EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 11, No. 3, 2018, 628-644
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Spectra of Local Cluster Flows on Open Chain of Contours

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Abstract. A dynamical system is considered. This dynamical system is a flow of clusters with the same length l on contours of unit length connected into open chain. A similar system, contours of which are connected into closed chain, was considered earlier. It has been found that, in the case of closed chain of contours, the dynamical system has a spectrum of velocity and mode periodicity consisted of more than one component. In this paper, it has been shown that, in the case of open chain, the spectrum of cluster velocity and mode periodicity contains only one component. The conditions of self-organization and the dependence of cluster velocity on load l is developed.

2010 Mathematics Subject Classifications: 47J10, 05C21, 11J06, 76B75

Key Words and Phrases: Dynamical System, Spectrum, Self-Organization, Contour Graph, Cluster Model

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1. Introduction. Chains, clusters, local flows

We consider a system of N contours $C_1, \dots, C_N$ $(N \geq 1)$. Each contour is a circle of unit length. On each contour, from the Eastern pole counterclockwise, a coordinate system is given. On the contour C_i, coordinates of points are $x_i \in [0, 1)$, $i = 1, \dots, N$. Contours $C_1, \dots, C_N$ form a *graph (chain).* The point of the contour $C_i(0)$ with coordinate 0 is identified with the point $C_{i+1}(1/2)$, $i = 1, \dots, N-1$. These points are called *nodes* of the contour network. We denote by (C_i, C_{i+1}) the common node of contours C_i and C_{i+1}. A fragment of a contour, which conserves the length and can move, is called a cluster. We consider a system with one cluster of length l on each contour.

A state of the system is *admissible* if, for this state, no node is covered by more than one cluster. In the general state, each cluster moves counterclockwise, uniformly with velocity equal to 1. We shall call such movement *local. Simultaneous movement* of more than one cluster through a node is forbidden. If cluster B comes to a node when the

DOI: https://doi.org/10.29020/nybg.ejpam.v11i3.3292

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cluster A of neighboring contour moves through the node, then the cluster B stops, and its movement does not continue while the cluster A covers the node. If two clusters come to the node (C_i, C_{i+1}) *simultaneously,* then the cluster of contour C_i (cluster Cl_i) moves through the node, and the cluster Cl_{i+1} stops (*left-priority* conflict resolution rule), i.e., the cluster Cl_{i+1} moves through the node first. The *delay* of the cluster A is the duration of time interval such that, in this interval, the cluster A waits the node release.

The *state of the system* at the time t is a vector $\alpha(t) = (\alpha_1(t), \ldots, \alpha_N(t))$, where $\alpha_i(t)$ is the coordinate of the frontal point of cluster Cl_i, $i = 1, \ldots, N$, Fig. 1. The back point of cluster Cl_i is located in the point with coordinate $\alpha_i(t) - l$ (substraction modulo 1), $i = 1, \ldots, N$.

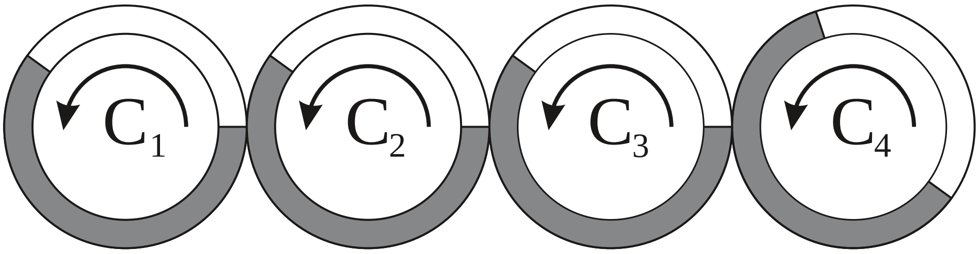

Figure 1: State $(0, 0, 0.9, 0.8)$

2. Formulation of problem. Spectra of flows on graphs

2.1. Wreaths of trajectories and spectra of velocities

The considered system is *deterministic.* The system behavior in the future is determined fully by the state $\alpha(t_0)$ at current time t_0. Each admissible state generates a trajectory $\alpha(t)$ in the space of admissible states. Two trajectories $\alpha_A(t)$ and $\alpha_B(t)$ can coincide at the some moments t_A and t_B. A *wreath* is a batch of trajectories such that any two trajectories join after a finite time interval. A pair of states on the wreath can be classified as *recurrent* if the system comes after a finite time from any of these states to the other state. A pair of states is called *dependent* if only from one of the states it is possible to come to the other state. A pair of states is called *independent* if from none of these states it is possible to come to the other state. *Any recurrent pair of states form a cycle.* The space of admissible states is divided into non-intersecting sets, which are trajectories, continuous modulo 1. These trajectories are piecewise linear functions with inclination 0 or 1. The system is in the *the state of free movement at the time* t_0 if at any time $t \geq t_0$ all clusters move without delays. *Self-organization* is the property of the dynamical system such that the system comes to the state of free movement after a finite time interval from *any admissible initial state.* It is interesting to study the average velocity of cluster on each contour and the average velocity of all clusters of the system. *The average velocity of the cluster* C_i is defined as the limit, if this limit exists,

$$v_i = \lim_{T \to \infty} \frac{S_i(T)}{T},$$

where $S_i(T)$ is the total distance such that the cluster passes this distance in the time interval $(0, T)$, $i = 1, \ldots, N$. The average value of clusters velocities is called *the average velocity of system clusters in time.*

In general case, *partial self-organization* takes place, [1]–[4]. The system comes to the state of free movement from some initial conditions. From the other admissible initial conditions, the free movement is not reached and the movement is characterized by the velocity less the 1. It is possible that the self-organization does not take place for any initial states. *The set of admissible velocity values,* for fixed values of system parameters and different initial states is called the *spectrum of system velocities.*

The following problems are interesting for research. Is the spectrum of velocities a countable or continual set? Whether the trajectory of the process in the system state space is repeated cyclicly, and therefore the average distance covered by clusters per time unit, reaches a limit value, or the trajectory of the process in the system state space is an attractor and the average distance covered by clusters per time unit tends to a limit value? Whether the average velocity of each cluster is the same or the average velocities of different clusters can be different? The same value of the average velocity can correspond to the different cyclic trajectories in the system state space. How many possible cyclic trajectories correspond to the fixed value of velocity?

Thus the study of systems, considered in [1]–[4], leads to the concept of the spectrum of limit cyclic trajectories in the state space and the spectrum of velocities corresponding to these trajectories.

2.2. Chain of discrete binary contours

A closed chain of contours was studied in [4]. Each contour has common points (nodes) with two neighboring contours. There are two cells (the lower cell - cell 0, and the upper cell - cell 1) and a particle on each contour. In every discrete moment, the particle is located in upper (lower) cell and, if this is allowed, moves counterclockwise to the lower (upper) cell, Fig. 2.

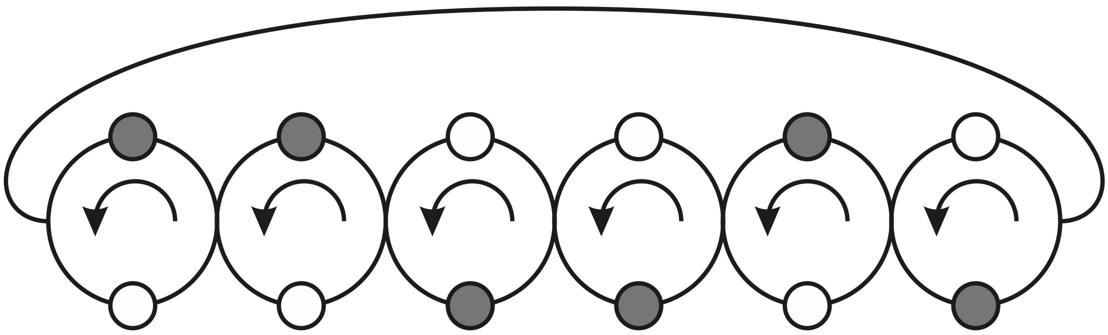

Figure 2: The binary vector 110010

Particles cannot move through the common node simultaneously. If the particle, located to the right of the node, tries to move from the upper cell to the lower cell, and the particle, located to the left of the node, tries to move from the lower cell to the upper cell, then there is *a conflict.* Assume that, *in the case of conflict, the particle, located to the left of the node,* moves, and the particle, located to the right of the node, does not move.

2.2.1. Open chain of contours with binary states

It is obvious that, in the case of *open chain of contours,* Fig. 3, for any of 2^N initial states (all possible states are admissible in this case), no more than after time $2N$ all particles will move without delays.

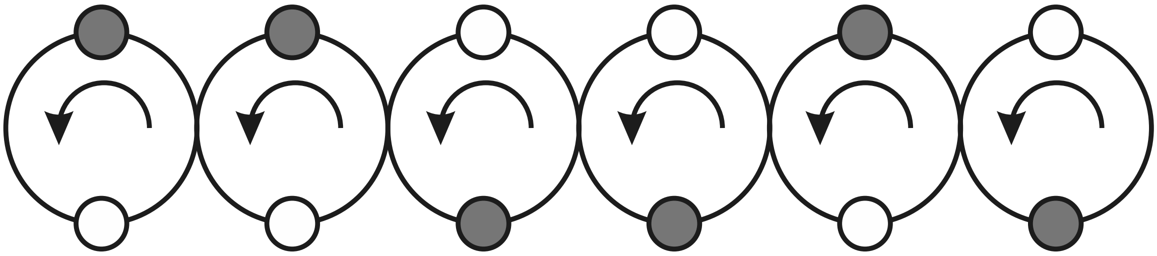

Figure 3: The state 110010 on an open chain

2.2.2. Closed chain with binary states

The following has been proved in [4].

(i) The dynamical system is equivalent to elementary cellular automaton CA 063 in terms of Wolfram classification, [5]. States of the system are cyclic vectors with N coordinates. The ith coordinate of vector equals 0 if the particle is in the cell 0, and equals 1 if the particle is in the cell 1. At any discrete moment, the value of each coordinate is changed except the case in which the value of this coordinate equals 1, and the value of the neighboring coordinate on the left equals 0.

(ii) The space of states is divided into the set of *recurrent* states and the set of *nonrecurrent* states. A state is non-recurrent *if and only if the vector of this state contains at least one coordinate such that the value of this coordinate is equal to 1, and the values of neighboring coordinates on the left and on the right are equal to 0.*

(iii) The system can be in a non-recurrent state only at the initial moment.

(iv) Each recurrent state is repeated after no more than $2N$ steps, Fig. 4.

(v) The vector of state is shifted onto one position to the right for every two steps.

(vi) The quantity of sign changes in the state vector, divided by 2, is called the system *variation.* The variation decreases if the system moves through a non-recurrent state to a recurrent state, and does not change if the system is in the recurrent state.

(vii) If the system is in a recurrent state, then the variation is not more than $[N/3]$ - integer part of $N/3$.

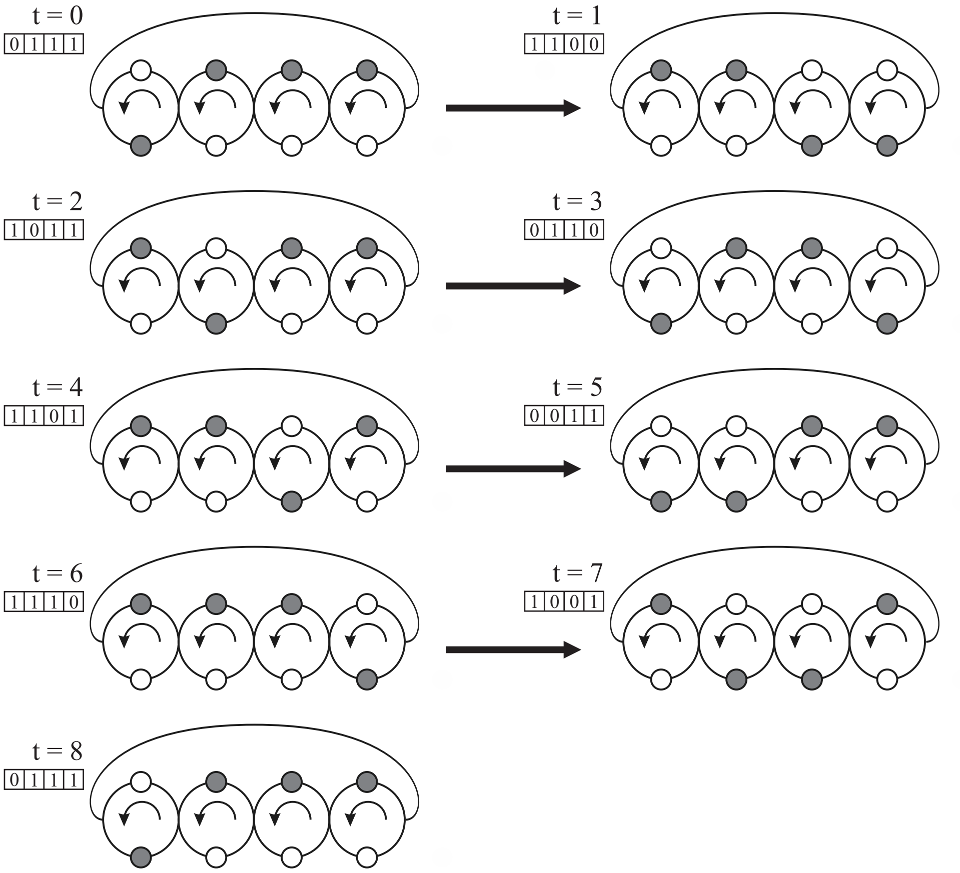

Figure 4: A cyclic trajectory in the state space

(viii) If the system is in a recurrent state at initial moment and the variation is equal to k, then the average velocity of particles is equal to $(N-k)/N$, $k = 0, 1, \dots, [N/3]$.

(ix) If the initial state is non-recurrent, then the average velocity of particles is equal to $(N-k)/N$, where k is the value of variation at the next moment, $k = 0, 1, \dots, [N/3]$.

(x) *If k is an integer value and satisfies the condition*

$$0 \leq k \leq [N/3],$$

then there exists an initial state such that the average velocity equals $(N-k)/N$.

(xi) *For any initial condition, the value of average velocity satisfies the above conditions.*

2.2.3. The open chain with two contours with discrete states

A two contour system with a common point (node) is studied in [1], [2]. There are fixed quantities of cells and particles on each contour. Particles are located in cells and move in accordance with a given rule. In particular, this system can be an open chain with two contours and one cluster of particles on each contour. If the contour length (the quantity of cells on the contour) is the same for the contours, then the velocity of particles does not depend on the initial state, i.e., if the system parameters are fixed, *the spectrum of possible states contains only one value.* It is noted in [2], that the average velocity of clusters can depend on initial state if the lengths of clusters are different.

2.2.4. Closed chain of three contours with continual clusters

A closed dynamical chain of contours is considered in [3]. A cluster of the same length moves on each contour. This system is characterized by continuous state space and time. Each contour has common nodes with two neighboring contours. It is found that, *after a finite moment the system is in a set of recurring states, and the average velocity of clusters depends, in general, on the set in which the system is.* In what set of recurring states system will be, depends on the initial state. It has been found that *the spectrum of velocity values is finite.* If the length of a cluster is more than a half, then, for any quantity of contours, all clusters stop (*collapse*) after a finite time. Therefore, in this case, the spectrum of velocities contains only the value 0. *In the case of three clusters, it has been proved that, if the length of cluster is not more than 1/6, then the system comes to the state of free movement from any initial state, i.e., the spectrum of velocities contains only the value 1. If the cluster length l satisfies the condition* $1/6 < l < 1/2$, *then the spectrum contains the value 1 and one value not equal to 1.*

Hypotheses, characterizing the motion on the closed chain of contours, have been formulated in [3]. If the quantity of contours is not less than 6, then the spectrum of velocities can contain more than one value. It has been proved that, for any arbitrarily small value, there exist values of the quantity of contours and the initial state such that the system does not come to the states of free movement.

2.2.5. Open chain of contours with uniform load

In this paper, we consider a one-dimensional system of contours. This system differs from the closed chain in that the leftmost (rightmost) contour has common node only with one neighboring contour (open chain of contours). We have proved that for an open chain, as for a closed chain, after a time interval, the same set of states is repeated with a period. For the open chain, this set is determined by the quantity of contours and the cluster length and does not depend on the initial states. Hence the spectrum of velocities of clusters contains only one value if the quantity of contours and the cluster length are fixed. We have found a formula for the velocity of movement for fixed quantity of contours and cluster length. Though contours differ in their location on the open chain, they move with the same average velocity, i.e, the velocity of the cluster does not depend on the distances to the ends of the chains.

The value

$$v = \frac{v_1 + \cdots + v_N}{N}$$

is called *the average velocity of movement in the network.* It is obvious that the lower bound of the average velocity on the time interval T is more than 0 for any l, $0 < l < 1$.

Proposition 1. *At every moment t, at least one cluster moves with velocity 1.*

Proof. If all clusters do not move at the moment t, then coordinates of their frontal points are equal to 0 or 1/2, and, in this pair, the left cluster moves. The first coordinate of the vector of the initial state is equal 0, and the last coordinate of the vector of the initial state is equal 1/2. Then there exists a pair 0, 1/2.

3. Formulation of maim results

Let us formulate main results that will be proved in this paper.

Theorem 1. *If*

$$l \leq \frac{1}{2},$$

then there is one point of the spectrum of the system

$$v = 1.$$

The system reaches the state of free movement after a finite time interval.

Theorem 2. *If*

$$l \geq \frac{1}{2},$$

then there is one point of the spectrum

$$v = \frac{1}{2(N-1)l - N + 2},$$

and, from a finite moment, a cyclic trajectory in the state space is repeated. The trajectory does not depend on the initial state. The point $(0, \ldots, 0)$ belongs to this trajectory.

4. Potential of delays and its properties

4.1. Definitions of delay potential and one-sided potential of delay

By definition, put

$$d_i(t) = \begin{cases} \frac{1}{2} + \alpha_i(t), & 0 \leq \alpha_i(t) < \frac{1}{2}, \\ \alpha_i(t) - \frac{1}{2}, & \frac{1}{2} \leq \alpha_i(t) < 1. \end{cases}$$

Now we introduce the following concept.

The measure of time inside the segment $t \in [0, 1]$ is called *the potential of delays in the node* (C_i, C_{i+1}) if, during this time, clusters Cl_i, Cl_{i+1} move through this node, provided that clusters can move through the node simultaneously.

Suppose that simultaneous movement of clusters through the node is possible.

By $H_{i,i+1}$ we denote the potential of delays in the node (C_i, C_{i+1})

It is readily seen that, dependent on values l, α_i, d_{i+1}, one of the following three available alternatives is realized.

(i) Clusters Cl_i, Cl_{i+1} move through the node simultaneously on one time segment belonging to the segment $[0,1]$.

(ii) Clusters Cl_i, Cl_{i+1} move through the node simultaneously on two time segments belonging to $[0,1]$. The first of these segments begins when one of these two clusters begins to move through the node. The second segment begins when the other cluster begins to move through the node.

(iii) Clusters Cl_i, Cl_{i+1} cannot move through the node simultaneously on the time segment $[0,1]$.

We shall give the following definition.

We assume again that simultaneous movement of clusters Cl_i and Cl_{i+1} through the node on the time segment $[0,1]$ is possible. The duration of time segment *the potential of the cluster Cl_i delay concerning the cluster Cl_{i+1}* if these clusters move through the node simultaneously during this segment such that, at the initial moment of this interval, the cluster Cl_i comes to the node, and the cluster Cl_{i+1} moves through the node, if there exists such time segment of non-zero duration. We denote *the potential of the cluster Cl_i delay concerning the cluster Cl_{i+1}* by $h_{i,i+1}$. If such time interval does not exist, then we suppose $h_{i,i+1} = 0$.

Similarly, the duration of time segment is called *the potential of the cluster Cl_i delay concerning the cluster Cl_{i+1}* if these clusters move through the node simultaneously during this segment such that, at the initial moment of this interval, either the cluster Cl_{i+1} comes to the node, and the cluster Cl_{i+1} moves through the node, or clusters Cl_i, Cl_{i+1} come to the node simultaneously, if there exists such time segment of non-zero duration. We denote *the potential of the cluster C_{i+1} delay concerning the cluster C_i* by $h_{i+1,i}$. If such time interval does not exist, then we suppose $h_{i+1,i} = 0$.

Now we shall give a formal definition of the delay potential and one-sided potentials.

Suppose, at the moment $t = 0$, coordinates of frontal points of clusters Cl_i, Cl_{i+1} are $\alpha_i(0) = \alpha_i$ and $d_{i+1} = d_{i+1}(0)$. Assume that clusters move free and can pass through the node (Cl_i, Cl_{i+1}) simultaneously.

If

$$0 \leq \alpha_i < l, \tag{1}$$

then the cluster Cl_i covers the node (C_i, C_{i+1}) on time segments $[0, l - \alpha_i]$ and $[1 - \alpha_i, 1]$.

If

$$l \leq \alpha_i < 1, \tag{2}$$

then the cluster Cl_i covers the node (C_i, C_{i+1}) on time segment $[1 - \alpha_i, 1 - \alpha_i + l]$.

If

$$0 \le d_{i+1} < l, \tag{3}$$

then the cluster Cl_i covers the node (C_i, C_{i+1}) on time segments $[0, l-d_{i+1}]$ and $[1-d_{i+1}, 1]$.

If

$$l \le d_{i+1} < 1, \tag{4}$$

then the cluster Cl_i covers the node (C_i, C_{i+1}) on the segments $[1 - d_{i+1}, 1 - d_{i+1} + l]$.

If (1) and (4) and conditions

$$d_{i+1} - \alpha_i > 1 - l, \tag{5}$$

$$d_{i+1} - \alpha_i \ge l \tag{6}$$

are fulfilled, then clusters Cl_i and Cl_{i+1} cover the node simultaneously on the segment $[1 - d_{i+1}, l - \alpha_i]$. In this case,

$$H_{i,i+1} = l + d_{i+1} - \alpha_i - 1.$$

$$h_{i,i+1} = 0,\ h_{i+1,i} = l + d_{i+1} - \alpha_i - 1,\ i = 1, \dots, N - 1.$$

If conditions (1), (4), (5) are fulfilled, and the condition (6) is not fulfilled (this case is possible only if $l > \frac{1}{2}$), then clusters Cl_i, Cl_{i+1} cover the node (C_i, C_{i+1}) simultaneously on time segment $[1 - d_{i+1}, l - \alpha_i]$ and $[1 - \alpha_i, 1 - d_{i+1} + l]$. In this case,

$$H_{i,i+1} = 2l - 1,$$

$$h_{i,i+1} = l - d_{i+1} + \alpha_i,\ h_{i+1,i} = l - \alpha_i + d_{i+1} - 1.$$

If conditions (1), (4), (6) are fulfilled, and the condition (5) is not fulfilled (this case is possible only if $l \le \frac{1}{2}$), then clusters Cl_i, Cl_{i+1} cannot move through the node simultaneously. In this case,

$$H_{i,i+1} = 0,$$

$$h_{i,i+1} = h_{i+1,i} = 0.$$

If conditions (1), (4) are fulfilled, and conditions (5), (6) are not fulfilled, then clusters Cl_i, Cl_{i+1} move through the node (C_i, C_{i+1}) on time segment $[1 - \alpha_i, 1 - d_{i+1} + l]$. In this case,

$$H_{i,i+1} = l - d_{i+1} + \alpha_i,$$

$$h_{i,i+1} = l - d_{i+1} + \alpha_i,\ h_{i+1,i} = 0.$$

If conditions (2), (3) and

$$\alpha_i - d_{i+1} > 1 - l \tag{7}$$

$$\alpha_i - d_{i+1} \geq l, \tag{8}$$

are fulfilled, then clusters Cl_i, Cl_{i+1} cover the node on the segment $[1-\alpha_i, l-d_{i+1}]$. In this case,

$$H_{i,i+1} = l + \alpha_i - d_{i+1} - 1,$$

$$h_{i+1,i} = 0, \ h_{i,i+1} = l + \alpha_i - d_{i+1} - 1.$$

If conditions (2), (3), (7) are fulfilled, and the condition (8), is not fulfilled (this is possible only if $l > \frac{1}{2}$), then clusters Cl_i, Cl_{i+1} cover the node (C_i, C_{i+1}) on time segments $[1-\alpha_i, l-d_{i+1}]$ $[1-d_{i+1}, 1-\alpha_i+l]$. In this case,

$$H_{i,i+1} = 2l - 1,$$

$$h_{i,i+1} = l - d_{i+1} + \alpha_i - 1, \ h_{i+1,i} = l - \alpha_i + d_{i+1}.$$

If conditions (2), (3), (8) are fulfilled, and condition (7) is not fulfilled (this is possible only if $l \leq \frac{1}{2}$), then clusters cannot cover the node simultaneously. In this case,

$$H_{i,i+1} = 0,$$

$$h_{i,i+1} = h_{i+1,i} = 0.$$

If conditions (2), (3) are fulfilled, and conditions (7), (8) are not fulfilled, then clusters Cl_i, Cl_{i+1} cover the node (C_i, C_{i+1}) on time segment $[1-d_{i+1}, 1-\alpha_i+l]$. In this case,

$$H_{i,i+1} = l - \alpha_i + d_{i+1},$$

$$h_{i,i+1} = 0, \ h_{i+1,i} = l - \alpha_i + d_{i+1}.$$

Suppose conditions (2), (4) and one of the equalities

$$\alpha_i \geq d_{i+1} + l$$

and

$$d_{i+1} \geq \alpha_i + l$$

are fulfilled. Then the clusters Cl_i and Cl_{i+1} cannot move through the node simultaneously. In this case,

$$H_{i,i+1} = 0,$$

$$h_{i,i+1} = h_{i+1,i} = 0.$$

If conditions (2), (4) and

$$\alpha_i < d_{i+1} < \alpha_i + l$$

are fulfilled, then clusters move through the node (C_i, C_{i+1}) simultaneously on the time segment $(1-\alpha_i, 1-d_{i+1}+l)$. In this case,

$$H_{i,i+1} = \alpha_i - d_{i+1} - l,$$

$$h_{i,i+1} = \alpha_i - d_{i+1} - l,\ h_{i+1,i} = 0.$$

If conditions (2), (4) and

$$d_{i+1} \le \alpha_i < d_{i+1} + l$$

are fulfilled, then clusters Cl_i and Cl_{i+1} move through the node simultaneously on time segment $[1 - d_{i+1}, 1 - \alpha_i + l]$. In this case,

$$H_{i,i+1} = d_{i+1} - \alpha_i - l,$$

$$h_{i,i+1} = 0,\ h_{i+1,i} = d_{i+1} - \alpha_i - l.$$

If the conditions (1), (3) are fulfilled simultaneously, then the state is not admissible.

4.2. Definition of system potential of delays

The sum of potentials of delays in the nodes

$$H(t) = \sum_{i=1}^{N-1} H_{i,i+1}(t) \tag{9}$$

is called *the potential of delays of the system.*

4.3. Properties of delay potential in node and one-sided potentials

Suppose, at time t_0, the potential of delay of the cluster Cl_i concerning the cluster Cl_{i+1} positive, $h_{i,i+1} > 0$. Then a delay of the cluster Cl_i at the node (C_i, C_{i+1}) begins at the time $t = 1 - \alpha_i(t_0)$ if there were no delays earlier, $i = 2, \ldots, N$. The duration of this delay is $h_{i,i+1}$.

Similarly, if $h_{i+1,i} > 0$, then a delay of the cluster Cl_{i+1} begins at the time $t = 1 - d_{i+1}(t_0)$ if there were no delays earlier, $i = 2, \ldots, N$. The duration of this delay is $h_{i+1,i}$.

Proposition 2. *(i) It is true for any $t \ge t_0$*

$$H_{ij}(t) = h_{ij}(t) + h_{ji}(t), \tag{10}$$

$$H(t) = \sum_{i=1}^{N-1} (h_{i,i+1}(t) + h_{i+1,i}(t)). \tag{11}$$

(ii) If $l \le \frac{1}{2}$, then at least one of the values $h_{i,i+1}(t)$ and $h_{i+1,i}(t)$ equals 0 for any $t \ge 0$, $i = 1, \ldots, N-1$.

Proof. Equation (10) follows from definitions of delay potential in a node and one-sided potential of delays. All versions of relations between system parameters are considered.

Equation (11) follows from (9) and (10).

The second statement of Proposition 2 is also proved on the basis of definition. Proposition 2 has been proved.

4.4. Properties of the system delay potential

Let us prove properties of the system delay potential.

Proposition 3. *If $H(t) = 0$, then the system is in the state of free movement at time t.*

Proof. If $H(t) = 0$, then $H_{i-1,i}(t) = 0$, $i = 1, \dots, N$. Proposition 3 follows from the definition and invariance of the clusters intersection measure with respect to the same shift.

Proposition 4. *If there is a delay in the node (C_{i-1}, C_i), then $H_{i-1,i}(t)$ does not increase at $t \geq t_0$. If, in addition $l \leq \frac{1}{2}$, then $H_{i-1,i}(t)$ decreases strictly in a neighborhood of $t \geq t_0$.*

Proof. Proposition 4 follows from the definitions of delay potential in node and one-sided potentials.

Proposition 5. *The potential of delays is non-increasing function of time for any value of l, $0 < l < 1$.*

Proof. Suppose $l \leq \frac{1}{2}$, and the function $H(t)$ increases at time t. It is possible only if at least one term on the right side of (11) increases. Assume that the term $h_{i_0,i_0+1}(t)$ increases. The term can increase only with velocity 1. This term can increase only if the cluster Cl_{i_0} moves and the cluster Cl_{i_0+1} does not move. If the cluster Cl_{i_0+1} is at the node (C_{i_0}, C_{i_0+1}), then the term $h_{i_0,i_0+1}(t)$ equals 0 and does not increase. Therefore the cluster Cl_{i_0+1} does not move and is located at the node (C_{i_0+1}, C_{i_0+2}). The cluster C_{i_0+2} moves through the node. Hence, $h_{i_0+1,i_0+2}(t)$ decreases with velocity 1 at time t. Similarly, it is proved that, if, at the moment t_0, $h_{i_0,i_0-1}(t_0)$ increases, then $h_{i_0-1,i_0-2}(t_0)$ decreases at this moment. Therefore each increasing term on the right side of (11) correspond to a term, decreasing with the same velocity, and different terms corresponds to different decreasing terms. Thus Proposition 5 is true in the case of $l \leq \frac{1}{2}$.

Assume that $l > \frac{1}{2}$. We shall prove that, in this case, the potential of delays does not increase in any node, and therefore the system potential $H(t)$ does not also increase. If at the time t clusters Cl_{i_0}, Cl_{i_0+1} move or both the clusters do not move, then the delay potential in the node (C_{i_0}, C_{i_0+1}) does not change at the time t, $i_0 = 1, \dots, N-1$. The potential of delays can change at moment t. The delay potential can change at the moment t only if one of two clusters move.

Assume that the cluster Cl_{i_0} moves, and the cluster Cl_{i_0+1} does not move (the case in that only the cluster Cl_{i_0+1} moves can be considered similarly). Then the coordinate of the frontal coordinate of the cluster C_{i_0+1} is equal to $\frac{1}{2}$ and 0. Suppose the coordinate of frontal point of the cluster Cl_{i_0+1} is equal to $\frac{1}{2}$. Since the cluster Cl_{i_0+1} does not move the value $\alpha_{i_0}(t_0)$ satisfies the condition $0 \leq \alpha_{i_0}(t_0) \leq l$. Without loss of generality we assume that $t_0 = 0$ and $\alpha_{i_0}(t_0) = \alpha_{i_0,0}$. If both the clusters move, then the cluster Cl_{i_0} move through the node (C_{i_0}, C_{i_0+1}) on time segments $(0, l - \alpha_{i_0,0})$ and $(1 - \alpha_{i_0,0}, 1)$, and the cluster Cl_{i_0+1} move through the node on the segment $(0, l)$. Hence the clusters Cl_{i_0}, Cl_{i_0+1} move through the common node simultaneously on the segment $(0, l - \alpha_{i_0,0})$ and, if $\alpha_{i_0,0} > 1-l$, on the segment $(1-\alpha_{i_0,0}, l)$ too. From this follows that the delay potential does not increase with respect to $\alpha_{i_0,0}$, and therefore the delay potential does not increase with respect to time.

Proposition 6. *Suppose $l \leq \frac{1}{2}$. If a delay of cluster C_1 or cluster Cl_N takes place in the time interval $(t, t+a)$, then $H(t+a) = H(t) - a$.*

Proof.

Assume that the cluster Cl_N does not move in the time interval $(t, t+a)$. The case in that the cluster Cl_1 does not move can be considered similarly. We have

$$h_{N,N-1}(t+a) = h_{N,N-1}(t) - a, \ h_{N-1,N}(t+a) = h_{N-1,N}(t) = 0. \tag{12}$$

In accordance with Proposition 5 the value of H does not increase if the system comes from the state

$$(\alpha_1(t), \dots, \alpha_N(t))$$

to the state

$$(\alpha_1(t+a), \dots, \alpha_N(t+a)).$$

Suppose the value of H does not change, i.e., $H(t+a) = H(t)$. Then, in accordance with (12),

$$\sum_{i=1}^{N-2}(h_{i,i+1}(t+a) + h_{i+1,i}(t+a)) > \sum_{i=1}^{N-2}(h_{i,i+1}(t) + h_{i+1,i}(t)). \tag{13}$$

Using (13), we obtain that, for a system with $N-1$ contours such that it differs from the system under consideration by the absence of the contour C_N, the potential of delays increases if this system comes from the state

$$(\alpha_1(t), \dots, \alpha_{N-1}(t))$$

to the state

$$(\alpha_1(t+a), \dots, \alpha_{N-1}(t+a)).$$

However, in accordance with Proposition 5, the potential of delays cannot increase. The contradiction proves that, if the cluster CL_N does not move in the time interval $(t, t+a)$, then the potential of delays increases in this interval with the unit velocity. In the case of the cluster Cl_1 movement, Proposition 5 is proved similarly.

Proposition 7. *Suppose $l < \frac{1}{2}$, and, at time t_0, a delay of the cluster Cl_{N-1} at the node (C_{N-1}, C_{N-2}) ends (or a delay of the cluster Cl_2 at the node (C_2, C_3) ends), and, at the moment $t_1 > t_0$, another delay of this cluster begins. The latter delay ends, at the moment $t = t_1 + a$, and the total duration of the cluster Cl_{N-1} (the delay of the cluster Cl_2) in the time interval (t_0, t_1) (denote this duration by b) is not more than $1-2l$. Then $H(t_1+a) \leq H(t_0) - \min(a+b, 1-2l)$.*

Proof. We have

$$\alpha_N(t_0) = \alpha_{N-1}(t_0) + l$$

(modulo 1). While the total delay of the cluster Cl_{N-1}, after the moment t_0, is not more than $1 - 2l$, the cluster Cl_{N-1} can be delayed at the node (C_{N-2}, C_{N-1}), and clusters $Cl_1, \dots, Cl_{N-1}$ behave in such a way that if the contour C_N is absent. Taking into account Proposition 6, we get Proposition 6 in the case of cluster Cl_{N-1}. Similarly, the proposition is proved in the case of the cluster Cl_2.

Remark 1. *Although the potential of delays is non-increasing function in time, the quantity of non-moving particles is not in general non-increasing function in time.*

5. Criterion for the system to enter the state of free movement

Theorem 3. *If*

$$l < \frac{1}{2}, \tag{14}$$

then the system comes to the state of free movement after a finite time interval from any initial state.

If

$$l > \frac{1}{2}, \tag{15}$$

then the system does not come to the state of free movement after a finite time interval from any initial state.

Proof. If (17) is fulfilled, then, in accordance with Theorem 4 (Section 6), the average velocity of clusters is not equal to 1, and therefore the system does not come to the state of free movement.

We shall prove by induction on N that (14) is sufficient for self-organization. The statement is true for $N = 1$.

Suppose that the statement is true for $N = K - 1$, $K \geq 2$. Consider the case $l < \frac{1}{2}$. The system comes to the state of free movement after a finite time. From Propositions 3–7 follows that either the potential of delays becomes equal to 0 and the system comes to the state of free movement or the total delay of the cluster Cl_{K-1}, on all infinite time interval from a moment, does not exceed $1 - 2l$, and, from this moment, the presence of the cluster Cl_K does not affect behavior of the system. Hence, in accordance with induction statement, the system comes to the state of free movement after a finite time interval. Thus Theorem 3 is true in the case $l < \frac{1}{2}$. Theorem 3 has been proved.

6. Behavior of system in case $l > \frac{1}{2}$

Theorem 4. *Let (15) be fulfilled. Then, from a finite time, the system passes, in the system state space, the same cyclic trajectory, containing the state* $(0, 0, \dots, 0)$, *and the* *system states are repeated with the period*

$$T = 2(N-1)l - N + 2. \tag{16}$$

After a moment such that at this moment periodic movement begins, the potential of delays equals

$$H(t) = 2(N-1)l - N + 1. \tag{17}$$

The average velocity equals

$$v_1 = \dots = v_N = \frac{1}{2(N-1)l - N + 2}. \tag{18}$$

Proof. If (16) is fulfilled, the clusters Cl_1 Cl_2 cannot move through the node (C_1, C_2) without delays. Indeed, if both the clusters move through the node (C_1, C_2) without delays, then each of clusters Cl_1, Cl_2 covers the node during l time units. Therefore the inequality $2l \leq 1$ is fulfilled. However this contradicts (17).

If, at the moment t_0, a delay of the cluster Cl_2, then there exists a moment $t_1 \geq t_0$ such that

$$\alpha_1(t_1) = l,\ \alpha_2(t_1) = \frac{1}{2}. \tag{19}$$

If a delay of the cluster Cl_1 begins at the moment t_0, then, at the moment of the end of this delay, the system comes to the state such that $\alpha_1(t_1) = 0$, $\alpha_2(t_1) = \frac{1}{2} + l$. After $1-l$ time units after the end of this delay, a delay of the cluster Cl_2 begins. At the moment of the end of the latter delay, the system comes to the state such that (19) is fulfilled.

Hence, from any initial state, the system comes to a state such that (19) is fulfilled.

Let the system is at the time t_1 in the state

$$\left(l, \frac{1}{2}, \alpha_3(t_1), \dots, \alpha_N(t_1)\right).$$

Since $\alpha_2(t_1) = l$, i.e., the cluster Cl_2 is at the node (C_2, C_3) at the time t_1, then, taking into account that $l > \frac{1}{2}$, we have that, at this moment, the cluster Cl_2 covers the node (C_2, C_3), the cluster Cl_3 covers the node (C_3, C_4), etc. Similarly, the cluster Cl_i covers the node (C_i, C_{i+1}) $i = 4, \dots, N-1$, at time t_1. For state to be admissible, it is necessary that

$$l - \frac{1}{2} \leq \alpha_i(t_1) \leq \frac{1}{2},\ i = 3, \dots, N. \tag{20}$$

In accordance with (20), the frontal point of the cluster Cl_i comes to the node (C_i, C_{i+1}), not earlier than at the time $t_1 + \frac{1}{2}$, and the cluster Cl_{i+1} comes to the node (C_i, C_{i+1}) not later than at the time $T_1 + \frac{1}{2} - l$. On the other hand, the cluster Cl_{i+1}, beginning to pass through the node (C_i, C_{i+1}) at the moment $t_2(i)$ such that $\alpha_{i+1} = \frac{1}{2} + l$, releases the node (C_i, C_{i+1}) later than the cluster Cl_i comes to this node, $i = 1, \dots, N-1$. Stopping at the node (C_i, C_{i+1}), the cluster Cl_i continues to hamper the movement of the cluster Cl_{i-1}. Therefore there exists a moment t_0 such that $\alpha_1(t) = \dots = \alpha_{N-1} = 0$, $\alpha_N \geq \frac{1}{2}$, and only the cluster Cl_N moves at the time t_0. Thus there exists a moment $t = a$ such that

$$\alpha_1(a) = \cdots = \alpha_{N-1}(a) = 0. \tag{21}$$

Since, only on the contour Cl_N, there is no node at the point 0, only the Cl_N moves in the time interval $\left(a, a + l - \frac{1}{2}\right)$. Movement of the cluster Cl_{N-1} resumes at the moment $t = a + l - \frac{1}{2}$. Movement of the cluster Cl_i resumes at the moment $t_2(i) = a + (N-i)(l - \frac{1}{2})$, $i = 1, 2, \ldots, N-1$. All clusters, except the cluster Cl_1, coming to the point $\frac{1}{2}$, stops and waits for the node release. The cluster Cl_1 comes to the point $\frac{1}{2}$ last at the time $t = a + \frac{1}{2} + (N-1)\left(l - \frac{1}{2}\right)$. At this moment, the system is in the state

$$\alpha_0\left(a + \frac{1}{2} + (N-1)\left(l - \frac{1}{2}\right)\right) = \alpha_1\left(a + \frac{1}{2} + (N-1)\left(l - \frac{1}{2}\right)\right) = \cdots =$$

$$= \alpha_{N-1}\left(a + \frac{1}{2} + (N-1)\left(l - \frac{1}{2}\right)\right) = \frac{1}{2}.$$

Hence, after the time $t = a$, all clusters pass half of the circle in time interval of duration $\frac{1}{2} + (N-1)\left(l - \frac{1}{2}\right)$. After new interval of the same duration, the system returns to the state (21), at which the system was at the time $t = a$, and, in this interval the movement of the clusters resumes in inverse order. Therefore the period is equal to $2\left(\frac{1}{2} + (N-1)\left(l - \frac{1}{2}\right)\right) = 2(N-1)l - N + 2$, i.e., (16) is fulfilled. Duration of the cluster Cl_i delay at the point 0 equals $(N-1-i)\left(l - \frac{1}{2}\right)$, $i = 1, \ldots, N-1$, and total delay of each cluster during the period is equal to $(N-1)(2l-1) = 2(N-1)l - N + 1$. Thus the average velocity of each cluster equals

$$v_1 = \cdots = v_N = 1 - \frac{2(N-1)l - N + 1}{2(N-1)l - N + 2} = \frac{1}{2(N-1)l - N + 2},$$

i.e., (18) is fulfilled.

It is proved by direct consideration that at the time $t = a$, and therefore, in accordance with Proposition 5, at any time $t \geq a$, the potential of delays is calculated by (17). This completes the proof of Theorem 4.

Remark 2. *In accordance with (18), if N is fixed, the average velocity of the cluster is a continuous function on l, and this velocity tends to $\frac{1}{N}$ as $l \to 1$. If l is fixed, then the average velocity of clusters tends to 0 as $N \to \infty$.*

7. Conclusion

A deterministic dynamical system is considered. This system is an open chain of N contours, on which clusters of length l move in accordance with specified rules.

In [3], a similar system was considered. The supporter of the system is a closed chain of contours. It has been found in [3] that the dynamical system has a spectrum of velocity and mode periodicity consisted of more than one component.

In this paper, it has been shown that, in the case of open chain, the spectrum of cluster velocity and mode periodicity contains only one component. If $l < 1/2$, then the system comes to the state of free movement after a finite time interval from any initial state. If $l < 1/2$, then the average velocity of clusters is less than 1. The dependence of this velocity on N and l has been found. Properties of delay potential function are studied. These properties are used in proof of self-organization conditions.

Acknowledgements

This work has been supported by the Russian Foundation for Basic Research. Grant No. 17-01-00821-a and No. 17-07-01358-a.

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