteq S$, and so $J = S$, which is a contradiction. This implies that $J = I$. In this case, we can conclude that I is the unique n-ideal of S

Case 2: Suppose that there exists $a \in S \setminus I$ such that $S \neq I_n(a)$. This implies that $I_n(a) \neq I$ and $I_n(a)$ is a maximal n-ideal of S by the fact that $I_n(a)$ is a proper n-ideal of S. By Lemma 5, we have that $I_n(a) \cup I$ is an n-ideal of S. Since I is a maximal n-ideal of S and $I \subsetneq I_n(a) \cup I$, we obtain that $I_n(a) \cup I = S$. By the maximality of an n-ideal $I_n(a)$ and by the fact that $I_n(a) \cap I \subsetneq I_n(a)$, we gain that $I_n(a) \cap I = \emptyset$. Next, we show that S has exactly two proper n-ideal I and $I_n(a)$. Suppose that M is a proper n-ideal of S. Then M is a maximal n-ideal of S by the hypothesis. Hence $M = M \cap S = M \cap (I_n(a) \cup I) = (M \cap I_n(a)) \cup (M \cap I)$. If $M \cap I \neq \emptyset$, then $M = I$ because M and I are both maximal n-ideals of S. If $M \cap I = \emptyset$, then $M = M \cap I_n(a)$. Then $M \cap I_n(a) \neq \emptyset$. This implies that $M = I_n(a)$ because M and $I_n(a)$ are both maximal n-ideals of S. In this case, we can conclude that S contains exactly two proper n-ideal I and $I_n(a)$, moreover, we obtain $I_n(a) \cup I = S$ and $I_n(a) \cap I = \emptyset$.

Conversely, if S contains exactly one proper n-ideal, then it is clearly that it is just a maximal n-ideal. Next, assume that S contains exactly two proper n-ideals I_1 and I_2 such that $I_1 \cup I_2 = S$ and $I_1 \cap I_2 = \emptyset$. Since $I_1 \cap I_2 = \emptyset$, we obtain that $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. Hence I_1 and I_2 are both maximal n-ideals of S.

Therefore, the proof is completed.

Theorem 6. *If S has a zero element and nonzero proper n-ideals, then every nonzero proper n-ideal of S is maximal if and only if S contains exactly one nonzero proper n-ideal or S contains exactly two nonzero proper n-ideals I_1 and I_2 such that $I_1 \cup I_2 = S$ and $I_1 \cap I_2 = \{0\}$.*

Proof. The proof of this theorem follows from the proof of Theorem 5 and the fact that every n-ideal of S contains a zero element.

Theorem 7. *Let I be a proper n-ideal of S. Then I is maximal n-ideal if and only if*

(1) *$S \smallsetminus I = \{a\}$ and $f(a, S^{n-2}, a) \subseteq I$ for some $a \in S$ or*

(2) *$S \smallsetminus I \subseteq f(S^{n-1}, a)$ for all $a \in S \smallsetminus I$.*

Proof. Assume that I is maximal n-ideal of S. We consider the following two cases:

Case 1: Suppose that there exists $a \in S \smallsetminus I$ such that $f(S^{n-1}, a) \subseteq I$. Then $f(a, S^{n-2}, a) \subseteq f(S^{n-1}, a) \subseteq I$. By Corollary 1, we obtain $I \cup \{a\} = (I \cup f(S^{n-1}, a)) \cup \{a\} = I \cup (f(S^{n-1}, a) \cup \{a\}) = I \cup I_n(a)$. This implies that $I \cup \{a\}$ is an n-ideal of S because $I \cup I_n(a)$ is an n-ideal of S. Since I is a maximal n-ideal of S and $I \subsetneq I \cup \{a\}$, we obtain that $I \cup \{a\} = S$. This implies that $S \smallsetminus I = \{a\}$. Hence we have that $S \smallsetminus I = \{a\}$ and $f(a, S^{n-2}, a) \subseteq I$ for some $a \in S$ as desire. In this case, the statement (1) is satisfied.

Case 2: Suppose that $f(S^{n-1}, a) \not\subseteq I$ for all $a \in S \smallsetminus I$. Let $a \in S \smallsetminus I$. Then $f(S^{n-1}, a) \not\subseteq I$. Moreover, we obtain that $f(S^{n-1}, a)$ is an n-ideal of S by Lemma 2. By Lemma 5, we gain that $I \cup f(S^{n-1}, a)$ is an n-ideal of S. Since I is a maximal n-ideal of S and $I \subsetneq I \cup f(S^{n-1}, a)$, we acquire that $I \cup f(S^{n-1}, a) = S$. Hence $a \in f(S^{n-1}, a)$ because $a \in S \smallsetminus I$. This implies that $S \smallsetminus I \subseteq f(S^{n-1}, a)$ for all $a \in S \smallsetminus I$. Hence, this case satisfies the statement (2).

Conversely, suppose that J is an n-ideal of S such that $I \subsetneq J$. Then $J \smallsetminus I \neq \emptyset$. If there exists $a \in S$ such that $S \smallsetminus I = \{a\}$ and $f(a, S^{n-2}, a) \subseteq I$, then $J \smallsetminus I \subseteq S \smallsetminus I = \{a\}$, and hence $J \smallsetminus I = \{a\}$. This implies that $J = I \cup \{a\} = S$. Hence we obtain that I is a maximal n-ideal of S. Next, if $S \smallsetminus I \subseteq f(S^{n-1}, a)$ for all $a \in S \smallsetminus I$, then $S \smallsetminus I \subseteq f(S^{n-1}, x) \subseteq f(S^{n-1}, J) \subseteq J$ for all $x \in J \smallsetminus I$. Hence $S = (S \smallsetminus I) \cup I \subseteq J \cup J = J \subseteq S$, and so $J = S$. Therefore, I is a maximal n-ideal of S.

Hence the proof of this theorem is completed.

Example 5. (1) Let $S = \mathbb{N}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 + x_2 + \ldots + x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $+$ is the usual addition of $\mathbb{N}$. Let $I = \mathbb{N} \setminus \{1\}$. Thus $S \setminus I = \{1\}$ and $f(1, S^{n-2}, 1) \subseteq I$. By Theorem 7(1), I is a maximal n-ideal of S.

(2) Let $S = \{0, -1, 1\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Let $I = \{0\}$. Then $S \setminus I \subseteq f(S^{n-1}, 1)$ and $S \setminus I \subseteq f(S^{n-1}, -1)$. By Theorem 7(2), I is a maximal n-ideal of S.

For an n-ary semigroup S, the notation $\mathcal{U}$ is assumed to be the union of all nonzero proper n-ideals of S if S has a zero element and the notation $\mathcal{U}$ is assumed to be the the union of all proper n-ideals of S if S has no a zero element, from now on.

Lemma 8. *$\mathcal{U} = S$ if and only if $I_n(a) \neq S$ for all $a \in S$.*

Proof. Assume that $\mathcal{U} = S$. If $I_n(a) = S$ for some $a \in S$. Then $a \notin I_\gamma$ for all proper n-ideal I_γ of S. Hence $a \notin \mathcal{U} = S$, which is a contradiction. Therefore, $I_n(a) \neq S$ for all $a \in S$. Conversely, suppose that $I_n(a) \neq S$ for all $a \in S$. This implies that $I_n(a)$ is a proper ideal for all $a \in S$, and so $S \subseteq \bigcup_{a \in S} I_n(a) \subseteq \mathcal{U} \subseteq S$. Therefore, we obtain that $\mathcal{U} = S$.

Theorem 8. *If S has no zero element, then the exactly one of the following statements is satisfied:*

(1) *S is n-simple.*

(2) *$I_n(a) \neq S$ for all $a \in S$.*

(3) *There exists $a \in S$ such that $I_n(a) = S, a \notin f(S^{n-1}, a), f(a, S^{n-2}, a) \subseteq \mathcal{U} = S \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal n-ideal of S.*

(4) *$S \setminus \mathcal{U} = \{a \in S \mid f(S^{n-1}, a) = S\}$ and $\mathcal{U}$ is the unique maximal n-ideal of S.*

Proof. Assume that S is not n-simple. This implies that there exists a proper n-ideal I of S. Hence $\mathcal{U}$ is an n-ideal of S. We divide into two cases:

Case 1: If $\mathcal{U} = S$, then $I_n(a) \neq S$ for all $a \in S$ by Lemma 8. In this case, the statement (2) is satisfied.

Case 2: If $\mathcal{U} \neq S$, then $\mathcal{U}$ is a maximal n-ideal of S. We would like to show that $\mathcal{U}$ is the unique maximal n-ideal of S. Suppose that I is a maximal n-ideal of S, and so I is a proper n-ideal of S. Hence $I \subseteq \mathcal{U} \subsetneq S$. Since I is a maximal n-ideal of S, we obtain $I = \mathcal{U}$. Therefore, $\mathcal{U}$ is the unique maximal n-ideal of S as desire. Furthermore, by Theorem 7, we acquire

(1) $S \setminus \mathcal{U} = \{a\}$ and $f(a, S^{n-2}, a) \subseteq \mathcal{U}$ for some $a \in S$ or

(2) $S \setminus \mathcal{U} \subseteq f(S^{n-1}, a)$ for all $a \in S \setminus \mathcal{U}$.

First, we assume that $S \setminus \mathcal{U} = \{a\}$ and $f(a, S^{n-2}, a) \subseteq \mathcal{U}$ for some $a \in S$. Since $S \setminus \mathcal{U} = \{a\}$, we have $f(a, S^{n-2}, a) \subseteq \mathcal{U} = S \setminus \{a\}$. Since $a \notin \mathcal{U}$, we have $I_n(a) = S$. If $a \in f(S^{n-1}, a)$, then $\{a\} \subseteq f(S^{n-1}, a)$, and hence $S = I_n(a) = f(S^{n-1}, a) \cup \{a\} = f(S^{n-1}, a)$ by Corollary 1. This implies that $a = f(s_1^{n-1}, a)$ and $s_1 = f(s_n^{2n-2}, a)$ for some $s_1, s_2, \ldots, s_{2n-2} \in S$. Hence $a = f(s_1^{n-1}, a) = f(s_1, s_2^{n-1}, a) = f(f(s_n^{2n-2}, a), s_2^{n-1}, a) = f(s_n^{2n-2}, f(a, s_2^{n-1}, a))$. Since $f(a, S^{n-2}, a) \subseteq \mathcal{U}$ and $\mathcal{U}$ is an n-ideal of S, we have that $a = f(s_n^{2n-2}, f(a, s_2^{n-1}, a)) \in \mathcal{U}$, which is a contradiction. Hence $a \notin f(S^{n-1}, a)$. In this case, the statement (3) is satisfied.

Finally, suppose that $S \setminus \mathcal{U} \subseteq f(S^{n-1}, a)$ for all $a \in S \setminus \mathcal{U}$. We would like to show that $S \setminus \mathcal{U} = \{a \in S \mid f(S^{n-1}, a) = S\}$. Let $a \in S \setminus \mathcal{U}$. By the hypothesis, we have that $a \in f(S^{n-1}, a)$, and so $\{a\} \subseteq f(S^{n-1}, a)$. Then $I_n(a) = f(S^{n-1}, a) \cup \{a\} = f(S^{n-1}, a)$ by Corollary 1. Since $a \notin \mathcal{U}$, we obtain $I_n(a) = S$. Hence $S = I_n(a) = f(S^{n-1}, a)$. Now, we get $S \setminus \mathcal{U} \subseteq \{a \in S \mid f(S^{n-1}, a) = S\}$. Conversely, let $a \in S$ be such that $S = f(S^{n-1}, a)$. If $a \in \mathcal{U}$, then $I_n(a) \subseteq \mathcal{U} \subsetneq S$. By Corollary 1, we have $I_n(a) = f(S^{n-1}, a) \cup \{a\} = S \cup \{a\} = S$, which is a contradiction. This implies that $a \in S \setminus \mathcal{U}$. This implies that $\{a \in S \mid f(S^{n-1}, a) = S\} \subseteq S \setminus \mathcal{U}$. Therefore, $S \setminus \mathcal{U} = \{a \in S \mid f(S^{n-1}, a) = S\}$, as desired. In this case, the statement (4) is satisfied.

Hence the proof is completed.

Example 6. (1) Let $S = \{-1, 1\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Then S is n-simple, this implies that $\mathcal{U} = \emptyset$. So, S satisfies the condition (1) of Theorem 8.

(2) Let $S = \mathbb{N} \setminus \{1\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. It is easy to verify that $I_n(a) \neq S$ for all $a \in S$. Hence S satisfies the condition (2) of Theorem 8.

(3) Consider $\mathbb{Z}_{2^{n+1}}$, let $S = \{\overline{0}, \overline{2}, \overline{2^n}\}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Thus $\mathcal{U} = \{\overline{0}, \overline{2^n}\}$. It is easy to verify that S satisfies the condition (3) of Theorem 8 by use $a = \overline{2}$.

(4) Let $S = \mathbb{N}$. Define $f : S^n \to S$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \ \ldots \ \cdot x_n$$

for all $x_1, x_2, \ldots, x_n \in S$ where $\cdot$ is the usual multiplication. Then $\mathcal{U} = S \setminus \{1\}$. It is easy to verify that S satisfies the condition (4) of Theorem 8.

Theorem 9. *If S has a zero element and $f(S^n) \neq \{0\}$, then the exactly one of the following statements is satisfied:*

(1) *S is 0-n-simple.*

(2) *$I_n(a) \neq S$ for all $a \in S$.*

(3) *There exists $a \in S$ such that $I_n(a) = S, a \notin f(S^{n-1}, a), f(a, S^{n-2}, a) \subseteq \mathcal{U} = S \setminus \{a\}$ and $\mathcal{U}$ is the unique maximal n-ideal of S.*

(4) *$S \setminus \mathcal{U} = \{a \in S \mid f(S^{n-1}, a) = S\}$ and $\mathcal{U}$ is the unique maximal n-ideal of S.*

Proof. This follows from Theorem 8.

6. Discussion

In this paper, we introduce many algebraic structures of n-ary semigroups and ones of those important are n-ideals, n-simple, 0-n-simple, minimal n-ideals, 0-minimal n-ideals, and maximal n-ideals. The concept of n-ideals (n-simple, 0-n-simple, minimal n-ideals, 0-minimal n-ideals, and maximal n-ideals, respectively) of n-ary semigroups that we studied supports the concept of left ideals (left-simple, left 0-simple, minimal left ideals, 0-minimal left ideals, maximal left ideals, respectively) of semigroups in case $n = 2$ and of ternary semigroups in case $n = 3$ that are investigated by several researchers before. Of course, the study of left ideals (left-simple, left 0-simple, minimal left ideals, 0-minimal left ideals, maximal left ideals, respectively) is always come together with the study of right ideals (right-simple, right 0-simple, minimal right ideals, 0-minimal right ideals, maximal right ideals, respectively) because they have the similar structures, no matter what we consider in groups, semigroups, or ternary semigroups, and so that many researchers usually show in the only one case between left and right. In case of right, the right ideals (right-simple, right 0-simple, minimal right ideals, 0-minimal right ideals, maximal right ideals, respectively) of semigroups/ternary semigroups are just the 1-ideals (1-simple, 0-1-simple, minimal 1-ideals, 0-minimal 1-ideals, and maximal 1-ideals, respectively) of 2-ary semigroups/3-ary semigroups. For any results of this research, we can place 1-ideal ,1-simple, 0-1-simple, minimal 1-ideal, 0-minimal 1-ideal, and maximal 1-ideal, respectively, instead of n-ideal, n-simple, 0-n-simple, minimal n-ideal, 0-minimal n-ideal, and maximal n-ideal, respectively, and then we will obtain the similar results.

Finally, we present some ideas for extending our results. We know that an n-ary semigroup is just a ternary semigroup if $n = 3$. There are many researchs about the lateral ideals of ternary semigroups which are just 2-ideals of 3-ary semigroups. For the more generally cases of our results that we let them be the open problems are the minimality and maximality of i-ideals where $1 < i < n$ in n-ary semigroups.

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