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Introducing Partial Transformation UP-Algebras*

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Abstract. The main aim of this paper is to introduce the notion of a partial transformation UP-algebra $P(X)$ induced by a UP-algebra X and prove that the set of all full transformations $T(X)$ is a UP-ideal of $P(X)$.

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1. Introduction and Preliminaries

Iampan [2] introduced a new algebraic structure, called a UP-algebra, which is a generalization of a KU-algebra. Many researchers have studied on UP-algebras such as [4, 6, 7]. Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Denote $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$ and $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting

$$A \cdot B = B \cap (A' \cup \Omega) \text{ for all } A, B \in \mathcal{P}_\Omega(X)$$

and a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting

$$A * B = B \cup (A' \cap \Omega) \text{ for all } A, B \in \mathcal{P}^\Omega(X).$$

Satirad et al. [5] proved that $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ and $(\mathcal{P}^\Omega(X), *, \Omega)$ are UP-algebras. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras.

In this paper, we introduce the notion of a partial transformation UP-algebra $P(X)$ induced by a UP-algebra X and prove that the set of all full transformations $T(X)$ is a UP-ideal of $P(X)$.

Now we will recall the definition of a UP-algebra from [2].

An algebra $X = (X, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in X$,

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(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

In a UP-algebra $X = (X, \cdot, 0)$, the following assertions are valid (see [2, 3]).

$$(\forall x \in X)(x \cdot x = 0), \tag{1.1}$$
$$(\forall x, y, z \in X)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0), \tag{1.2}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0), \tag{1.3}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0), \tag{1.4}$$
$$(\forall x, y \in X)(x \cdot (y \cdot x) = 0), \tag{1.5}$$
$$(\forall x, y \in X)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x), \tag{1.6}$$
$$(\forall x, y \in X)(x \cdot (y \cdot y) = 0), \tag{1.7}$$
$$(\forall a, x, y, z \in X)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0), \tag{1.8}$$
$$(\forall a, x, y, z \in X)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0), \tag{1.9}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0), \tag{1.10}$$
$$(\forall x, y, z \in X)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0), \tag{1.11}$$
$$(\forall x, y, z \in X)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0), \text{and} \tag{1.12}$$
$$(\forall a, x, y, z \in X)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0). \tag{1.13}$$

From now on, X will always denote a UP-algebra $(X, \cdot, 0)$.

Definition 1. *[2] A subset S of X is called a UP-subalgebra of X if the constant 0 of X is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.*

Iampan [2] proved the useful criteria that a nonempty subset S of a UP-algebra X is a UP-subalgebra of X if and only if S is closed under the $\cdot$ multiplication on X.

Definition 2. *[2, 8] A subset S of X is called*

(1) a UP-filter of X if it satisfies the following properties:

(i) the constant 0 of X is in S, and

(ii) for any $x, y \in X, x \cdot y \in S$ and $x \in S$ imply $y \in S$.

(2) a UP-ideal of X if it satisfies the following properties:

(i) the constant 0 of X is in S, and

(ii) for any $x, y, z \in X, x \cdot (y \cdot z) \in S$ and $y \in S$ imply $x \cdot z \in S$.

(3) a strongly UP-ideal of X if it satisfies the following properties:

(i) the constant 0 *of* X *is in* S*, and*

(ii) for any $x, y, z \in X, (z \cdot y) \cdot (z \cdot x) \in S$ *and* $y \in S$ *imply* $x \in S$*.*

Guntasow et al. [1] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra X is the only one strongly UP-ideal of itself.

2. Main Results

We denote

$B(X)$	the set of all binary relations on X,
$P(X)$	the set of all partial transformations on X,
$T(X)$	the set of all full transformations on X.

Then $T(X) \subseteq P(X) \subseteq B(X)$. If $\alpha \in B(X)$ and $x \in X$, then $x\alpha = \{y \in X \mid (x, y) \in \alpha\}$. Thus $x\alpha$ is the set of all elements that are α-related to x. Define a function O from X to X by $O(x) = 0$ for all $x \in X$, that is, $O \in T(X)$. Define a binary operation $\bullet$ on $B(X)$ by: for all $\alpha, \beta \in B(X)$,

$$(x, y) \in \alpha \bullet \beta \Leftrightarrow \begin{cases} x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta \text{ and } y = y_{x\alpha} \cdot y_{x\beta} \text{ for } y_{x\alpha} \in x\alpha \text{ and } y_{x\beta} \in x\beta, \text{ or} \\ x \notin \operatorname{dom}\alpha \text{ and } y = 0. \end{cases}$$

We can redefine a binary operation $\bullet$ on $P(X)$ by: for all $\alpha, \beta \in P(X)$,

$$(\alpha \bullet \beta)(x) = \begin{cases} \alpha(x) \cdot \beta(x) & \text{if } x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta, \\ 0 & \text{if } x \notin \operatorname{dom}\alpha. \end{cases}$$

We see that

- for all $\alpha, \beta \in B(X)$,
$$\operatorname{dom}(\alpha \bullet \beta) = (\operatorname{dom}\alpha - \operatorname{dom}\beta)', \tag{2.1}$$

- the empty function $\emptyset \in P(X)$ and for all $\alpha \in P(X)$,
$$\emptyset \bullet \alpha = O \text{ and } \alpha \bullet \emptyset = O|_{(\operatorname{dom}\alpha)'}. \tag{2.2}$$

Theorem 1. $B(X) = (B(X), \bullet, O)$ *is an algebra of type* $(2, 0)$ *satisfying (UP-2) and (UP-3).*

Proof. Let $\alpha \in B(X)$. Then

$$\begin{aligned}(x, y) \in O \bullet \alpha &\Leftrightarrow x \in X \cap \operatorname{dom}\alpha \text{ and } y = \mathbf{r}_{xO} \cdot y_{x\alpha} \text{ for some } y_{x\alpha} \in x\alpha \qquad (\operatorname{dom} O = X) \\ &\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } y = O(x) \cdot y_{x\alpha} \text{ for some } y_{x\alpha} \in x\alpha\end{aligned}$$

$$\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } y = 0 \cdot y_{x\alpha} \text{ for some } y_{x\alpha} \in x\alpha$$
$$\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } y = y_{x\alpha} \text{ for some } y_{x\alpha} \in x\alpha \qquad \text{((UP-2))}$$
$$\Leftrightarrow (x, y) \in \alpha.$$

Hence, $O \bullet \alpha = \alpha$, so (UP-2) is holding.

Let $\alpha \in B(X)$ and $x \in X$. Then
Case 1: $x \notin \operatorname{dom}\alpha$. Then $(x, 0) \in (\alpha \bullet O) \Leftrightarrow (x, 0) \in O$.
Case 2: $x \in \operatorname{dom}\alpha$. Then

$$(x, y) \in \alpha \bullet O \Leftrightarrow x \in \operatorname{dom}\alpha \cap X \text{ and } y = \mathbf{r}_{x\alpha} \cdot y_{xO} \text{ for some } y_{xO} \in xO \qquad (\operatorname{dom} O = X)$$
$$\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } y = \mathbf{r}_{x\alpha} \cdot O(x)$$
$$\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } y = \mathbf{r}_{x\alpha} \cdot 0$$
$$\Leftrightarrow x \in \operatorname{dom}\alpha \text{ and } y = 0 \qquad \text{((UP-3))}$$
$$\Leftrightarrow (x, y) \in O.$$

Hence, $\alpha \bullet O = O$, so (UP-3) is holding.

Therefore, $B(X) = (B(X), \bullet, O)$ is an algebra of type (2,0) satisfying (UP-2) and (UP-3).

Theorem 2. *$P(X) = (P(X), \bullet, O)$ is a UP-algebra and we shall call it the partial transformation UP-algebra induced by a UP-algebra X.*

Proof. Let $\alpha, \beta, \gamma \in P(X)$ and let $x \in X$.
Case 1: $x \notin \operatorname{dom}\alpha$. Then $(\alpha \bullet \beta)(x) = 0 = (\alpha \bullet \gamma)(x)$, so $x \in \operatorname{dom}(\alpha \bullet \beta) \cap \operatorname{dom}(\alpha \bullet \gamma)$. Thus

$$\begin{aligned} ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))(x) &= (\alpha \bullet \beta)(x) \cdot (\alpha \bullet \gamma)(x) \\ &= 0 \cdot 0 \\ &= 0, && \text{((UP-2))} \end{aligned}$$

so $x \in \operatorname{dom}((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))$.
Case 1.1: $x \notin \operatorname{dom}(\beta \bullet \gamma)$. Then $((\beta \bullet \gamma) \bullet ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma)))(x) = 0 = O(x)$.
Case 1.2: $x \in \operatorname{dom}(\beta \bullet \gamma)$. Then $x \in \operatorname{dom}(\beta \bullet \gamma) \cap \operatorname{dom}((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))$. Thus

$$\begin{aligned} ((\beta \bullet \gamma) \bullet ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma)))(x) &= (\beta \bullet \gamma)(x) \cdot ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))(x) \\ &= (\beta \bullet \gamma)(x) \cdot 0 \\ &= 0 && \text{((UP-3))} \\ &= O(x). \end{aligned}$$

Case 2: $x \in \operatorname{dom}\alpha$.

Case 2.1: $x \notin \operatorname{dom}\beta$. Then $x \in \operatorname{dom}\alpha - \operatorname{dom}\beta$, so $(\beta \bullet \gamma)(x) = 0$ and $(\alpha \bullet \beta)(x)$ is not defined. Thus $x \notin \operatorname{dom}(\alpha \bullet \beta)$, so $((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))(x) = 0$. Thus $x \in \operatorname{dom}(\beta \bullet \gamma) \cap ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))$, so

$$\begin{aligned}((\beta \bullet \gamma) \bullet ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma)))(x) &= (\beta \bullet \gamma)(x) \cdot ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))(x) \\ &= 0 \cdot 0 \\ &= 0 && \text{((UP-2))} \\ &= O(x).\end{aligned}$$

Case 2.2: $x \in \operatorname{dom}\beta$. If $x \notin \operatorname{dom}\gamma$, then $x \in \operatorname{dom}\beta - \operatorname{dom}\gamma$. Thus $(\beta \bullet \gamma)(x)$ is not defined, so $x \notin \operatorname{dom}(\beta \bullet \gamma)$. Thus $((\beta \bullet \gamma) \bullet ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma)))(x) = 0 = O(x)$. If $x \in \operatorname{dom}\gamma$, then we conclude that

$$\begin{aligned}((\beta \bullet \gamma) \bullet ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma)))(x) &= (\beta \bullet \gamma)(x) \cdot ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma))(x) \\ &= (\beta \bullet \gamma)(x) \cdot ((\alpha \bullet \beta)(x) \cdot (\alpha \bullet \gamma)(x)) \\ &= (\beta(x) \cdot \gamma(x)) \cdot ((\alpha(x) \cdot \beta(x)) \cdot (\alpha(x) \cdot \gamma(x))) \\ &= 0 \\ &= O(x).\end{aligned}$$

Hence, $(\beta \bullet \gamma) \bullet ((\alpha \bullet \beta) \bullet (\alpha \bullet \gamma)) = O$, so (UP-1) is holding.

Let $\alpha \in P(X)$ and let $x \in X$.

Case 1: $x \notin \operatorname{dom}\alpha$. Then $x \in \operatorname{dom} O - \operatorname{dom}\alpha$. Thus $\alpha(x)$ and $(O \bullet \alpha)(x)$ are not defined.

Case 2: $x \in \operatorname{dom}\alpha$. Then $x \in \operatorname{dom} O \cap \operatorname{dom}\alpha$. Thus $(O \bullet \alpha)(x) = O(x) \cdot \alpha(x) = 0 \cdot \alpha(x) = \alpha(x)$.

Hence, $O \bullet \alpha = \alpha$, so (UP-2) is holding.

Let $\alpha \in P(X)$ and let $x \in X$.

Case 1: $x \notin \operatorname{dom}\alpha$. Then $(\alpha \bullet O)(x) = 0 = O(x)$.

Case 2: $x \in \operatorname{dom}\alpha$. Then $x \in \operatorname{dom}\alpha \cap \operatorname{dom} O$. Thus $(\alpha \bullet O)(x) = \alpha(x) \cdot O(x) = \alpha(x) \cdot 0 = 0 = O(x)$.

Hence, $\alpha \bullet O = O$, so (UP-3) is holding.

Let $\alpha, \beta \in P(X)$ be such that $\alpha \bullet \beta = O$ and $\beta \bullet \alpha = O$. Let $x \in X$. Then $(\alpha \bullet \beta)(x) = O(x) = 0$ and $(\beta \bullet \alpha)(x) = O(x) = 0$. If $x \in \operatorname{dom}\alpha - \operatorname{dom}\beta$, then $(\alpha \bullet \beta)(x)$ is not defined which is a contradiction. If $x \in \operatorname{dom}\beta - \operatorname{dom}\alpha$, then $(\beta \bullet \alpha)(x)$ is not defined which is a contradiction. If $x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$, then $0 = (\alpha \bullet \beta)(x) = \alpha(x) \cdot \beta(x)$ and $0 = (\beta \bullet \alpha)(x) = \beta(x) \cdot \alpha(x)$. By (UP-4), we have $\alpha(x) = \beta(x)$. If $x \notin \operatorname{dom}\alpha$ and $x \notin \operatorname{dom}\beta$, then $\alpha(x)$ and $\beta(x)$ are not defined. Hence, $\alpha = \beta$, so (UP-4) is holding.

Therefore, $(P(X), \bullet, O)$ is a UP-algebra.

Theorem 3. *$T(X)$ is a UP-ideal of $P(X)$ and we shall call it the full transformation UP-algebra induced by a UP-algebra X.*

Proof. Clearly, $O \in T(X)$. Let $\alpha, \beta, \gamma \in P(X)$ be such that $\alpha \bullet (\beta \bullet \gamma) \in T(X)$ and $\beta \in T(X)$. Then $\operatorname{dom}(\alpha \bullet (\beta \bullet \gamma)) = X$ and $\operatorname{dom}\beta = X$ and so by (2.1), $X = \operatorname{dom}(\alpha \bullet (\beta \bullet \gamma)) = (\operatorname{dom}\alpha - \operatorname{dom}(\beta \bullet \gamma))'$. Thus $\operatorname{dom}\alpha - \operatorname{dom}(\beta \bullet \gamma) = \emptyset$ and so by (2.1), $\emptyset = \operatorname{dom}\alpha - \operatorname{dom}(\beta \bullet \gamma) = \operatorname{dom}\alpha - (\operatorname{dom}\beta - \operatorname{dom}\gamma)' = \operatorname{dom}\alpha - (X - \operatorname{dom}\gamma)' = \operatorname{dom}\alpha - ((\operatorname{dom}\gamma)')' = \operatorname{dom}\alpha - \operatorname{dom}\gamma$. By (2.1), $\operatorname{dom}(\alpha \bullet \gamma) = (\operatorname{dom}\alpha - \operatorname{dom}\gamma)' = \emptyset' = X$. That is, $\alpha \bullet \gamma \in T(X)$. Hence, $T(X)$ is a UP-ideal of $P(X)$ and also a UP-filter and a UP-subalgebra.

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