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On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras

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Abstract. In this research article, we study some properties of doubt bipolar fuzzy H-ideals in $BCK/$ BCI-algebras. Doubt bipolar fuzzy H-ideals are connected with doubt bipolar fuzzy subalgebras and doubt bipolar fuzzy ideals. Moreover, doubt bipolar fuzzy H-ideals are characterized using doubt positive t-level cut set, doubt negative s-level cut set and H-Artin BCK/BCI-algebras.

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1. Introduction

In 1965, Zadeh [30] introduced the concept of fuzzy set to handle the uncertainties in our daily life. Fuzzy sets are extremely useful to solve many problems in applied mathematics, information sciences and decision making. After that many generalizations of fuzzy sets are presented, for example, interval valued fuzzy sets [31] and intuitionistic fuzzy sets [7]. Lee [22] introduced the notion of bipolar fuzzy sets which is an extension of fuzzy sets. Fuzzy sets give a degree of membership of an element in a given set, whereas bipolar fuzzy sets give both a positive membership degree belongs to the interval [0, 1] and a negative membership degree belongs to the interval [-1, 0]. In the case of bipolar fuzzy sets, the membership degrees range is enlarged from the interval [0, 1] to the interval [-1, 1]. Recently, the theory of bipolar fuzzy sets becomes a vigorous area of research in different domains such as group theory, semigroup theory, ring theory, semiring theory, graph theory, engineering, physics, statics, medical science, social science, artificial intelligent, computer networks, expert systems, decision making and so on.

BCK-algebras introduced by Imai and Iséki [11] as a generalization of notion of the concept of set theoretic difference and propositional calculus and then Iséki [12] introduced

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the notion of BCI-algebras which is a generalization of BCK-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971 by Rosenfeld [26] and later these ideas have been applied to other algebraic structures such as semigroups, rings, semirings, hemirings, ideals, modules and vector spaces. In 1991, Xi [29] applied the concept of fuzzy sets to BCK-algebras. After that, Jun [15] and Ahmad [1] applied the concept of fuzzy sets to BCI-algebras. Jun [14] provided characterizations of Noetherian BCK-algebras in terms of fuzzy ideals. Fuzzy H-ideals of BCI-algebras introduced in [20] by Khalid and Ahmad and the concept of H-Noetherian BCK-algebras was studied in [33] by Zhan and Tan. Huang [9] fuzzified BCI-algebras in little different ways. Jun [16] renamed Huanǵs definition as doubt fuzzy ideals in BCK/BCI-algebras and introduced the concepts of doubt fuzzy subalgebras and doubt fuzzy ideals in BCK/BCI-algebras. Zhan and Tan [32] introduced the concept of doubt fuzzy H-ideals and provided characterizations of H-Artin BCK-algebras in terms of doubt fuzzy H-ideals. Muhiuddin and Aldhafeeri [24] introduced the notions of uni-hesitant fuzzy algebras and uni-hesitant fuzzy (closed) ideals in BCK-algebras and BCI-algebras. Muhiuddin et al. [25] introduced hesitant fuzzy translations and extensions of subalgebras and ideals in BCK/BCI-algebras. Also, Jun et al. [17–19] studied the notions of subalgebras and ideals of BCK/BCI-algebras based on hesitant fuzzy soft sets, double-framed soft sets and cubic soft sets.

In 2009, Lee [21] applied the concept of bipolar fuzzy set theory to BCK/BCI-algebras, and introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. Recently, the notion of bipolar fuzzy set theory was applied to BCK/BCI-algebras [4, 23] and other algebraic structures such that Lie algebras [2], Lie superalgebras [3], hemirings [8] and BF-algebras [27, 28], etc. Al-Masarwah and Ahmad [5] introduced the notions of doubt bipolar fuzzy subalgebras and doubt bipolar fuzzy ideals in BCK/BCI-algebras. Also, they introduced the concept of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras and investigated some interesting properties [6].

This paper is a continuation of the papers [5] and [6]. We study some properties of doubt bipolar fuzzy H-ideals in $BCK/$ BCI-algebras. We provide relations between a doubt bipolar fuzzy H-ideal and a doubt bipolar fuzzy ideal. We give conditions for a doubt bipolar fuzzy ideal to be a doubt bipolar fuzzy H-ideal. We investigate characterizations of doubt bipolar fuzzy H-ideals by means of doubt positive t-level cut set, doubt negative s-level cut set and H-Artin BCK/BCI-algebras.

2. Preliminaries

We first recall some elementary aspects which are used to present the paper. A BCK/BCI-algebra is an important class of logical algebras introduced by Imai and Iséki [11, 12] and was extensively investigated by several researchers. This algebra is defined as follows.

By a BCI-algebra, we mean an algebra $(X; *, 0)$ of type (2, 0) satisfying the following axioms for all $x, y, z \in X$:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

If a BCI-algebra X satisfies $0 * x = 0$, then X is called a BCK-algebra. In a BCK/BCI-algebra, $x * 0 = x$ hold. A BCI-algebra is said to be associative if $(x*y)*z = x*(y*z)$ for all $x, y, z \in X$. A partial ordering $\leq$ on a BCK/BCI-algebra X can be defined by $x \leq y$ if and only if $x*y = 0$. Any BCK/BCI-algebra X satisfies the following axioms for all $x, y, z \in X$:

(1) $x * 0 = x$,

(2) $(x * y) * z = (x * z) * y$,

(3) $x * y \leq x$,

(4) $(x * y) * z \leq (x * z) * (y * z)$,

(5) $x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x$.

Definition 1. *[29] A non-empty subset S of a BCK/BCI-algebra X is called an ideal of X if*

(i) $0 \in S$

(ii) $x * y \in S$ *and* $y \in S$ *then* $x \in S$, *for all* $x, y \in X$.

Definition 2. *[20] A non-empty subset S of a BCK/BCI-algebra X is called an H-ideal of X if*

(i) $0 \in S$

(ii) $x * (y * z) \in S$ *and* $y \in S$ *then* $x * z \in S$, *for all* $x, y, z \in X$.

We refer the reader to [10, 13] for further information regarding BCK/BCI-algebras. In what follows, we use $(X; *, 0)$ to denote a BCK/BCI-algebra unless otherwise specified. For the sake of brevity, we call X a BCK/BCI-algebra.

Definition 3. *[16] A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ in X is called a doubt fuzzy ideal of X if*

(i) $\mu_A(0) \leq \mu_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$, *for all* $x, y \in X$.

Definition 4. *[32] A fuzzy set $A = \{(x, \mu_A(x)) \mid x \in X\}$ in X is called a doubt fuzzy H-ideal of X if* *(i)* $\mu_A(0) \leq \mu_A(x)$,

(ii) $\mu_A(x * z) \leq \max\{\mu_A(x * (y * z)), \mu_A(y)\}$, *for all* $x, y, z \in X$.

The proposed work is done on a bipolar fuzzy set. The formal definition of a bipolar fuzzy set is given below:

Definition 5. *[22] Let* X *be a non-empty set. A bipolar fuzzy set* A *in* X *is an object having the form*

$$A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$$

where $\mu_A^P : X \longrightarrow [0, 1]$ *and* $\mu_A^N : X \longrightarrow [-1, 0]$ *are mappings.*

We use the positive membership degree $\mu_A^P(x)$ to denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set A, and the negative membership degree $\mu_A^N(x)$ to denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set A. If $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A. If $\mu_A^P(x) = 0$ and $\mu_A^N(x) \neq 0$, it is the situation that x does not satisfy the property of A but somewhat satisfies the counter property of A. It is possible for an element x to be such that $\mu_A^P(x) \neq 0$ and $\mu_A^N(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

For the sake of simplicity, we shall use the symbol $A = (\mu_A^P, \mu_A^N)$ for the bipolar fuzzy set $A = \{(x, \mu_A^P(x), \mu_A^N(x)) | x \in X\}$.

Definition 6. *[22] Let* $A = (\mu_A^P(x), \mu_A^N(x))$ *and* $B = (\mu_B^P(x), \mu_B^N(x))$ *be two bipolar fuzzy sets in* X*. Then* $A \subseteq B$ *if and only if* $\mu_A^P(x) \leq \mu_B^P(x)$ *and* $\mu_A^N(x) \geq \mu_B^N(x)$, *for all* $x \in X$.

Doubt bipolar fuzzy subalgebras and doubt bipolar fuzzy ideals are extensions of doubt fuzzy subalgebras and doubt fuzzy ideals which are defined by Al-Masarwah and Ahmad [5] as follows:

Definition 7. *[5] A bipolar fuzzy set* $A = (\mu_A^P, \mu_A^N)$ *in* X *is called a doubt bipolar fuzzy subalgebra of* X *if it satisfies:*

(i) $\mu_A^P(x * y) \leq \max\{\mu_A^P(x), \mu_A^P(y)\}$,

(ii) $\mu_A^N(x * y) \geq \min\{\mu_A^N(x), \mu_A^N(y)\}$, *for all* $x, y \in X$.

Definition 8. *[5] A bipolar fuzzy set* $A = (\mu_A^P, \mu_A^N)$ *in* X *is called a doubt bipolar fuzzy ideal of* X *if it satisfies:*

(i) $\mu_A^P(0) \leq \mu_A^P(x)$ *and* $\mu_A^N(0) \geq \mu_A^N(x)$,

(ii) $\mu_A^P(x) \leq \max\{\mu_A^P(x * y), \mu_A^P(y)\}$,

(iii) $\mu_A^N(x) \geq \min\{\mu_A^N(x * y), \mu_A^N(y)\}$, *for all* $x, y \in X$.

3. Doubt bipolar fuzzy H-ideals

In this section, the concepts of doubt bipolar fuzzy H-ideals were introduced by Al-Masarwah and Ahmad [6] will be used to study and investigate several properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras.

Definition 9. *[6] Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of X, then A is called a doubt bipolar fuzzy H-ideal of X if it satisfies:*

(i) $\mu_A^P(0) \leq \mu_A^P(x)$ *and* $\mu_A^N(0) \geq \mu_A^N(x)$,

(ii) $\mu_A^P(x * z) \leq \max\{\mu_A^P(x * (y * z)), \mu_A^P(y)\}$,

(iii) $\mu_A^N(x * z) \geq \min\{\mu_A^N(x * (y * z)), \mu_A^N(y)\}$, *for all* $x, y, z \in X$.

Definition 10. *Let M be a nonempty subset of X. A bipolar fuzzy set $\widetilde{C}_M = (\widetilde{C}_M^P, \widetilde{C}_M^N)$ expressed by*

$$\widetilde{C}_M^P(x) = \begin{cases} 0, & x \in M, \\ 1, & x \notin M, \end{cases} \quad \text{and} \quad \widetilde{C}_M^P(x) = \begin{cases} 0, & x \in M, \\ -1, & x \notin M. \end{cases}$$

is called a doubt bipolar fuzzy characteristic function.

Lemma 1. *Let M be a nonempty subset of X. Then the constant 0 of X is in M if and only if $\widetilde{C}_M^P(0) \leq \widetilde{C}_M^P(x)$ and $\widetilde{C}_M^N(0) \geq \widetilde{C}_M^N(x)$, for all $x \in X$.*

Proof. If $0 \in M$, then $\widetilde{C}_M^P(0) = 0$ and $\widetilde{C}_M^N(0) = 0$. Thus, $\widetilde{C}_M^P(0) = 0 \leq \widetilde{C}_M^P(x)$ and $\widetilde{C}_M^N(0) = 0 \geq \widetilde{C}_M^N(x)$, for all $x \in X$.

Conversely, assume that $\widetilde{C}_M^P(0) \leq \widetilde{C}_M^P(x)$ and $\widetilde{C}_M^N(0) \geq \widetilde{C}_M^N(x)$, for all $x \in X$. Since M is a nonempty subset of X, we have $m \in M$ for some $m \in X$. Then $\widetilde{C}_M^P(0) \leq \widetilde{C}_M^P(m) = 0$ and $\widetilde{C}_M^N(0) \geq \widetilde{C}_M^P(m) = 0$. Thus, $\widetilde{C}_M^P(0) = 0$ and $\widetilde{C}_M^N(0) = 0$. So $0 \in M$.

Theorem 1. *Let M be a nonempty subset of X. Then M is an H-ideal of X if and only if the doubt bipolar fuzzy characteristic function $\widetilde{C}_M = (\widetilde{C}_M^P, \widetilde{C}_M^N)$ is a doubt bipolar fuzzy H-ideal of X.*

Proof. Assume that M is an H-ideal of X. Since $0 \in M$, it follows from Lemma 1 that $\widetilde{C}_M^P(0) \leq \widetilde{C}_M^P(x)$ and $\widetilde{C}_M^N(0) \geq \widetilde{C}_M^N(x)$, for all $x \in X$. Next, let $x, y, z \in X$. Then we have the following cases:

Case(1). Suppose that $x * (y * z) \in M$ and $y \in M$, then $\widetilde{C}_M^P(x * (y * z)) = 0, \widetilde{C}_M^P(y) = 0, \widetilde{C}_M^N(x * (y * z)) = 0$, and $\widetilde{C}_M^N(y) = 0$. Therefore,

$$\max\{\widetilde{C}_M^P(x * (y * z)), \widetilde{C}_M^P(y)\} = \max\{0, 0\} = 0, \text{ and}$$
$$\min\{\widetilde{C}_M^N(x * (y * z)), \widetilde{C}_M^N(y)\} = \min\{0, 0\} = 0.$$

Since $x*(y*z) \in M$ and $y \in M$, we have $x*z \in M$. So $\widetilde{C}_M^P(x*z) = 0$ and $\widetilde{C}_M^N(x*z) = 0$. Therefore,

$$\widetilde{C}_M^P(x*z) = 0 \leq 0 = \max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\}, \text{ and}$$
$$\widetilde{C}_M^N(x*z) = 0 \geq 0 = \min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\}.$$

Case(2). Suppose that $x*(y*z) \notin M$ and $y \notin M$, then $\widetilde{C}_M^P(x*(y*z)) = 1, \widetilde{C}_M^P(y) = 1, \widetilde{C}_M^N(x*(y*z)) = -1$, and $\widetilde{C}_M^N(y) = -1$. So,

$$\max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\} = \max\{1,1\} = 1, \text{ and}$$
$$\min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\} = \min\{-1,-1\} = -1.$$

Therefore,

$$\widetilde{C}_M^P(x*z) \leq 1 = \max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\}, \text{ and}$$
$$\widetilde{C}_M^N(x*z) \geq -1 = \min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\}.$$

Case(3). Suppose that $x*(y*z) \in M$ or $y \in M$. Then we have two subcases:

Subcase (3a). If $x*(y*z) \in M$ and $y \notin M$, then $\widetilde{C}_M^P(x*(y*z)) = 0, \widetilde{C}_M^P(y) = 1, \widetilde{C}_M^N(x*(y*z)) = 0$, and $\widetilde{C}_M^N(y) = -1$. So,

$$\max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\} = \max\{0,1\} = 1, \text{ and}$$
$$\min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\} = \min\{0,-1\} = -1.$$

Therefore,

$$\widetilde{C}_M^P(x*z) \leq 1 = \max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\}, \text{ and}$$
$$\widetilde{C}_M^N(x*z) \geq -1 = \min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\}.$$

Subcase (3b). If $x*(y*z) \notin M$ and $y \in M$, then $\widetilde{C}_M^P(x*(y*z)) = 1, \widetilde{C}_M^P(y) = 0, \widetilde{C}_M^N(x*(y*z)) = -1$, and $\widetilde{C}_M^N(y) = 0$. So,

$$\max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\} = \max\{1,0\} = 1, \text{ and}$$
$$\min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\} = \min\{-1,0\} = -1.$$

Therefore,

$$\widetilde{C}_M^P(x*z) \leq 1 = \max\{\widetilde{C}_M^P(x*(y*z)), \widetilde{C}_M^P(y)\}, \text{ and}$$
$$\widetilde{C}_M^N(x*z) \geq -1 = \min\{\widetilde{C}_M^N(x*(y*z)), \widetilde{C}_M^N(y)\}.$$

Hence, $\widetilde{C}_M = (\widetilde{C}_M^P, \widetilde{C}_M^N)$ is a doubt bipolar fuzzy H-ideal of X.

Conversely, assume that $\widetilde{C}_M = (\widetilde{C}_M^P, \widetilde{C}_M^N)$ is a doubt bipolar fuzzy H-ideal of X. Since $\widetilde{C}_M^P(0) \leq \widetilde{C}_M^P(x)$ and $\widetilde{C}_M^N(0) \geq \widetilde{C}_M^N(x)$, for all $x \in X$. It follows that from Lemma 1 that

$0 \in M$. Next, let $x, y, z \in X$ such that $x * (y * z) \in M$ and $y \in M$. To show that $x * z \in M$, assume that $x * z \notin M$. Then $\widetilde{C}_M^P(x * z) = 1$ and $\widetilde{C}_M^N(x * z) = -1$. So

$$\begin{aligned} 1 = \widetilde{C}_M^P(x * z) &\leq \max\{\widetilde{C}_M^P(x * (y * z)), \widetilde{C}_M^P(y)\}, \text{ and} \\ -1 = \widetilde{C}_M^N(x * z) &\geq \min\{\widetilde{C}_M^N(x * (y * z)), \widetilde{C}_M^N(y)\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\max\{\widetilde{C}_M^P(x * (y * z)), \widetilde{C}_M^P(y)\} = 1, \text{ and} \\ &\min\{\widetilde{C}_M^N(x * (y * z)), \widetilde{C}_M^N(y)\} = -1. \end{aligned}$$

This implies that $\widetilde{C}_M^P(x*(y*z)) = 1$ or $\widetilde{C}_M^P(y) = 1$ and $\widetilde{C}_M^N(x*(y*z)) = -1$ or $\widetilde{C}_M^N(y) = -1$. So, $x * (y * z) \notin M$ or $y \notin M$, a contradiction. Hence, $x * z \in M$, and thus M is an H-ideal of X.

Theorem 2. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of associative BCK/BCI-algebras X. If the inequality $x * y \leq z$ holds in X, then $\mu_A^P(x * y) \leq \mu_A^P(z)$ and $\mu_A^N(x * y) \geq \mu_A^N(z)$ for all $x, y, z \in X$.*

Proof. Let $x, y, z \in X$ such that $x * y \leq z$. Then $(x * y) * z = 0$ and since A is a doubt bipolar fuzzy H-ideal of X, so

$$\begin{aligned} \mu_A^P(x * y) &\leq \max\{\mu_A^P(x * (z * y)), \mu_A^P(z)\} \\ &= \max\{\mu_A^P((x * z) * y), \mu_A^P(z)\} \\ &= \max\{\mu_A^P((x * y) * z), \mu_A^P(z)\} \\ &= \max\{\mu_A^P(0), \mu_A^P(z)\} \\ &= \mu_A^P(z). \end{aligned}$$

Therefore, $\mu_A^P(x * y) \leq \mu_A^P(z)$ for all $x, y, z \in X$. Again,

$$\begin{aligned} \mu_A^N(x * y) &\geq \min\{\mu_A^N(x * (z * y)), \mu_A^N(z)\} \\ &= \min\{\mu_A^N((x * z) * y), \mu_A^N(z)\} \\ &= \min\{\mu_A^N((x * y) * z), \mu_A^N(z)\} \\ &= \min\{\mu_A^N(0), \mu_A^N(z)\} \\ &= \mu_A^N(z). \end{aligned}$$

Therefore, $\mu_A^N(x * y) \geq \mu_A^N(z)$ for all $x, y, z \in X$.

Proposition 1. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X. If the inequality $x \leq y$ holds in X, then $\mu_A^P(x) \leq \mu_A^P(y)$ and $\mu_A^N(x) \geq \mu_A^N(y)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$ such that $x \leq y$. Then $x * y = 0$. Now

$$\begin{aligned}\mu_A^P(x) = \mu_A^P(x * 0) &\leq \max\{\mu_A^P(x * (y * 0)), \mu_A^P(y)\} \\ &= \max\{\mu_A^P(x * y), \mu_A^P(y)\} \\ &= \max\{\mu_A^P(0), \mu_A^P(y)\} \\ &= \mu_A^P(y).\end{aligned}$$

Therefore, $\mu_A^P(x) \leq \mu_A^P(y)$ for all $x, y \in X$. Again,

$$\begin{aligned}\mu_A^N(x) = \mu_A^N(x * 0) &\geq \min\{\mu_A^N(x * (y * 0)), \mu_A^N(y)\} \\ &= \min\{\mu_A^N(x * y), \mu_A^N(y)\} \\ &= \min\{\mu_A^N(0), \mu_A^N(y)\} \\ &= \mu_A^N(y).\end{aligned}$$

Therefore, $\mu_A^N(x) \geq \mu_A^N(y)$ for all $x, y \in X$.

Proposition 2. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of a BCK-algebra X, then $\mu_A^P(0 * (0 * x)) \leq \mu_A^P(x)$ and $\mu_A^N(0 * (0 * x)) \geq \mu_A^N(x)$ for all $x \in X$.*

Proof. Note that

$$\begin{aligned}\mu_A^P(0 * (0 * x)) &\leq \max\{\mu_A^P(0 * (x * (0 * x))), \mu_A^P(x)\} \\ &= \max\{\mu_A^P(0 * (x * 0)), \mu_A^P(x)\} \\ &= \max\{\mu_A^P(0 * x), \mu_A^P(x)\} \\ &= \max\{\mu_A^P(0), \mu_A^P(x)\} \\ &= \mu_A^P(x), \text{ for all } x \in X.\end{aligned}$$

Therefore, $\mu_A^P(0 * (0 * x)) \leq \mu_A^P(x)$ for all $x \in X$. Again,

$$\begin{aligned}\mu_A^N(0 * (0 * x)) &\geq \min\{\mu_A^N(0 * (x * (0 * x))), \mu_A^N(x)\} \\ &= \min\{\mu_A^N(0 * (x * 0)), \mu_A^N(x)\} \\ &= \min\{\mu_A^N(0 * x), \mu_A^N(x)\} \\ &= \min\{\mu_A^N(0), \mu_A^N(x)\} \\ &= \mu_A^N(x), \text{ for all } x \in X.\end{aligned}$$

Therefore, $\mu_A^N(0 * (0 * x)) \geq \mu_A^N(x)$ for all $x \in X$.

Theorem 3. *Every doubt bipolar fuzzy H-ideal of X is both a doubt bipolar fuzzy subalgebra of X and a doubt bipolar fuzzy ideal of X.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X, then for any $x, y \in X$, we have

$$\begin{aligned} \mu_A^P(x * y) &\leq \max\{\mu_A^P(x * (y * y)), \mu_A^P(y)\} \\ &= \max\{\mu_A^P(x * 0), \mu_A^P(y)\} \\ &= \max\{\mu_A^P(x), \mu_A^P(y)\}, \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x * y) &\geq \min\{\mu_A^N(x * (y * y)), \mu_A^N(y)\} \\ &= \min\{\mu_A^N(x * 0), \mu_A^N(y)\} \\ &= \min\{\mu_A^N(x), \mu_A^N(y)\}. \end{aligned}$$

Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X. Also, since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X. Then $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$. Now, since $x * 0 = x$ for all $x \in X$, we obtain

$$\begin{aligned} \mu_A^P(x) = \mu_A^P(x * 0) &\leq \max\{\mu_A^P(x * (y * 0)), \mu_A^P(y)\} \\ &= \max\{\mu_A^P(x * y), \mu_A^P(y)\}, \end{aligned}$$

and

$$\begin{aligned} \mu_A^N(x) = \mu_A^N(x * 0) &\geq \min\{\mu_A^N(x * (y * 0)), \mu_A^N(y)\} \\ &= \min\{\mu_A^N(x * y), \mu_A^N(y)\}. \end{aligned}$$

Therefore, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal of X.

The converse of Theorem 3 is not true. That is every doubt bipolar fuzzy subalgebra of X and doubt bipolar fuzzy ideal of X is not necessarily to be a doubt bipolar fuzzy H-ideal of X. It can be verified by the following example:

Example 1. *Let $X = \{0, a, b\}$ be a BCI-algebra with the Cayley table which is appeared in Table 1.*

Table 1: Cayley table for the $*$-operation.

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X as follows:

$$\mu_A^P(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.8, & \text{if } x = a,\ b, \end{cases} \quad \text{and} \quad \mu_A^N(x) = \begin{cases} -0.2, & \text{if } x = 0 \\ -0.4, & \text{if } x = a,\ b. \end{cases}$$

Then, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X and a doubt bipolar fuzzy ideal of X. But $A = (\mu_A^P, \mu_A^N)$ is not a doubt bipolar fuzzy H-ideal of X, since $\mu_A^P(a * b) = 0.8 \nleq \max\{\mu_A^P(a * (0 * b)), \mu_A^P(0)\} = \mu_A^P(0) = 0$.

In the following example, we have a doubt bipolar fuzzy subalgebra of X but it is neither a doubt bipolar fuzzy ideal of X nor a doubt bipolar fuzzy H-ideal of X.

Example 2. *Let $X = \{0, a, b, c\}$ be a BCK-algebra with the Cayley table which is appeared in Table 2.*

Table 2: Cayley table for the $*$-operation.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X as follows:

$$\mu_A^P(x) = \begin{cases} 0.5, & \text{if } x = 0, a, c \\ 0.6, & \text{if } x = b, \end{cases}$$

and

$$\mu_A^N(0) = \mu_A^N(a) = \mu_A^N(b) = \mu_A^N(c) = -0.5.$$

Then by routine calculation we know that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy subalgebra of X. But, it is not a doubt bipolar fuzzy ideal of X, since $\mu_A^P(b) = 0.6, \mu_A^P(b) = 0.6 \nleq 0.5 = \max\{\mu_A^P(b * a), \mu_A^P(a)\}$, and hence it is not a doubt bipolar fuzzy H- ideal of X, since $\mu_A^P(b * c) = \mu_A^P(b) = 0.6$, $\mu_A^P(b) = 0.6 \nleq 0.5 = \max\{\mu_A^P(b * (a * c)), \mu_A^P(a)\}$.

Now, we give a condition for the bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$, which is a doubt bipolar fuzzy ideal of X to be a doubt bipolar fuzzy H-ideal of X.

Theorem 4. *In associative BCK/BCI-algebras X, every doubt bipolar fuzzy ideal is a doubt bipolar fuzzy H-ideal of X.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy ideal of X. Then $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$, for all $x \in X$. Now, since X is an associative, then $x * (y * z) = (x * y) * z$, for $x, y, z \in X$. Now,

$$\begin{aligned} \max\{\mu_A^P(x * (y * z)), \mu_A^P(y)\} &= \max\{\mu_A^P((x * y) * z), \mu_A^P(y)\} \\ &= \max\{\mu_A^P((x * z) * y), \mu_A^P(y)\} \\ &\geq \mu_A^P(x * z). \end{aligned}$$

Therefore, $\mu_A^P(x * z) \leq \max\{\mu_A^P(x * (y * z)), \mu_A^P(y)\}$ for all $x, y, z \in X$. Again,

$$\begin{aligned} \min\{\mu_A^N(x * (y * z)), \mu_A^N(y)\} &= \min\{\mu_A^N((x * y) * z), \mu_A^N(y)\} \\ &= \min\{\mu_A^N((x * z) * y), \mu_A^N(y)\} \\ &\leq \mu_A^N(x * z). \end{aligned}$$

Therefore, $\mu_A^N(x*z) \geq \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}$ for all $x, y, z \in X$. Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X.

Example 3. *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the Cayley table which is appeared in Table 3.*

Table 3: Cayley table for the $*$-operation.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

Here, X is an associative BCK-algebra. Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X as follows:

$$\mu_A^P(x) = \begin{cases} 0, & \text{if } x = 0 \\ 0.6, & \text{if } x = a \\ 0.4, & \text{if } x = b \\ 0.8, & \text{if } x = c \\ 0.9, & \text{if } x = d, \end{cases}$$

and

$$\mu_A^N(0) = \mu_A^N(a) = \mu_A^N(b) = \mu_A^N(c) = \mu_A^N(d) = r, \text{ where } r \in [-1, 0].$$

Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy ideal as well as a doubt bipolar fuzzy H-ideal of X.

Corollary 1. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X. Then the sets $D_{\mu_A^P} = \{x \in X : \mu_A^P(x) = \mu_A^P(0)\}$ and $D_{\mu_A^N} = \{x \in X : \mu_A^N(x) = \mu_N^P(0)\}$ are H-ideals of X.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X. Obviously, $0 \in D_{\mu_A^P}$ and $0 \in D_{\mu_A^N}$. Now, let $x, y, z \in D_{\mu_A^P}$ such that $x*(y*z), y \in D_{\mu_A^P}$. Then $\mu_A^P(x*(y*z)) = \mu_A^P(0) = \mu_A^P(y)$. Now, $\mu_A^P(x*z) \leq \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\} = \mu_A^P(0)$. Again, since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of $X, \mu_A^P(0) \leq \mu_A^P(x*z)$. Therefore, $\mu_A^P(0) = \mu_A^P(x*z)$. It follows that $x*z \in D_{\mu_A^P}$, for all $x, y, z \in X$. Therefore, $D_{\mu_A^P}$ is an H-ideal of X. Also, let $x, y, z \in D_{\mu_A^N}$ such that $x*(y*z), y \in D_{\mu_A^N}$. Then $\mu_A^N(x*(y*z) = \mu_A^N(0) = \mu_A^N(y)$. Now, $\mu_A^N(x*z) \geq \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\} = \mu_A^N(0)$. Again, since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of $X, \mu_A^N(0) \geq \mu_A^N(x*z)$. Therefore, $\mu_A^N(0) = \mu_A^N(x*z)$. It follows that $x*z \in D_{\mu_A^N}$, for all $x, y, z \in X$. Therefore, $D_{\mu_A^N}$ is an H-ideal of X.

Lemma 2. *Let μ be a fuzzy set in X. Then the following statements holds, for all $x, y \in X$,* *(1)* $1 - \max\{\mu(x), \mu(y)\} = \min\{1 - \mu(x), 1 - \mu(y)\}$,

(2) $1 - \min\{\mu(x), \mu(y)\} = \max\{1 - \mu(x), 1 - \mu(y)\}$.

Proof. (1) If $\max\{\mu(x), \mu(y)\} = \mu(x)$, then $\mu(y) \leq \mu(x)$. Thus, $1 - \mu(y) \geq 1 - \mu(x)$, so $\min\{1-\mu(x), 1-\mu(y)\} = 1-\mu(x) = 1-\max\{\mu(x), \mu(y)\}$. Similarly, if $\max\{\mu(x), \mu(y)\} = \mu(y)$, then

$$\min\{1 - \mu(x), 1 - \mu(y)\} = 1 - \mu(y) = 1 - \max\{\mu(x), \mu(y)\}.$$

(2) If $\min\{\mu(x), \mu(y)\} = \mu(x)$, then $\mu(x) \leq \mu(y)$. Thus, $1 - \mu(x) \geq 1 - \mu(y)$, so $\max\{1 - \mu(x), 1 - \mu(y)\} = 1 - \mu(x) = 1 - \min\{\mu(x), \mu(y)\}$. Similarly, if $\min\{\mu(x), \mu(y)\} = \mu(y)$, then

$$\max\{1 - \mu(x), 1 - \mu(y)\} = 1 - \mu(y) = 1 - \min\{\mu(x), \mu(y)\}.$$

Remark 1. *$A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set defined on any universe set X if and only if μ_A^P and $-\mu_A^N$ are fuzzy subsets of X.*

Theorem 5. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X if and only if the fuzzy subsets μ_A^P and $-\mu_A^N$ are doubt fuzzy H-ideals of X.*

Proof. Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X. Then clearly μ_A^P is a doubt fuzzy H-ideal of X. Also, $\mu_A^N(0) \geq \mu_A^N(x)$ and $\mu_A^N(x*z) \geq \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}$, implies that, $-\mu_A^N(0) \leq -\mu_A^N(x)$ and

$$\begin{aligned} -\mu_A^N(x * z) &\leq -\min\{\mu_A^N(x * (y * z)), \mu_A^N(y)\} \\ &= \max\{-\mu_A^N(x * (y * z)), -\mu_A^N(y)\}. \end{aligned}$$

Therefore, $-\mu_A^N$ is a doubt fuzzy H-ideal of X.

Conversely, assume that μ_A^P and $-\mu_A^N$ are doubt fuzzy H-ideals of X. So that $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^P(x*z) \leq \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\}$, for all $x, y, z \in X$. Now, we prove that

$$\mu_A^N(0) \geq \mu_A^N(x) \text{ and } \mu_A^N(x * z) \geq \min\{\mu_A^N(x * (y * z)), \mu_A^N(y)\} \text{ for all } x, y, z \in X.$$

Since $-\mu_A^N$ is a doubt fuzzy H-ideal of X, so that $-\mu_A^N(0) \leq -\mu_A^N(x)$ and

$$\begin{aligned} -\mu_A^N(x * z) &\leq \max\{-\mu_A^N(x * (y * z)), -\mu_A^N(y)\} \\ &= -\min\{\mu_A^N(x * (y * z)), \mu_A^N(y)\}, \end{aligned}$$

implies that, $\mu_A^N(0) \geq \mu_A^N(x)$ and $\mu_A^N(x*z) \geq \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}$ for all $x, y, z \in X$. Therefore, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X.

Theorem 6. *A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X if and only if $\triangle A = (\mu_A^P, -\mu_A^P)$ and $\triangledown A = (-\mu_A^N, \mu_A^N)$ are also doubt bipolar fuzzy H-ideals of X.*

Proof. $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X if and only if the fuzzy subsets μ_A^P and $-\mu_A^N$ are doubt fuzzy H-ideals of X by Theorem 5. That is, if and only if $\triangle A = (\mu_A^P, -\mu_A^P)$ and $\triangledown A = (-\mu_A^N, \mu_A^N)$ are also doubt bipolar fuzzy H-ideals of X by definition of $\triangle A$ and $\triangledown A$.

4. Characterizations of doubt bipolar fuzzy H-ideals

In this section, we define a doubt positive t-level cut set and a doubt negative s-level cut set of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras. We investigate characterizations of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras by means of doubt positive t-level cut set, doubt negative s-level cut set and H-Artin BCK/BCI-algebras.

Definition 11. *Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of a BCK/BCI-algebra X, and $(s,t) \in [-1,0] \times [0,1]$. Then the doubt positive t-level cut set and the doubt negative s-level cut set of A are as follows:*

$$A_t^P = \{x \in X : \mu_A^P(x) \leq t\} \text{ and } A_t^N = \{x \in X : \mu_A^N(x) \geq s\}.$$

The set $S_{(s,t)} = \{x \in X : \mu_A^P(x) \leq t$ and $\mu_A^N(x) \geq s\}$ is called a doubt (s,t)-level cut set of A. For every $\gamma \in [0,1]$, the set $A_\gamma^P \cap A_{-\gamma}^N$ is called a doubt γ-level cut set of A.

From Definition 11, we can easily obtained the relation between a doubt bipolar fuzzy H-ideal and H-ideal in BCK/BCI-algebras.

Theorem 7. *For a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X, the following are equivalent:*

1. *$A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X.*
2. *$A = (\mu_A^P, \mu_A^N)$ satisfies the following assertions:*
   - *i.* $(\forall t \in [0,1])(A_t^P \neq \emptyset \Rightarrow A_t^P = \{x \in X : \mu_A^P(x) \leq t\}$ *is an H-ideal of X).*
   - *ii.* $(\forall s \in [-1,0])(A_t^N \neq \emptyset \Rightarrow A_s^N = \{x \in X : \mu_A^N(x) \geq s\}$ *is an H-ideal of X).*

Proof. $(1 \Rightarrow 2)$ Let $A = (\mu_A^P, \mu_A^N)$ be a doubt bipolar fuzzy H-ideal of X. Let $t \in [0,1]$ and $s \in [-1,0]$ such that $A_t^P \neq \emptyset$ and $A_s^N \neq \emptyset$. Then there exists $a \in A_t^P$ and $b \in A_s^N$, that is $\mu_A^P(a) \leq t$ and $\mu_A^N(b) \geq s$. Since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X, we have $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$, for all $x \in X$. Thus, $\mu_A^P(0) \leq \mu_A^P(a) \leq t$ and $\mu_A^N(0) \geq \mu_A^N(b) \geq s$, so $0 \in A_t^P$ and $0 \in A_s^N$. Let $x, y, z \in X$ such that $x * (y * z) \in A_t^P$ and $y \in A_t^P$. Then $\mu_A^P(x * (y * z)) \leq t$ and $\mu_A^P(y) \leq t$. Using Definition 9, we have

$$\begin{aligned} \mu_A^P(x * z) &\leq \max\{\mu_A^P(x * (y * z)), \mu_A^P(y)\} \\ &\leq \max\{t, t\} = t, \end{aligned}$$

so $x*z \in A_t^P$. Hence, A_t^P is an H-ideal of X. Finally, Let $x, y, z \in X$ such that $x*(y*z) \in A_s^N$ and $y \in A_s^N$. Then $\mu_A^N(x * (y * z) \geq s$ and $\mu_A^N(y)) \geq s$. It follows that

$$\begin{aligned} \mu_A^N(x * z) &\geq \min\{\mu_A^N(x * (y * z)), \mu_A^N(y)\} \\ &\geq \min\{s, s\} = s, \end{aligned}$$

so $x * z \in A_s^N$. Hence, A_s^N is an H-ideal of X.

($2 \Rightarrow 1$) Suppose that $A_t^P \neq \emptyset$ and $A_t^N \neq \emptyset$ are H-ideals of X for all $t \in [0,1]$ and $s \in [-1,0]$.
Assume that there exists $a \in X$ such that $\mu_A^P(0) > \mu_A^P(a)$ and $\mu_A^N(0) < \mu_A^N(a)$. Taking

$$t_o = \frac{1}{2}(\mu_A^P(0) + \mu_A^P(a)),$$
$$s_o = \frac{1}{2}(\mu_A^N(0) + \mu_A^N(a)),$$

implies that $\mu_A^P(a) < t_o < \mu_A^P(0)$ and $\mu_A^N(a) > s_o > \mu_A^N(0)$. This shows that $0 \notin A_t^P$ and $0 \notin A_s^N$, which leads to a contradiction. Therefore, $\mu_A^P(0) \leq \mu_A^P(x)$ and $\mu_A^N(0) \geq \mu_A^N(x)$ for all $x \in X$. Now, suppose that there are $a, b, c \in X$ such that $\mu_A^P(a * c) > \max\{\mu_A^P(a * (b * c)), \mu_A^P(b)\}$. Then by taking

$$t_1 = \frac{1}{2}(\mu_A^P(a * c) + \max\{\mu_A^P(a * (b * c)), \mu_A^P(b)\}),$$

we have $\max\{\mu_A^P(a*(b*c), \mu_A^P(b)\} < t_1 < \mu_A^P(a*c)$. Hence $a*c \notin A_{t_1}^P, a*(b*c)) \in A_{t_1}^P$ and $b \in A_{t_1}^P$, that is $A_{t_1}^P$ is not H- ideal of X, which a contradiction. Therefore, $\mu_A^P(x * y) \leq \max\{\mu_A^P(x * (y * z)), \mu_A^P(y)\}$ for all $x, y, z \in X$. Finally, assume that $p, q, r \in X$ such that $\mu_A^N(p * r) < \min\{\mu_A^N(p * (q * r)), \mu_A^N(q)\}$. Taking

$$s_1 = \frac{1}{2}(\mu_A^N(p * r) + \min\{\mu_A^N(p * (q * r)), \mu_A^N(q)\}),$$

then $\mu_A^N(p*r) < s_1 < \min\{\mu_A^N(p*(q*r)), \mu_A^N(q)\}$. Therefore, $p*(q*r) \in A_{s_1}^N$ and $q \in A_{t_1}^N$ but $p*r \notin A_{t_1}^N$. Again a contradiction. Thus, $\mu_A^N(x*z) \geq \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}$ for all $x, y, z \in X$. Hence, $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X.

Example 4. *Let $X = \{0, a, b, c\}$ be a BCK-algebra with the Cayley table which is appeared in Table 4.*

Table 4: Cayley table for the $*$-operation.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	a	0	0
c	c	a	c	0

Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X as follows:

$$\mu_A^P(x) = \begin{cases} 0.2, & \text{if } x = 0 \\ 0.6, & \text{if } x = a \\ 0.8, & \text{if } x = b \\ 0.7, & \text{if } x = c, \end{cases} \quad \text{and} \quad \mu_A^N(x) = \begin{cases} -0.3, & \text{if } x = 0, a, c \\ -0.5, & \text{if } x = b, \end{cases}$$

which is not a doubt bipolar fuzzy H- ideal of X, *since* $\mu_A^P(b*0) = \mu_A^P(b) = 0.8 \nleq \max\{\mu_A^P((b*(a*0))), \mu_A^P(a)\} = \max\{\mu_A^P(a), \mu_A^P(a)\} = 0.6$. *Now, for* $t = 0.75$ *and* $s = -0.45$, *we get* $A_t^P = A_s^N = \{0, a, c\}$ *which are not H-ideals of* X, *since* $a \in \{0, a, c\}$ *and* $b*(a*0) = b*a = a \in \{0, a, c\}$, *but* $b*0 = b \notin \{0, a, c\}$.

Corollary 2. *If* $A = (\mu_A^P, \mu_A^N)$ *is a doubt bipolar fuzzy H-ideal of* X, *then the doubt* γ*-level cut set of* $A = (\mu_A^P, \mu_A^N)$ *is a doubt bipolar fuzzy H-ideal of* X, *for all* $\gamma \in [0, 1]$.

Corollary 3. *If* $A = (\mu_A^P, \mu_A^N)$ *is a doubt bipolar fuzzy H-ideal of* X. *Then* $S_{(s,t)}$ *is an H-ideal of* X *for all* $(s, t) \in [-1, 0] \times [0, 1]$. *In particular, the nonempty doubt* γ*-level cut set of* $A = (\mu_A^P, \mu_A^N)$ *is an H-ideal of* X *for all* $\gamma \in [0, 1]$.

Theorem 8. *If* $A = (\mu_A^P, \mu_A^N)$ *is a doubt bipolar fuzzy H-ideal of* X *and* $\mu_A^P(z) + \mu_A^N(z) \leq 0$ *for all* $z \in X$, *then* $A_\gamma^P \cup A_{-\gamma}^N$ *is an H- ideal of* X *for all* $\gamma \in [0, 1]$.

Proof. Given that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X and $\mu_A^P(z) + \mu_A^N(z) \leq 0$ for all $z \in X$. Assume that A_γ^P and $A_{-\gamma}^N$ are nonempty for all $\gamma \in [0, 1]$. Then by Theorem 7, A_γ^P and $A_{-\gamma}^N$ are H-ideals of X.

Let $x, y, z \in X$ such that $x*(y*z) \in A_\gamma^P \cup A_{-\gamma}^N$ and $y \in A_\gamma^P \cup A_{-\gamma}^N$. Here we have four cases to prove the theorem:

(i) $x*(y*z) \in A_\gamma^P$ and $y \in A_\gamma^P$,

(ii) $x*(y*z) \in A_\gamma^P$ and $y \in A_{-\gamma}^N$,

(iii) $x*(y*z) \in A_{-\gamma}^N$ and $y \in A_\gamma^P$,

(iv) $x*(y*z) \in A_{-\gamma}^N$ and $y \in A_{-\gamma}^N$.

Case(i). If $x*(y*z) \in A_\gamma^P$ and $y \in A_\gamma^P$, implies that $\mu_A^P(x*(y*z)) \leq \gamma$ and $\mu_A^P(y) \leq \gamma$. Since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X, it follows that

$$\mu_A^P(x*z) \leq \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\} \leq \gamma.$$

Therefore, $x*z \in A_\gamma^P \subseteq A_\gamma^P \cup A_{-\gamma}^N$.

Case(ii). If $x*(y*z) \in A_\gamma^P$ and $y \in A_{-\gamma}^N$, implies that $\mu_A^P(x*(y*z)) \leq \gamma$ and $\mu_A^N(y) \geq -\gamma$. Since $\mu_A^P(y) + \mu_A^N(y) \leq 0$, so $\mu_A^P(y) \leq -\mu_A^N(y) \leq \gamma$, it follows that

$$\begin{aligned} \mu_A^P(x*z) &\leq \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\} \\ &\leq \max\{\mu_A^P(x*(y*z)), -\mu_A^N(y)\} \leq \gamma. \end{aligned}$$

Therefore, $x*z \in A_\gamma^P \subseteq A_\gamma^P \cup A_{-\gamma}^N$.

Case(iii). If $x*(y*z) \in A_{-\gamma}^N$ and $y \in A_\gamma^P$, implies that $\mu_A^N(x*(y*z)) \geq -\gamma$ and $\mu_A^P(y) \leq \gamma$. Since $\mu_A^P(x*(y*z)) + \mu_A^N(x*(y*z)) \leq 0$, so $\mu_A^P(x*(y*z)) \leq -\mu_A^N(x*(y*z)) \leq \gamma$, it follows that

$$\mu_A^P(x*z) \leq \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\}$$

$$\leq \max\{-\mu_A^N(x*(y*z)), \mu_A^P(y)\} \leq \gamma.$$

Therefore, $x*z \in A_\gamma^P \subseteq A_\gamma^P \cup A_{-\gamma}^N$.

Case(iv). If $x*(y*z) \in A_{-\gamma}^N$ and $y \in A_{-\gamma}^N$, implies that $\mu_A^N(x*(y*z)) \geq -\gamma$ and $\mu_A^N(y) \geq -\gamma$. Since $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X, it follows that

$$\mu_A^N(x*z) \geq \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\} \geq -\gamma.$$

Therefore, $x*z \in A_{-\gamma}^N \subseteq A_\gamma^P \cup A_{-\gamma}^N$. Hence, $A_\gamma^P \cup A_{-\gamma}^N$ is an H-ideal of X.

Definition 12. *[33] A BCK/BCI-algebra X is said to satisfy the H-ascending (resp. H-descending) chain condition (briefly, H-ACC (resp. H-DCC)) if for every ascending (resp. descending) sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ (resp. $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$) of H-ideals of X there exists a natural number n such that $I_n = I_k$ for all $n \geq k$. If X satisfies H-DCC, we say that X is an H-Artin BCK/BCI-algebras.*

In the next two theorems, we investigate characterizations of H-Artin BCK/BCI-algebras in terms of doubt bipolar fuzzy H-ideals.

Theorem 9. *Let X be a BCK/BCI-algebra satisfying H-DCC and $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X. If a sequence of elements of $Im(\mu_A^P)$ is strictly decreasing and a sequence of elements of $Im(\mu_A^N)$ is strictly increasing, then $A = (\mu_A^P, \mu_A^N)$ has finite number of values.*

Proof. Let $\{t_n\}$ be a strictly decreasing sequence of $Im(\mu_A^P)$, then $0 \leq ... < t_2 < t_1 \leq 1$. Define $A_{t_r}^P = \{x \in X \mid \mu_A^P(x) \leq t_r\}, r = 1, 2, 3, ...$. Then $A_{t_r}^P$ is an H-ideal by Theorem 7. Let $x \in A_{t_r}^P$, then $\mu_A^P(x) \leq t_r < t_{r-1}$, which implies that $x \in A_{t_{r-1}}^P$. Hence, $A_{t_r}^P \subseteq A_{t_{r-1}}^P$. Since $t_{r-1} \in Im(\mu_A^P)$, there exists $x_{r-1} \in X$ such that $\mu_A^P(x_{r-1}) = t_{r-1}$. It follows that $x_{r-1} \in A_{t_{r-1}}^P$, but $x_{r-1} \notin A_{t_r}^P$. Thus, $A_{t_r}^P \subset A_{t_{r-1}}^P$, and so we obtain a strictly decreasing sequence $A_{t_1}^P \supset A_{t_2}^P \supset A_{t_3}^P \supset ...$ of H-ideals of X which is not terminating. a contradiction. Similar for $Im(\mu_A^N)$. This completes the proof.

Now we consider the converse of Theorem 9.

Theorem 10. *Let X be a BCK/BCI-algebra. If every doubt bipolar fuzzy H-ideal of X has finite number of values, then X satisfies H-DCC.*

Proof. Suppose that X does not satisfy H-DCC, then there exists a strictly descending chain $I_\circ \supset I_1 \supset I_2 \supset ...$ of H-ideals of X. Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ in X by

$$\mu_A^P(x) = \begin{cases} \frac{1}{n+1}, & \text{if } x \in I_n - I_{n+1}, n = 0, 1, 2, ... \\ 0, & \text{if } x \in \bigcap_{n=0}^{\infty} I_n, \end{cases}$$

$$\mu_A^N(x) = -\mu_A^P(x).$$

Where $I_\circ$ stands for X.

We prove that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal of X. Clearly, $\mu_A^P(0) = 0 \leq \mu_A^P(x)$ and $\mu_A^N(0) = 0 \geq \mu_A^N(x)$ for all $x \in X$. Let $x, y, z \in X$. Assume that $x*(y*z) \in I_n - I_{n+1}$ and $y \in I_k - I_{k+1}$ for $n = 0, 1, 2, ...; k = 0, 1, 2, ...$. Without loss of generality, we may assume that $n \leq k$. Then clearly $y \in I_n$. Since I_n is an H-ideal, we have $x*z \in I_n$. Hence,

$$\mu_A^P(x*z) \leq \tfrac{1}{n+1} = \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\}$$

and

$$\mu_A^N(x*z) \geq \tfrac{-1}{n+1} = \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}.$$

If $x*(y*z), y \in \bigcap_{n=0}^{\infty} I_n$, then $x*z \in \bigcap_{n=0}^{\infty} I_n$. Thus,

$$\mu_A^P(x*z) = 0 = \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\}$$

and

$$\mu_A^N(x*z) = 0 = \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}.$$

If $x*(y*z) \notin \bigcap_{n=0}^{\infty} I_n$ and $y \in \bigcap_{n=0}^{\infty} I_n$, then there exists $k \in \mathbb{N}$ such that $x*(y*z) \notin I_k - I_{k+1}$. It follows that $x*z \in I_k$, so that

$$\mu_A^P(x*z) \leq \tfrac{1}{k+1} = \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\}$$

and

$$\mu_A^N(x*z) \geq \tfrac{-1}{k+1} = \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}.$$

Finally, assume that $x*(y*z) \in \bigcap_{n=0}^{\infty} I_n$ and $y \notin \bigcap_{n=0}^{\infty} I_n$, then $y \in I_r - I_{r+1}$ for some $r \in \mathbb{N}$. It follows that $x*z \in I_r$ and hence

$$\mu_A^P(x*z) \leq \tfrac{1}{r+1} = \max\{\mu_A^P(x*(y*z)), \mu_A^P(y)\}$$

and

$$\mu_A^N(x*z) \geq \tfrac{-1}{r+1} = \min\{\mu_A^N(x*(y*z)), \mu_A^N(y)\}.$$

Consequently, we find that $A = (\mu_A^P, \mu_A^N)$ is a doubt bipolar fuzzy H-ideal and $A = (\mu_A^P, \mu_A^N)$ has infinite number of different values. This is a contradiction and the proof is complete.

5. Conclusions

In the study of a BCK/BCI-algebra, we know that doubt bipolar fuzzy H-ideals with special properties always play a vital role in the structure theory of a BCK/BCI-algebra. In this paper, we have studied some properties of doubt bipolar fuzzy H-ideals in $BCK/$ BCI-algebras. Also, we have discussed relations between a doubt bipolar fuzzy H-ideal and a doubt bipolar fuzzy ideal and we have provided conditions for a doubt bipolar fuzzy ideal to be a doubt bipolar fuzzy H-ideal. In addition, we have investigated characterizations of doubt bipolar fuzzy H-ideals by means of doubt positive t-level cut set, doubt negative s-level cut set and H-Artin BCK/BCI-algebras. In the future study of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of BCK/BCI-algebras by using this notion;

(2) To apply this notion to some other algebraic structures for example, BCH-algebras, BCC-algebras, B-algebras, BRK-algebras, semigroups, semirings and lattice implication algebras, etc.

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