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A common best proximity point theorem for ϕ-dominated pair

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Abstract. In the present research, an interesting common best proximity point theorem for pairs of non-self-mappings is presented. It satisfies a weakly contraction-like condition, thereby producing common optimal approximate solutions of certain simultaneous fixed point equations.

2010 Mathematics Subject Classifications: 47H09, 47H10

Key Words and Phrases: Common best proximity point, ϕ-dominated pair, ϕ-contraction, fixed point

1. Introduction

Fixed point theory is indispensable $Tx = x$ for self-mappings T on subsets of metric space or normed space. Let A and B be non-empty subsets of metric space (X, d) and let $T : A \longrightarrow B$ be non-self mapping. If the equation $Tx = x$ does not possess solution, then $d(x, Tx) > 0$. In this case, it is important that we find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense. For example, the best approximation problem and best proximity problem are investigated in this regard (see [2] and [5]). An element $x \in A$ is said to be a best proximity point of T if $d(x, Tx) = d(A, B)$ where

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

It is easy to check that if T is self-mapping the best proximity problem reduces to fixed point problem. There are several various of contractions that guarantee the existence of a best proximity point (see [2], [5], and [11]).

Suppose that A and B be nonempty subsets of metric space (X, d). Let $T : A \longrightarrow B$ and $S : A \longrightarrow B$ be nonself mapping. Let considering the fact S and T are nonself-mappings, it is possible that the equations $Tx = x$ and $Sx = x$ have a common solution, considered as a common fixed point of the mappings T and S . When the equations have no common solution, one thinks to find an element x that is in near proximity to

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Tx and Sx in the sense that $d(x,Tx)$ and $d(x,Sx)$ are minimal. In fact, one investigates the existence of such optimal approximate solutions, known as common best proximity points, to the equations $Sx = x$ and $Tx = x$. Further, one can comprehend that the real valued functions $x \longrightarrow d(x,Tx)$ and $x \longrightarrow d(x,Sx)$ approximate the value of the error of proximate solution of the equations $Tx = x$ and $Sx = x$. In view of the fact that $d(A,B) \leq d(x,Tx)$ and $d(A,B) \leq d(x,Sx)$, a common best proximity point theorem determines global minimum of both functions $x \longrightarrow d(x,Tx)$ and $x \longrightarrow d(x,Sx)$ by limiting a common approximate solution of the equations $Tx = x$ and $Sx = x$ to attain the requirement that $d(x,Sx) = d(A,B)$ and $d(x,Tx) = d(A,B)$.
Common best proximity point problem was studied by many mathematicians (see [7], [8] and [11]).

2. Preliminary Concepts

Definition 1. *An element $x \in X$ is said to be common best proximity point of the non-self-mappings $S : A \longrightarrow B$ and $T : A \longrightarrow B$ if it satisfies the condition that*

$$d(x,Sx) = d(x,Tx) = d(A,B).$$

Definition 2. *[4] A function $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is called a comparison if it satisfies the following conditions:*

- *ϕ is increasing,*
- *the sequence $(\phi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to +\infty$, for all $t \in [0,+\infty)$.*

We recall that a self-mapping T on a metric space (X,d) is said to be $\phi-$contraction if

$$d(T(x),T(y)) \leq \phi(d(x,y))$$

for any $x,y \in X$; where ϕ is comparison function.

Remark 1. *If ϕ is comparison function then*

- *$\phi(t) < t$ for any $t \in (0,+\infty)$,*
- *$\phi(t) = 0$ if and only if $t = 0$.*

Lemma 1. *[4] Let (X,d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $d(x_{n+1},x_n) \to 0$. If $(x_n)_{n\in\mathbb{N}}$ is not Cauchy sequence then there exists $\epsilon > 0$ and sequences $(n(k))$ and $(m(k))$ of positive integers such that the following sequences tend to ϵ as $k \to +\infty$:*

$$d(x_{m(k)},x_{n(k)}),\ \ d(x_{m(k)},x_{n(k)+1}),\ \ d(x_{m(k)-1},x_{n(k)}),$$
$$d(x_{m(k)-1},x_{n(k)+1}),\ \ d(x_{m(k)+1},x_{n(k)+1}),\ \ d(x_{m(k)+1},x_{n(k)}).$$

Definition 3. *Let $S : A \longrightarrow B$, $T : A \longrightarrow B$ and $F : B \longrightarrow A$ be given. F is said to commute with the pair (S,T) if $SFT = TFS$.*

Definition 4. *Let $S : A \longrightarrow B$, $T : A \longrightarrow B$, $F : B \longrightarrow A$ and $G : B \longrightarrow A$ be mappings. The pair (F, S) is said to be $\phi-$dominated by the pair (G, T) if for any $x \in A$ and $y \in B$ it satisfies the condition that*

$$d(FSx, FSy) \leq \phi(d(GTx, GTy))$$

where ϕ is comparison function.

3. Main Results

From here throughout this paper, X denotes a complete metric space and A and B are its nonempty subsets.

Now, we are ready to present our main result.

Theorem 1. *Let A and B be closed. Moreover, assume that $S : A \longrightarrow B$, $T : A \longrightarrow B$, $F : B \longrightarrow A$ and $G : B \longrightarrow A$ are continuous functions satisfying the following conditions:*

(1) FS commutes with GT and SF commutes with TG.

(2) (F, S) is $\phi-$dominated by (G, T) and (S, F) is $\psi-$dominated by (T, G), where ϕ and ψ are comparison functions.

(3) $FS(A) \subset GT(A)$ and $SF(B) \subset TG(B)$.

(4) S and T commute with the pair (F, G), and F and G commute with the pair (S, T).

(5) There is a non-negative number $\alpha < 1$ such that for all $x \in A$

$$d(Sx, FSx) \leq \alpha d(Tx, GTx) + (1 - \alpha)d(A, B).$$

Then, there exists $u \in A$ and $v \in B$ such that
$d(u, Su) = d(u, Tu) = d(A, B)$
$d(v, Fv) = d(v, Gv) = d(A, B)$
$d(u, v) = d(A, B)$.
If (I, S) is $\phi-$dominated by (I, T), where I is the identity mapping on B, then

$$2d(A, B) + \phi(2d(A, B) + d(a, a')) - d(a, a') \geq 0$$

whevever a' is another common best proximity point of S and T.

Proof. Let x_0 be an element in A. Since $FS(A) \subset GT(A)$, there exists an element $x_1 \in A$ such that $FS(x_0) = GT(x_1)$. Again by $FS(A) \subset GT(A)$, we can choose an element $x_2 \in A$ such that $FS(x_1) = GT(x_2)$. By continuing this process, we can construct a sequence (x_n) such that $FS(x_n) = GT(x_{n+1})$. By condition (2) there exists continuous non-decreasing function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$, with $\lim_{n \Rightarrow \infty} \phi_n(t) = 0$ for all $t \in [0, +\infty)$, such that

$$d(FSx_n, FSx_{n+1}) \leq \phi(d(GTx_n, GTx_{n+1})).$$

Since $FS(x_n) = GT(x_{n+1})$ so,

$$\begin{aligned} d(FSx_n, FSx_{n+1}) &\leq \phi(d(GTx_n, GTx_{n+1})) = \phi(d(FSx_{n-1}, FSx_n)) \\ &\leq \phi^2(d(GTx_{n-1}, GTx_n)) = \phi^2(d(FSx_{n-2}, GTx_{n-1})) \\ &\leq \phi^3(d(FSx_{n-3}, FSx_{n-2})) \\ &\leq \ldots \leq \phi^n(d(FSx_0, FSx_1)). \end{aligned}$$

Hence,

$$d(FSx_n, FSx_{n+1}) \leq \phi^n(d(FSx_0, FSx_1))$$

taking $n \longrightarrow +\infty$ we have

$$\lim_{n \to +\infty} d(FSx_n, FSx_{n+1}) = 0.$$

Clime: (FSx_n) is a cauchy sequence.
Let (FSx_n) is not Cauchy. Then there exists $\epsilon > 0$ and two sequences $\big(n(k)\big)$ and $\big(m(k)\big)$ of positive integers such that

$$n(k) > m(k) > k,\ d(FSx_{m(k)}, FSx_{n(k)-1}) < \epsilon \ \text{ and } \ d(FSx_{m(k)}, FSx_{n(k)}) \geq \epsilon. \quad (1)$$

Using condition (2) and the fact that ϕ is a comparison function we obtain that

$$d(FSx_{m(k)+1}, FSx_{n(k)}) \leq \phi\big(d(GTx_{m(k)+1}, GTx_{n(k)})\big) = \phi\big(d(FSx_{m(k)}, FSx_{n(k)-1})\big) \leq \phi(\epsilon) < \epsilon. \quad (2)$$

Taking $n \longrightarrow +\infty$ in (2) we get

$$\epsilon = \lim_{k \to +\infty} d(FSx_{m(k)+1}, FSx_{n(k)}) \leq \phi(\epsilon) < \epsilon.$$

This is a contradiction. Hence (FSx_n) is Cauchy sequence. Obviosly, (GTx_n) is also Cauchy. Because of the completeness of the space, there exists an element $x \in A$ such that $FSx_n \longrightarrow x$. By continuity of FS and GT;

$$\begin{aligned} (GT)(FS)x_n &\longrightarrow GTx \\ (FS)(GT)x_n &\longrightarrow FSx. \end{aligned}$$

By condition (1), it follows that $GTx = FSx$. Put $a = GTx = FSx$. Then, by condition (1) and (2),

$$\begin{aligned} d(FSa, a) &= d(FSa, FSx) \\ &\leq \phi(d(GTa, GTx)) = \phi(d(GT(FSx), a)) \\ &= \phi(d(FS(GTx), a)) = \phi(d(FSa, a)). \end{aligned}$$

Since ϕ is comparison by Remark 1 we get

$$d(FSa, a) = 0 \Longrightarrow FSa = a.$$

Further, $GTa = (GT)(FSx) = (FS)(GTx) = FSa = a$. A similar argument can be given to assert that there exists an element $b \in B$ such that $SFb = TGb = b$. Also, since T commutes with the pair (F, G), $GTFb = FTGb = Fb$. So,

$$d(a, Fb) = d(FSa, FS(Fb)) \leq \phi(d(GTa, GT(Fb))) = \phi(d(a, Fb)).$$

Since ϕ is a comparison by Remark 1, it follows that $Fb = a$. By the same argument we can show that $Gb = a$, $Sa = b$ and $Ta = b$. Consequently, by condition (5) there exists $\alpha \in [0, 1)$ such that

$$d(a, b) = d(Sa, FSa) \leq \alpha d(Ta, GTa) + (1 - \alpha)d(A, B) = \alpha d(a, b) + (1 - \alpha)d(a, b).$$

So $d(a, b) \leq d(A, B)$ and hence

$$d(a, b) = d(A, B).$$

Therefore,

$$d(a, Sa) = d(a, Ta) = d(a, b) = d(A, B)$$

$$d(b, Fb) = d(b, Gb) = d(a, b) = d(A, B).$$

If (I, S) is $\phi-$dominated by (I, T) and a' is another common best proximity point of S and T, then

$$\begin{aligned} d(a, a') &\leq d(a, Sa) + d(Sa, Sa') + d(a', Sa') \\ &\leq 2d(A, B) + \phi(d(Ta, Ta')) \\ &\leq 2d(A, B) + \phi(2d(A, B) + d(a, a')) \end{aligned}$$

and hence

$$2d(A, B) + \phi(2d(A, B) + d(a, a')) - d(a, a') \geq 0.$$

Example 1. *Consider the space of real numbers with the Euclidean meteric. Let* $A = [3, +\infty)$ *and* $B = (-\infty, -3]$ *Suppose that* $S, T : A \to B$, $F, G : B \to A$ *and* $\phi, \psi : [0, +\infty) \to [0, +\infty)$ *are defined by*

$$S(x) = -3; \quad T(x) = -x; \quad F(y) = \begin{cases} 3 & y \in \mathbb{Z} \\ 4 & y \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

$$; G(y) = -y \quad \text{and} \quad \phi(x) = \psi(x) = \frac{x}{1 + x}.$$

It is easy to check that $d(A, B) = 6$ *and the mapping* S, T, F *and* G *are satisfied the conditions in Theorem 1 and*

$$d(3, S(3)) = d(3, T(3)) = d(A, B)$$

$$d(-3, F(-3)) = d(-3, G(-3)) = d(A, B)$$

$$d(3, -3) = d(A, B).$$

If S and T are self-mappings on X and F and G are identity mappings on X, then Theorem 1 yields the following common fixed point theorem for pairs of commuting self-mappings.

Corollary 1. *Let X be a complete metric space. Moreover, assume that $S : X \longrightarrow X$, $T : X \longrightarrow X$ are continuous functions satisfying the following conditions:*

(1) S commutes with T.

(2) There exists comparison function ϕ such that

$$d(Sx, Sy) \leq \phi(d(Tx, Ty)) \quad \text{for all } x, y \in X.$$

(3) $S(X) \subset T(X)$.

Then the pair (S, T) has a unique common fixed point.

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