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The Proofs of the Arithmetic-Geometric Mean Inequality Through Both the Product and Binomial Inequalities

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Abstract. In this paper, we show new ways of proving the arithmetic-geometric mean AGM inequality through the first product and the second product inequalities. In addition, we prove the AGM inequality through the binomial inequalities. These methods are alternative ways of proving AGM inequalities.

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1. Introduction

The importance of inequalities cannot be underestimated as they play central role in mathematical analysis. In the 21^{st} century, the AGM inequality has received much attention and has been applied in the areas of statistics and engineering. The AGM inequality was first introduced by Lagrange (as cited in [1]). Since then the AGM inequality has used to establish the relationships between the areas and perimeters of geometrical plane figures, the so-called isoperimetric inequalities, for example, see authors in [2, 3]. However, a substantial progress has been made to increase the understanding of the AGM inequality by the researchers across the globe. In [4], the author proved the AGM inequality using the heuristic method. The author in [5] observed that the mutatis mutandis' method for proving the AGM inequality was similar to the result obtained by Jacobsthal and Rado, for

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example, see [6]. In [7], the author proved the AGM inequality through Taylor's theorem about $x = \frac{1}{2}$ by setting the function $f(x)$ equals to the Heinz mean. Thus,

$$f(x) = \frac{a^x b^{1-x} + a^{1-x} b^x}{2}, \quad \forall\, 0 \le x \le 1 \ \textit{and} \ f \in C^2[0,1].$$

In a similar development, another refinement of the AGM inequality was given by the authors in [8]. They obtained their result through Taylor's theorem about $x = \frac{1}{2}$, by setting

$$f(x) = \frac{a^x + a^{1-x}}{2}, \quad \forall\, 0 \le x \le 1 \ \textit{and} \ f \in C^k(0,\infty).$$

Notwithstanding, in [9], the authors proved the AGM inequality with the use of second derivative test by setting

$$f(x) = \frac{(x-a)^2}{a(x+\max\{x,a\})} + \ln x, \quad \forall\, a > 0 \ \textit{and} \ f \in C^2(0,\infty).$$

Some researchers have applied AGM inequality to solve matrix algebra. For example, see authors in [10, 11]. The author in [12] extended the AGM inequality to include the harmonic mean called AGHM inequality.

In this paper, the AGM inequality is proved through the first product inequality in a closed interval $[0, 2]$, then through the second product inequality in a half open ended interval $[2,\infty)$ and finally, through the binomial inequalities of rational numbers.

Definition 1. *Let A be a linear vector space defined over the real number field $\mathbf{R}$. A scalar-valued function $p : A \times A \to \mathbf{R}$ that associates with each pair a_1, a_2 of vectors in A a scalar, denoted (a_1, a_2), is called an inner product on A if and only if*

(i) $(a_1, a_2) > 0$ *whenever $a \neq 0$, and* $(a_1, a_1) = 0$ *if and only if* $a_1 = 0$

(ii) $(a_1, a_2) = (a_2, a_1)$, $\forall\, a_1,\ a_2 \in A$

(iii) $(\alpha a_1 + \beta a_2, a_3) = \alpha(a_1, a_3) + \beta(a_2, a_3)$, $\forall\, \alpha,\ \beta \in \mathbf{R}$, *and* $a_1, a_2, a_3 \in V$. *See* [13]

Definition 2. *Let A be a linear space over $\mathbf{R}$. A norm on A is a real-valued function $\|\cdot\| : A \to [0,\infty)$ such that for any $a_1, a_2 \in A$ and $\alpha \in \mathbf{R}$ the following conditions are met:*

$$\begin{aligned} \|a\| &\geq 0, \ \textit{and} \ \|a\| = 0, \ iff\ a = 0 \\ \|\alpha a\| &= |\alpha|\|ua\|, \ \forall\, a \in A \ \textit{and} \ \alpha \in \mathbf{R} \\ \|a_1 \pm a_2\| &\leq \|a_1\| + \|a_2\|, \ \forall\, a_1, a_2 \in A, \end{aligned}$$

See [14].

Definition 3 (First and Second Product Inequalities)**.** *Let a_1 and a_2 be any two positive real numbers, then*

$$(i)\|a_1\|\|a_2\| \leq \|a_1\|+\|a_2\|, \quad \forall a_1,a_2\in[0,2]. \tag{1}$$

$$(ii)\|a_1\|+\|a_2\| \leq \|a_1\|\|a_2\|, \quad \forall a_1,a_2\in[2,\infty). \tag{2}$$

See [15].

1.1. The Proof of the AGM Inequality through the First Product Inequality

In this section, we obtain the AGM inequality through both the first and second product inequalities by induction as follows. Multiplying both sides of inequality (1) by $(1-p)$ yields

$$(1-p)\Big(\|a_1\|\|a_2\|\Big) \leq (1-p)\Big\{\|a_1\|+\|a_2\|\Big\}, \ \forall \quad p\in[0,1]. \tag{3}$$

But, we see that:

$$\Big(\|a_1\|\|a_2\|\Big)^{(1-p)} \leq (1-p)\Big(\|a_1\|\|a_2\|\Big). \tag{4}$$

Substituting inequality (4) into inequality (3) yields

$$\Big(\|a_1\|\|a_2\|\Big)^{(1-p)} \leq (1-p)\Big(\|a_1\|+\|a_2\|\Big). \tag{5}$$

Setting $p=\frac{1}{2}$ into inequality (5) yields

$$\begin{aligned}\Big(\|a_1\|\|a_2\|\Big)^{\frac{1}{2}} &\leq \frac{1}{2}\Big(\|a_1\|+\|a_2\|\Big)\\ \Rightarrow \Big(\prod_{i=1}^{2}a_i\Big)^{\frac{1}{2}} &\leq \frac{1}{2}\sum_{i=1}^{2}a_i.\end{aligned}$$

We can see that for any three positive real numbers $n=3$, the following inequality holds.

$$\begin{aligned}\|a_1\|\|a_2\|\|a_3\| &\leq \|a_1\|+\|a_2\|+\|a_3\|, \quad \forall a_1,a_2,a_3\in[0,2]\\ \Rightarrow (1-p)\Big(\|a_1\|\|a_2\|\|a_3\|\Big) &= (1-p)\Big(\|a_1\|+\|a_2\|+\|a_3\|\Big)\\ \Rightarrow \Big(\|a_1\|\|a_2\|\|a_3\|\Big)^{(1-p)} &\leq (1-p)\Big(\|a_1\|+\|a_2\|+\|a_3\|\Big).\end{aligned}$$

Setting $p=\frac{2}{3}$ into the above inequality, we obtain

$$\Big(\|a_1\|\|a_2\|\|a_3\|\Big)^{\frac{1}{3}} \leq \frac{1}{3}\Big(\|a_1\|+\|a_2\|+\|a_3\|\Big)$$

$$\Rightarrow \Big(\prod_{i=1}^{3} a_i\Big)^{\frac{1}{3}} \leq \frac{1}{3}\sum_{i=1}^{3} a_i.$$

For any number of positive real numbers n, the following inequalities are observed:

$$\begin{aligned} \|a_1\|\|a_2\|\ldots\|a_n\| &\leq \|a_1\|+\|a_2\|+\ldots+\|a_n\| \quad \forall a_1,a_2,\ldots,a_n \in [0,2] \\ \Rightarrow (1-p)\Big(\|a_1\|\|a_2\|,\ldots,\|a_n\|\Big) &\leq (1-p)\Big(\|a_1\|+\|a_2\|+\ldots+\|a_n\|\Big) \\ \Rightarrow \Big(\|a_1\|\|a_2\|,\ldots,\|a_n\|\Big)^{(1-p)} &\leq (1-p)\Big(\|a_1\|+\|a_2\|+\ldots+\|a_n\|\Big). \end{aligned}$$

Setting $p = \frac{(n-1)}{n}$ into the above inequality yields

$$\begin{aligned} \Big(\|a_1\|\|a_2\|,\ldots,\|a_n\|\Big)^{\frac{1}{n}} &\leq \frac{1}{n}\Big(\|a_1\|+\|a_2\|+\ldots+\|a_n\|\Big) \\ \Rightarrow \Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}} &\leq \frac{1}{n}\sum_{i=1}^{n} a_i \ \ \forall \ a_1,a_2,\ldots,a_n \in [0,2]. \end{aligned}$$

1.2. The Proof of the AGM Inequality through the Second Product Inequality

In a similar development, we prove the AGM inequality through the second product inequality. The AGM inequality is obtained by induction. Multiplying both sides of inequality (2) by p yields

$$p\Big(\|a_1\|+\|a_2\|\Big) \leq p\Big\{\|a_1\|\|a_2\|\Big\}, \ \ \forall \ \ p \in [0,1]. \tag{6}$$

But, we see that:

$$\Big(\|a_1\|\|a_2\|\Big)^{p} \leq p\Big(\|a_1\|\|a_2\|\Big). \tag{7}$$

Substituting inequality (7) into inequality (6), we get

$$\Big(\|a_1\|\|a_2\|\Big)^{p} \leq p\Big(\|a_1\|+\|a_2\|\Big). \tag{8}$$

Setting $p = \frac{1}{2}$ into inequality (8) yields

$$\begin{aligned} \Big(\|a_1\|\|a_2\|\Big)^{\frac{1}{2}} &\leq \frac{1}{2}\Big(\|a_1\|+\|a_2\|\Big) \\ \Rightarrow \Big(\prod_{i=1}^{2} a_i\Big)^{\frac{1}{2}} &\leq \frac{1}{2}\sum_{i=1}^{2} a_i. \end{aligned}$$

Again, we observed that:

$$\|a_1\|+\|a_2\|+\|a_3\| \leq \|a_1\|\|a_2\|\|a_3\| \ \ \forall a_1,a_2,a_3 \in [2,\infty)$$

$$\Rightarrow p\Big(\|a_1\| + \|a_2\| + \|a_3\|\Big) \quad \leq \quad p\Big(\|a_1\|\|a_2\|\|a_3\|\Big). \tag{9}$$

We observed that:

$$\Big(\|a_1\|\|a_2\|\|a_3\|\Big)^p \leq p\Big(\|a_1\|\|a_2\|\|a_3\|\Big). \tag{10}$$

Substituting inequality (10) into inequality (9) yields

$$\Big(\|a_1\|\|a_2\|\|a_3\|\Big)^p \leq p\Big(\|a_1\| + \|a_2\| + \|a_3\|\Big).$$

Setting $p = \frac{1}{3}$ into the above inequality, we obtain

$$\begin{aligned} \Big(\|a_1\|\|a_2\|\|a_3\|\Big)^{\frac{1}{3}} &\leq \frac{1}{3}\Big(\|a_1\| + \|a_2\| + \|a_3\|\Big) \\ \Rightarrow \Big(\prod_{i=1}^{3} a_i\Big)^{\frac{1}{3}} &\leq \frac{1}{3}\sum_{i=1}^{3} a_i. \end{aligned}$$

We observed for any number of positive real numbers n, we have:

$$\begin{aligned} \|a_1\| + \|a_2\| + \ldots + \|a_n\| &\leq \|a_1\|\|a_2\| \ldots \|a_n\| \quad \forall a_1, a_2, \ldots, a_n \in [2, \infty) \\ \Rightarrow p\Big(\|a_1\| + \|a_2\| + \ldots + \|a_n\|\Big) &\leq p\Big(\|a_1\|\|a_2\| \ldots \|a_n\|\Big). \end{aligned} \tag{11}$$

But we see that:

$$\Big(\|a_1\|\|a_2\| \ldots \|a_n\|\Big)^p \leq p\Big(\|a_1\|\|a_2\| \ldots \|a_n\|\Big). \tag{12}$$

Substituting inequality (12) into inequality (11) yields

$$\Big(\|a_1\|\|a_2\| \ldots \|a_n\|\Big)^p \leq p\Big(\|a_1\| + \|a_2\| + \ldots + \|a_n\|\Big).$$

Setting $p = \frac{1}{n}$ in the above equation yields

$$\Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i \;\; \forall \;\; a_1, a_2, \ldots, a_n \in [2, \infty).$$

1.3. The Proof of the AGM Inequality through the Binomial Inequalities

In this section, the AGM inequality is proved through new binomial inequalities of rational numbers. We can see that $n = 2$, the following inequality holds.

$$\begin{aligned} (\sqrt{a_1} + \sqrt{a_2})^2 &\geq 0 \\ \Rightarrow \Big(\|a_1 a_2\|\Big)^{\frac{1}{2}} &\leq \frac{1}{2}\Big(\|a_1\| + \|a_2\|\Big) \end{aligned}$$

$$\begin{aligned} \Rightarrow \|a_1\|^{\frac{1}{2}}\|a_2\|^{\frac{1}{2}} &\leq \frac{1}{2}\Big(\|a_1\| + \|a_2\|\Big) \\ \Rightarrow \Big(\prod_{i=1}^{2} a_i\Big)^{\frac{1}{2}} &\leq \frac{1}{2}\sum_{i=1}^{2} a_i. \end{aligned}$$

Also, let a_1, a_2 and a_3 be three positive real numbers, then

$$\begin{aligned} & (\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3})^2 \geq 0 \\ \Rightarrow\ & -2\Big(\sqrt{a_1a_2} + \sqrt{a_1a_3} + \sqrt{a_2a_3}\Big) \leq (a_1 + a_2 + a_3) \\ \Rightarrow\ & \frac{-2}{3}\Big(\sqrt{a_1a_2} + \sqrt{a_1a_3} + \sqrt{a_2a_3}\Big) = \frac{(a_1 + a_2 + a_3)}{3} \\ \Rightarrow\ & \frac{-4}{3}\Big\{\frac{1}{2}\Big(\sqrt{a_1a_2} + \sqrt{a_3}\Big)\Big\} \leq \frac{(a_1 + a_2 + a_3)}{3} \\ \Rightarrow\ & \|\frac{-4}{3}\Big\{\frac{1}{2}\Big(\sqrt{a_1a_2} + \sqrt{a_3}\Big)\Big\}\| \leq \|\frac{(a_1 + a_2 + a_3)}{3}\| \\ \Rightarrow\ & \frac{4}{3}\Big\|\Big\{\frac{1}{2}\Big(\sqrt{a_1a_2} + \sqrt{a_3}\Big)\Big\}\Big\| \leq \frac{1}{3}(\|a_1\| + \|a_2\| + \|a_3\|) \\ \Rightarrow\ & \frac{4}{3}\Big(\|a_1\|\|a_2\|\|a_3\|\Big)^{\frac{1}{4}} \leq \frac{1}{3}\Big(\|a_1\| + \|a_2\| + \|a_3\|\Big). \end{aligned} \tag{13}$$

We see that:

$$\Big(\|a_1\|\|a_2\|\|a_3\|\Big)^{\frac{1}{3}} \leq \frac{4}{3}\Big(\|a_1\|\|a_2\|\|a_3\|\Big)^{\frac{1}{4}}. \tag{14}$$

Substituting inequality (13) into inequality (14) yields

$$\Rightarrow \Big(\prod_{i=1}^{3} a_i\Big)^{\frac{1}{3}} \leq \frac{1}{3}\sum_{i=1}^{3} a_i.$$

Similarly, we can see that:

$$\begin{aligned} & (\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} + \sqrt{a_4})^2 \geq 0 \\ \Rightarrow\ & (a_1 + a_2 + a_3 + a_4) \geq -2\Big(\sqrt{a_1a_2} + \sqrt{a_3a_4} + \sqrt{a_1a_3} + \sqrt{a_1a_4} + \sqrt{a_2a_3} + \sqrt{a_2a_4}\Big) \\ \Rightarrow\ & -\frac{1}{2}\Big(\sqrt{a_1a_2} + \sqrt{a_3a_4} + \sqrt{a_1a_3} + \sqrt{a_1a_4} + \sqrt{a_2a_3} + \sqrt{a_2a_4}\Big) \leq \frac{(a_1 + a_2 + a_3 + a_4)}{4} \\ \Rightarrow\ & -\frac{1}{2}\Big(\sqrt{a_1a_2} + \sqrt{a_3a_4}\Big) \leq \frac{(a_1 + a_2 + a_3 + a_4)}{4} \\ \Rightarrow\ & \| - \frac{1}{2}\Big(\sqrt{a_1a_2} + \sqrt{a_3a_4}\Big)\| = \|\frac{(a_1 + a_2 + a_3 + a_4)}{4}\| \\ \Rightarrow\ & \frac{1}{2}\|\Big(\sqrt{a_1a_2} + \sqrt{a_3a_4}\Big)\| \leq \frac{1}{4}(\|a_1\| + \|a_2\| + \|a_3\| + \|a_4\|) \\ \Rightarrow\ & \Big(\|a_1\|\|a_2\|\|a_3\|\|a_4\|\Big)^{\frac{1}{4}} \leq \frac{1}{4}(\|a_1\| + \|a_2\| + \|a_3\| + \|a_4\|) \end{aligned}$$

$$\Rightarrow \quad \Big(\prod_{i=1}^{4} a_i\Big)^{\frac{1}{4}} \leq \frac{1}{4}\sum_{i=1}^{4} a_i.$$

By the principle of induction, we see that for any n number of real numbers, we have:

$$\Big(\prod_{i=1}^{n} a_i\Big)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} a_i.$$

This completes the prove.

2. Conclusion

In a nutsell, we have provided the new ways of proving the AGM inequality through the product and binomial inequalities.

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