EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 740-750
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

The Product-Normed Linear Space

Benedict Barnes[1]*, I. A. Adjei[2], C. Sebil[3], E. Harris[4]

[1,2,3,4] *Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana*

Abstract. In this paper, both the product-normed linear space $P-NLS$ (product-Banach space) and product-semi-normed linear space (product-semi-Banch space) are introduced. These normed linear spaces are endowed with the first and second product inequalities, which have a lot of applications in linear algebra and differential equations. In addition, we showed that $P-NLS$ admits functional properties such as completeness, continuity and the fixed point.

2010 Mathematics Subject Classifications: 4B47, 44B49

Key Words and Phrases: product-normed linear space, product-semi-normed linear space, completeness, fixed point theorem

---

1. Introduction

In recent times, the solution space of a mathematical problem has become necessary condition for its existence. Without the solution space, the qualitative, as well as, quantitative property of the mathematical structure cannot be established. In [1], the author introduced the $n-$metrics. About 30 decades afterwards, the author in [2] narrowed the $n-$metric space to two-dimensional space and also, obtained some topological properties associated with this space.

Definition 1 (Norm)**.** *Let V be a linear space over the real number field $\mathbf{R}$. A norm on V is a real-valued function*

$$\|\cdot\| : V \to [0,\infty)$$

such that for any $u, v \in V$ and $\alpha \in \mathbf{R}$ the following conditions are met:

$$\begin{aligned} 1.\|u\| &\geq 0,\ and\ \|u\| = 0,\ iff\ u = 0 \\ 2.\|\alpha u\| &= |\alpha|\|u\|,\ \ \forall\ u \in V\ and\ \alpha \in \mathbf{R} \\ 3.\|u+v\| &\leq \|u\| + \|v\|,\ \ \forall\ u, v \in V \end{aligned}$$

See [3].

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3284

Email addresses: `ewiekwamina@gmail.com` `bbarnes.cos@knust.edu.gh` (B. Barnes)

In [4], the author generalized the normed linear space by replacing the third axiom of the norm, triangle inequality, with tetrahedral inequality.

The authors in [5] and [6] independently extended the linear normed space to quasi-normed linear space. They replaced the third axiom of the normed linear space with quasi-norm

$$\|u+v\| \leq K\Big(\|u\|+\|v\|\Big), \quad \forall\ u,v \in V,$$

where $K \geq 1$ is the quasi-number. In [7], the authors extended the quasi-normed linear space to the complex version and also, included a parameter $p \in (0,\infty)$ in this space, which is used to establish $p-$ uniformly $PL-$ convex. The author in [8], observed that the quasi-normed linear space becomes the $p-$normed linear space if the quasi number $K = 2^{\frac{1}{p}-1}$. In [9], the author established that the quasi-normed linear space is endowed with a unique fixed point. In [10], he narrowed the quasi-normed linear space to quasi-2-normed linear space Q-2-NLS. Addition properties of the Q-2-NLS such as the convergence of the two Cauchy sequences and a pseudo-quasi-2-norm of the Q-2-NLS were observed by the authors in [11].

Undoubtedly, the normed linear space is called a $p-$normed linear if

$$\|u+v\|^p \leq \|u\|^p + \|v\|^p, \quad \forall\ u,v \in V,\ and\ 0 < p \leq 1,$$

is included to the three axioms of the linear normed space. For example, see the author in [12]. There are, however, situations where it is more convenient to work with a subspace than the entire functional space. For example, the imposing of absolute value of a variable x as an inhomogeneous boundary condition of the Laplace equation does not yield a solution in the Hilbert space, but in Sobolev spaces. Thus, the continuity of the partial differential operator is not guaranteed in the Hilbert space. In order for a functional space to admit such a solution, we impose some continuity conditions on the partial differential operator which lead to analysis in Sobolev spaces. Again, much attention has been paid to the extension of the triangle inequality, the third axiom of the normed linear space, to obtain the different normed linear spaces with their functional properties.

In this paper, we include another axiom to the normed linear space as a fourth axiom, first product inequality, from which we coined the name product-normed linear space $P-NLS$, $\|.\|_{pn}$. This functional space narrows the normed linear space which allows the vectors to be defined on $[0,2]$. The $P-NLS$ admits the first product inequality, see [13], and useful properties which we shall establish them later in this paper. Also, the product-semi-normed linear space $\|.\|_{pn}$ is introduced in this paper.

2. Some Preliminary Results

In this section, we give the theorems and definitions regarding to the product-normed linear space and product-semi-normed linear space.

Definition 2 (Inner Product Space). *Let V be a linear vector space defined over the real number field $\mathbf{R}$. A scalar-valued function $p: V \times V \to \mathbf{R}$ that associates with each pair*

u, v of vectors in V a scalar, denoted (u, v), is called an inner product on V if and only if

(i) $(u, u) > 0$ *whenever* $u \neq 0$*, and* $(u, u) = 0$ *if and only if* $u = 0$

(ii) $(u, v) = (v, u),\ \forall\, u,\ v \in V$

(iii) $(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v),\ \ \forall\, \alpha,\ \beta \in \mathbf{R}$*, and* $u_1, u_2, v \in V$*. See* [14]

Definition 3 (Subspace)**.** *Let V be a vector space and Ω a nonempty subset of V. If Ω is a vector space with respect to the operations in V, then Ω is called a subspace of V. See* [15]*.*

Definition 4 (Continuity at a Point)**.** *Let U and V be two normed linear spaces. An operator $A : U \to V$ is continuous at $u_o \in U$, if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, u_o)$ such that*

$$\|Au - Au_o\| < \epsilon, \ \ \text{whenever} \ \ \|u - u_o\| < \delta.$$

See [16]*.*

Definition 5 (Convergent Sequence)**.** *A sequence $\{U_n\}_{n=1}^{\infty}$ in a normed linear space $\Big(\mathbf{R}^n, \|\cdot\|\Big)$ converges to a point $u \in \mathbf{R}^n$, if and only if, for every real number $\epsilon > 0$, there exists an integer N such that*

$$\|U_n - u\| < \epsilon, \quad n > N.$$

See [17]*.*

Definition 6 (Closed set)**.** *A set $u = [0, 2]$ in a normed linear space $\Big(\mathbf{R}^n, \|\cdot\|\Big)$ is closed if and only if every sequence of points $\{U_n\}_{n=1}^{\infty} \subset U$ convergent in $\mathbf{R}^n$ has a limit in U. For example, see* [18]

Definition 7 (Contractive Mapping)**.** *Let $T : U \to U$ be a mapping from a complete normed linear space U into itself. The Lipschitz continuity on T is said to be contraction if*

$$\|T(u) - T(v)\| \leq \lambda\|u - v\|, \quad \forall\, u,\ v \in U, \ \text{and}\ \lambda < 1.$$

See [19]*.*

3. Main Result

In this section, we introduce product-normed linear space and product-semi-normed linear space. The product-normed linear space is presented as follows.

Theorem 1 (Product-normed Linear Space). *Let U be a linear space over $[0,2] \subseteq \mathbf{R}$. A vector space with a product-norm $u \to \|U\|$ satisfying real-valued function $\|\cdot\|$,*

$$\|\cdot\|_{pn} : U \to [0,\infty),$$

such that for arbitrary $u, v \in U$, $\alpha \in [0,2]$, the following conditions are satisfied:

$$\begin{aligned} P1.\ \|u\| &\geq 0,\ and\ \|u\| = 0,\ if\ and\ only\ if\ u = 0 \\ P2.\ \|\alpha u\| &= |\alpha|\|u\|, \quad \alpha \in [0,2], and\ u \in U \\ P3.\ \|u+v\|_{pn} &\leq \|u\|_{pn} + \|v\|_{pn}, \ \ \forall u, v \in U \end{aligned}$$

We call $\|\cdot\|_{pn}$ a product norm if in addition to $P1 - P3$, the u and v satisfy

$$\begin{aligned} P4(a).\ \|u\|_{pn}\|y\|_{pn} &\leq \|u\|_{pn} + \|v\|_{pn} \quad \forall u, v \in [0,2] \ \ (first\ product\ inequality) \\ P4(b).\|u\| + \|v\| &\leq \|u\|\|v\|, \quad \forall u, v \in [2,\infty) \qquad (second\ product\ inequality). \end{aligned}$$

Proof: We start at $P1$. Suppose that $u = 0$, then

$$\begin{aligned} \|u\|_{pn}^2 &= \langle 0, 0\rangle \\ \Rightarrow \|u\|_{pn}^2 &= 0 \\ \Rightarrow \|u\|_{pn} &= 0. \end{aligned}$$

On the other hand, we suppose that $\|u\|_{pn} = 0$. But,

$$\begin{aligned} \|u\|_{pn}^2 &= \langle u, u\rangle \\ \Rightarrow 0 &= \langle uu\rangle. \end{aligned}$$

The above equality holds if $u = 0$. Hence, $\|u\| = 0$ if and only if $u = 0$. Without loss of generality, let the

$$\|u\|_{pn} = |u|_{pn},$$

then $|u|_{pn} = 0$, which implies that $\|u\|_{pn} = 0$. But

$$\|u\|_{pn}^2 = \langle u, u\rangle.$$

Since the $\langle u, u\rangle$ cannot be negative. This implies that $\|u\|_{pn} > 0$. Thus, $P1$ is satisfied.

In order to prove axiom $P2$, we take the expression on the left hand side yields

$$\begin{aligned} \|\alpha u\|^2 &= \langle \alpha u, \alpha u\rangle \\ \Rightarrow \|\alpha u\|^2 &= |\alpha|^2\langle u, u\rangle \\ \Rightarrow \|\alpha u\|^2 &= |\alpha|^2\|u\|^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \|\alpha u\|^2 &= \Big(|\alpha|\|u\|\Big)^2 \\ \Rightarrow \|\alpha u\| &= |\alpha|\|u\|. \end{aligned}$$

Also, the axiom $P3$ is proved as follows.

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle \\ \Rightarrow \|u+v\|^2 &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ \Rightarrow \|u+v\|^2 &= \langle u,u\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} + \langle v,v\rangle \\ \Rightarrow \|u+v\|^2 &= \langle u,u\rangle + 2|\langle u,v\rangle| + \langle v,v\rangle \\ \Rightarrow \|u+v\|^2 &= \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to the middle term on the right hand side, we obtain

$$\begin{aligned} \|u+v\|^2 &\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\ \Rightarrow \|u+v\|^2 &= \Big(\|u\| + \|v\|\Big)^2 \\ \Rightarrow \|u+v\| &\leq \|u\| + \|v\|. \end{aligned}$$

The axiom $P4$ has been discussed by the authors in [13]. We present the proof as follows.

$$\begin{aligned} & (u+v)^n \geq 0 \\ \Rightarrow\ & \langle u,u\rangle^{\frac{n}{2}} +^n C_1\langle u,v\rangle\langle u,u\rangle^{\frac{n-2}{2}} +^n C_2\langle u,u\rangle^{\frac{n-2}{2}}\langle v,v\rangle +^n C_3\langle u,u\rangle^{\frac{n-4}{2}}\langle v,v\rangle\langle u,v\rangle \\ +\ & {}^nC_4\langle u,u\rangle^{\frac{n-4}{2}}\langle v,v\rangle^2 +^n C_5\langle u,u\rangle^{\frac{n-6}{2}}\langle v,v\rangle^2\langle u,v\rangle +^n C_6\langle u,u\rangle^{\frac{n-6}{2}}\langle v,v\rangle^3 \\ +\ & {}^nC_7\langle u,u\rangle^{\frac{n-8}{2}}\langle v,v\rangle^3\langle u,v\rangle + \ldots +^n C_{\frac{n}{2}}\langle u,u\rangle^{\frac{n}{4}}\langle v,v\rangle^{\frac{n}{4}} + \ldots + \langle u,u\rangle^{\frac{n}{2}} \geq 0 \\ \Rightarrow\ & -{}^nC_{\frac{n}{2}}\langle u,u\rangle^{\frac{n}{4}}\langle v,v\rangle^{\frac{n}{4}} \leq \{\langle u,u\rangle^{\frac{n}{2}} +^n C_1\langle u,v\rangle\langle u,u\rangle^{\frac{n-2}{2}} +^n C_2\langle u,u\rangle^{\frac{n-2}{2}}\langle v,v\rangle \\ +\ & {}^nC_3\langle u,u\rangle^{\frac{n-4}{2}}\langle v,v\rangle\langle u,v\rangle +^n C_4\langle u,u\rangle^{\frac{n-4}{2}}\langle v,v\rangle^2 +^n C_5\langle u,u\rangle^{\frac{n-6}{2}}\langle v,v\rangle^2\langle u,v\rangle \\ +\ & {}^nC_6\langle u,u\rangle^{\frac{n-6}{2}}\langle v,v\rangle^3 +^n C_7\langle u,u\rangle^{\frac{n-8}{2}}\langle v,v\rangle^3\langle u,v\rangle + \ldots + \langle u,u\rangle^{\frac{n}{2}}\} \\ \Rightarrow\ & -{}^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} \leq \Big\{(u,u)^{\frac{n}{2}} +^n C_1(u,v)(u,u)^{\frac{n-2}{2}} +^n C_2(u,u)^{\frac{n-2}{2}}(v,v) \\ +\ & {}^nC_3(u,u)^{\frac{n-4}{2}}(v,v)(u,v) +^n C_4(u,u)^{\frac{n-4}{2}}(v,v)^2 +^n C_5(u,u)^{\frac{n-6}{2}}(v,v)^2(u,v) \\ +\ & {}^nC_6(u,u)^{\frac{n-6}{2}}(v,v)^3 +^n C_7(u,u)^{\frac{n-8}{2}}(v,v)^3(u,v) + \ldots +^n C_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} + \ldots + (u,u)^{\frac{n}{2}}\Big\} \\ \Rightarrow\ & -{}^nC_{\frac{n}{2}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}}\frac{1}{{}^nC_{\frac{n}{2}}}(u,u)^{\frac{n}{4}}(v,v)^{\frac{n}{4}} \leq (u+v)^n \\ \Rightarrow\ & \|-(u,u)^{\frac{n}{2}}(v,v)^{\frac{n}{2}}\| = \|(u+v)^n\| \\ \Rightarrow\ & \{\|u\|\|v\|\}^n = \|(u+v)\|^n \\ \Rightarrow\ & \|u\|\|v\| \leq \|u\| + \|v\|, \quad \forall\ u,v \in [0,2]. \end{aligned}$$

The proof of $P4(b)$, see [13].

We show that the product-normed space is valid for $1-$ norm, $2-$norm, $p-$ norms and $\infty-$norm. We give the proof for $p-$norm which can be used to generate the other norms.

Theorem 2. *Suppose that $f(x)$ and $g(x)$ are measurable functions over the domain $[0,2]$, then*

$$\|f\|_p\|g\|_p \leq \|f\|_p + \|g\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: We see that

$$(\|f\|\|g\|)^p = (\|f\|\|g\|)^{p-1}(\|f\|\|g\|)$$

Applying the first product inequality to the right hand side of the above equation yields

$$\begin{aligned}
(\|f\|\|g\|)^p &\leq (\|f\| + \|g\|)^{p-1}(\|f\| + \|g\|)\\
(\|f\|\|g\|)^p &= (\|f\| + \|g\|)^{p-1}\|f\| + (\|f\| + \|g\|)^{p-1}\|g\|
\end{aligned}$$

Taking the Lebesgue integral of both sides of the above equation with respect μ, we obtain

$$\begin{aligned}
\int (\|f\|\|g\|)^p d\mu &= \int \Big\{(\|f\| + \|g\|)^{p-1}\|f\| + (\|f\| + \|g\|)^{p-1}\|g\|\Big\} d\mu\\
\Rightarrow \int (\|f\|\|g\|)^p d\mu &= \int (\|f\| + \|g\|)^{p-1}\|f\| d\mu + \int (\|f\| + \|g\|)^{p-1}\|g\| d\mu\\
\Rightarrow \Big(\int (\|f\|\|g\|)^p\Big)^{\frac{1}{p}} d\mu &= \Big[\int (\|f\| + \|g\|)^{p-1}\|f\| d\mu + \int (\|f\| + \|g\|)^{p-1}\|g\| d\mu\Big]^{\frac{1}{p}}\\
\Rightarrow \|f\|_p\|g\|_p &= \Big[\Big(\int ((\|f\| + \|g\|)^{p-1}\|f\|)^p d\mu\Big)^{\frac{1}{p}}\\
&+ \Big(\int ((\|f\| + \|g\|)^{p-1}\|g\|)^p d\mu\Big)^{\frac{1}{p}}\Big]^{\frac{1}{p}}\\
\Rightarrow \|f\|_p\|g\|_p &\leq \Big[\Big\{\Big(\int \|f\|^p d\mu\Big)^{\frac{1}{p}} + \Big(\int \|g\|^p d\mu\Big)^{\frac{1}{p}}\Big\}\Big\{\Big(\int (\|f\| + \|g\|)^p d\mu\Big)^{\frac{1}{p}}\Big\}^{p-1}\Big]^{\frac{1}{p}}\\
\Rightarrow \|f\|_p\|g\|_p &\leq \Big[\Big(\|f\|_p + \|g\|_p\Big)\Big(\|f\|_p + \|g\|_p\Big)^{p-1}\Big]^{\frac{1}{p}}\\
\Rightarrow \|f\|_p\|g\|_p &= \Big(\|f\|_p + \|g\|_p\Big)^{\frac{1}{p}}\Big(\|f\|_p + \|g\|_p\Big)^{\frac{p-1}{p}}\\
\Rightarrow \|f\|_p\|g\|_p &\leq \|f\|_p + \|g\|_p. \qquad (1)
\end{aligned}$$

Choosing $p = 1$ and $p = 2$ in inequality (1), we obtain the following

$$\|f\|_1\|g\|_1 \leq \|f\|_1 + \|g\|_1$$

and

$$\|f\|_2\|g\|_2 \leq \|f\|_2 + \|g\|_2,$$

respectively. But if, $p \to \infty$ then the inequality (1) becomes

$$\|f\|_\infty \|g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Theorem 3 (Completeness of the Product-normed linear space)**.** *The product-normed linear space* $(U, \|\cdot\|_{pn})$ *is complete.*

Proof: Let $\{U_n\}_{n=1}^{\infty}$ be a sequence in $(U, \|\cdot\|_{pn})$ and $u \in U$. For every $\epsilon > 0$, there exists an integer $n_o > \mathbf{N}$ such that $\|U_n - u\| < \epsilon$, for all $n > n_o$, then $\{U_n\}_{n=1}^{\infty}$ converges u. Thus, $\lim_{n\to\infty} \|U_n - u\|_{pn} = 0$. Also, we see that for every $\epsilon > 0$, there exists integers $m, n > \mathbf{N}$ such that $\lim_{m,n\to\infty} \|U_m - U_n\| = 0$. In this case, we see that every Cauchy sequence converges to a point $(U, \|\cdot\|_{pn})$, then $(U, \|\cdot\|_{pn})$ is a complete product-normed linear space. Alternatively, we the following result shows the product-normed linear space is complete.

Theorem 4 (Completeness of the Product-normed linear space Using Closed Set)**.** *Let* U *be a linear space over* $[0, 2] \subseteq \mathbf{R}$*. A subspace* $\Big(U, \|\cdot\|_{pn}\Big)$ *of a complete normed linear space* $\Big(\mathbf{R}^n, \|\cdot\|_{pn}\Big)$ *is complete if and only if* U *is a closed set.*

Proof: Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of points of U. We see that $\{U_n\}_{n=1}^{\infty}$ has limit u in U, since U is closed. Thus,

$$\lim_{n\to\infty} \|U_n - u\|_{pn} = 0.$$

Since $u \in U$ then $\Big(U, \|\cdot\|_{pn}\Big)$ is complete.

Conversely, assume that $\Big(U, \|\cdot\|_{pn}\Big)$ is complete. Let u be an accumulation point of U. Then each open ball centred at u, $B_{1/n}(u)$ contains a point $u_n \in U$. That is,

$$\|U_n - u\|_{pn} < \frac{1}{n}.$$

This implies that $\{U_n\}_{n=1}^{\infty}$ converges to u. However, $\Big(U, \|\cdot\|_{pn}\Big)$ is complete on the grounds that u is in U. Hence, U is closed.

Theorem 5 (Continuity)**.** *Let* $(U, \|\cdot\|_{pn1})$ *and* $(U, \|\cdot\|_{pn2})$ *be two product normed linear spaces, and* $T : U_1 \to U_2$ *be a linear operator. Then* T *is bounded if and only if* T *is continuous.*

Proof: Let A be a bounded linear operator. Then there exists a constant M such that

$$\|A(u)\|_{U_2} \leq M\|x\|_{U_1}, \quad \forall \;\; u \in U_1.$$

Let $\{U_n\}_{n=1}^{\infty}$ be a sequence in U_1 such that

$$\lim_{n\to\infty} \|U_n - u\|_{U_1} = 0.$$

Now,

$$\begin{aligned}
\|A(U_n) - A(u)\|_{U_2} &= \Big\|A\Big(U_n - u\Big)\Big\|_{U_2} \\
\|A(U_n) - A(u)\|_{U_2} &\leq M\Big\|U_n - u\Big\|_{U_1} \\
\lim_{n\to\infty} \|A(U_n) - A(u)\|_{U_2} &\leq 0 \\
\lim_{n\to\infty} A(U_n) - \lim_{n\to\infty} A(u) &= 0 \\
\lim_{n\to\infty} A(U_n) - A(u) &= 0 \\
A(\lim_{n\to\infty} U_n) &\to A(u).
\end{aligned}$$

This implies that A is continuous on U_1.

Conversely, suppose that A is a continuous linear operator. Then A is continuous at 0. Thus, for every $\delta > 0$, $\|u\|_{U_1}$, there exists an $\epsilon > 0$ such that

$$\|Au\| \leq \epsilon. \tag{2}$$

Setting $u_o \neq 0$, be an arbitrary element in U_1 and $u = \alpha u_o$, where $\alpha = \frac{\delta}{\|u_o\|_{U_1}}$. We can see that

$$\begin{aligned}
\|Au\|_{U_2} &= \|A(\alpha u_o)\|_{U_2} \\
\|Au\|_{U_2} &= |\alpha|\|Au_o\|_{U_2}, \quad \forall\, u_o \in U_1. \qquad (3)
\end{aligned}$$

Comparing (2) and (3), we obtain

$$\begin{aligned}
|\alpha|\|Au_o\|_{U_2} &\leq \epsilon \\
\Rightarrow \|Au_o\|_{U_2} &\leq \frac{\epsilon}{|\alpha|} \\
\Rightarrow \|Au_o\|_{U_2} &\leq \frac{\epsilon}{|\delta|}\|u_o\|_{U_1}.
\end{aligned}$$

Hence A is bounded. This completes the proof.

Theorem 6 (Unique Fixed Point)**.** *Let $\Big(U, \|\cdot\|_{pn}\Big)$ be a product-normed linear space and setting $T : U \to U$ be a contractive mapping. Then T has a unique fixed point $\bar{u} \in U$.*

Proof: Letting $u_o \in U$, then

$$\begin{aligned}
Un &= T(U_n) \\
\Rightarrow Un &= T_n(u_o), \quad \forall\, n \in \mathbf{N}.
\end{aligned}$$

Setting $\{U\}_n^\infty$ a Cauchy sequence in $\Big(U, \|\cdot\|_{pn}\Big)$. Since $\Big(U, \|\cdot\|_{pn}\Big)$ is a complete, which implies that $u \in U$ such that

$$\lim_{n\to\infty} U_n = u.$$

Again, we see that T is a contractive mapping, then it is continuous. Thus,

$$\lim_{n\to\infty} T(U_n) = Tu.$$

We can see that

$$\begin{aligned} Tu &= \lim_{n\to\infty} T(U_n) \\ \Rightarrow Tu &= \lim_{n\to\infty} U_{n+1} \\ \Rightarrow Tu &= u. \end{aligned}$$

Hence, u is a fixed point of the mapping T.

In addition, we show that the fixed point is unique in $\Big(U, \|\cdot\|_{pn}\Big)$. Setting $u_1, u_2 \in U$ be the fixed points of T, then

$$\|T(u_1) - T(u_2)\|_{pn} = \|u_1 - u_2\|_{pn}. \tag{2}$$

Also, we see that:

$$\|T(u_1) - T(u_2)\|_{pn} \leq \lambda \|u_1 - u_2\|_{pn}, \quad \forall\, \lambda < 1 \tag{3}$$

Substituting equation (2) into inequality (3) yields

$$\begin{aligned} \Big(1-\lambda\Big)\|u_1 - u_2\|_{pn} &\leq 0 \\ \Rightarrow \|u_1 - u_2\|_{pn} &= 0. \end{aligned}$$

The above equation holds if $u_1 = u_2$. Hence, the fixed point x is unique in the product-normed linear space.

Another normed linear space is introduced as:

Theorem 7 (Product Semi-Normed Linear Space)**.** *Let U be a linear space over* $[2, \infty] \subseteq$ **R**. *A product norm on U is a real-valued function $\|\cdot\|$,*

$$\|\cdot\| : U \to [0, \infty),$$

such that for arbitrary $u, v \in U$, $\alpha \in [0,2]$, the following conditions are satisfied:

$$\begin{aligned} 1.\ \|u\| &> 0,\ \textit{and}\ \|u\| = 0,\ \textit{if and only if}\ u = 0 \\ 2.\ \|\alpha u\| &= |\alpha|\|u\|, \quad \alpha \in [0,2], \textit{and}\ u \in U \\ 3.\ \|u+v\| &\leq \|u\| + \|v\|, \quad \forall u, v \in U \\ 4.\ \|u\| + \|v\| &\leq \|u\|\|v\| \ \ \forall u, v \in V \ \ (\textit{product inequality}). \end{aligned}$$

4. Conclusion

In a nutshell, we have introduced the product-normed linear space and product-semi-normed linear space (product-semi-Banch space) which are endowed with some functional space. In addition, $P - NLS$ is endowed with fixed point.

References

[1] I. K. Menger, Untersuchungen veber allgeine matrik, *Mathematische Annalen,* **vol. 100**, (1928).

[2] V. S. Gähler, Lineare 2−normierte räume *Diese Nachr,* **28, 1-2**, (1963).pp.1 - 43

[3] J. T. Scheick, Linear Algebra with Applications, Prentice-Hall, Inc, New York; 1997.

[4] A. Branciari, A fixed point theorem of Banach-Caccippoli type on a class of generalized metric spaces, *Publ. Math. Debrecen* **57** (2000). pp 31-37.

[5] M. Pavel, On quasi normed spaces *Bull. Acad. Polon Sci. ,* **CI III, 5, no. 5**, (1957).pp. 479-487

[6] S. Rolewiez, On a certain class of linear metric space *Bull. Acad. Polon Sci. ,* **CI III, 5, no. 5**, (1957).pp. 471-473

[7] W. J. Davis, D. J. H. Garling and N. Tomczak-Jaegermann, The complex of quasi-normed linear spaces *Journal of Functional Analysis,* **55** , (1984). pp. 110-150.

[8] S. Rolewicz, Metric linear spaces, PMN, Warsaw(1972).

[9] G. Rano and T. Bag, Bounded linear operators in quasi-normed linear space, *Journal of the Egyptian Mathematical Society,* **23** , (2015).pp. 303-308.

[10] C. Park, Generalized quasi-Banach spaces and quasi-(2, p) normed space, *Journal of the Chungcheong matematical society,* **19, No. 2** (2006).

[11] M. Kir and M. Acikgoz, A study involving the completion of a quasi-2-normed space, *International Journal of Analysis* , (2013). $http://dx.doi.org/$10.1155/2013/512372.

[12] M. Mastylo, On interpolation of some quasi-Banach spaces, *Journal of Mathematical Analysis and Applications,* **147** , (1990).pp. 403-419.

[13] B. Barnes, E. D. J. Owusu-Ansah, S. K. Amponsah and C. Sebil, The proofs of product inequalities in a generalized vector spaces, *European journal of pure and applied mathematics,* **11, No. 2**(2018). pp. 375-389.

[14] B. Kolman and D. R. Hill, Elementary Linear Algebra, Prentice-Hall, Inc, New Jersey; 2000.

[15] H. Royden and P. Fitxpatrick, Real Analysis. Pearson Education, Inc, 4^{th} ed., Upper saddle River; 2010.

[16] N. Dunford and J. T. Schwartz, Linear operators. Vol 1, Interscience, New York; 1958.

[17] B. D. MacCluer, Graduate texts in mathematics: elementary functional analysis, Springer science + business media, New York; 2009.

[18] G. Bachman and L. Narici, Functional Analysis, Academic Press, New York and London; 1966.

[19] J. T. Oden, Applied Functional Analysis: A first course for students of mechanics and engineering science, Prentice-Hall, Inc, New Jersey; 1979.