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Numerical computation of Lower bounds of Structured Singular Values

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Abstract. In this article we have considered numerical approximation of lower bounds of Structured Singular Values, SSV. The SSV is a well-known mathematical quantity which is widely used to analyse and syntesize the robust stability and instability analysis of linear feedback systems in control theory. The SSV establishes a link between numerical linear algebra and system theory. The computation of lower bounds of SSV by means of ordinary differential equations based technique is presented. The obtained numerical results for the lower bounds of SSV are compared with the well-known MATLAB function mussv available in MATLAB control toolbox.

1. Introduction

The Structured Singular Values known as μ-value is a well-known mathematical tool in control, introduced by J. C. Doyle around 1980's [12]. This tool can be used to discuss both stability and instability analysis of linear systems when subject to a certain perturbations. For more applications on SSV, the interested reader can consult [13] which describe the engineering motivation for μ-values. Due it's computational complexity, the approximation of an exact value of SSV appears to be NP-hard see [2]. In fact, the computation of bounds of SSV, especially the computation of upper bounds when certain properties are under consideration appers to be a NP-hard problem [14].

There has been done an extensive amount of research in order to develop new numerical algorithms which are very efficient and provides tighter bounds for μ-value. The power method [9] approximate the lower bound SSV while taking pure complex perturbations into an account. But unfortunately power method fail to converge; this happens when pure real uncertainties are under consideration, for more detail see [15]. The structured perturbations addressed by μ-tool is very generic and it allows to cover all types of parametric perturbations that can be incorporated into the linear control system theory by help of both real and complex Linear Fractional Transformations (LFT's). For more detail please see [1, 3, 5–8, 10] and the references therein for the applications of SSV. The

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message from the approximation of an upper bound of μ-tool provides conditions which guarantee the stability of feedback linear systems. The well-known Matlab function mussv available in MATLAB controlboox approximates an upper bounds for SSV by means of diagonal balancing technique and Linear Matrix Inequlaity (LMI) techniques [4].

1.1. Overview of the article

Section 2 provides the basic framework. In particular, it explain how the computation of the SSV can be addressed by an inner-outer algorithm, where the outer algorithm determines the perturbation level ϵ and the inner algorithm determines a (local) extremizer of the structured spectral value set. In Section 3 it is explain that how the inner algorithm works for the case of pure complex structured perturbations. An important characterization of extremizers shows that one can restrict himself to a manifold of structured perturbations with normalized and low-rank blocks. In Section 4, we construct a gradient system of ordinary differential equations in order to solve the local optimization problem. Finally, Section 5 presents a range of numerical experiments to compare the quality of the lower bounds to those obtained with mussv.

2. Framework

Consider $A \in \mathbb{C}^{r,r}$ or $A \in \mathbb{R}^{r,r}$ and let $\Theta_{\mathbb{B}'}$, a perturbation set defined as

$$\Theta_{\mathbb{B}'} = \{diag(\alpha_i I_{r_i},\ \Gamma_s) : \alpha_i \in \mathbb{C}(\mathbb{R}),\ \Gamma_s \in \mathbb{C}^{m_s,m_s}(\mathbb{R}^{m_s,m_s})\}.$$

Here, I_i is an identity matrix with dimension i.

Definition 2.1. [8]. The structured singular value for an operator $A \in \mathbb{C}^{r,r}$ or $A \in \mathbb{R}^{r,r}$ w.r.t $\Theta_{\mathbb{B}'}$ is defined as follows:

$$\mu_{\Theta_{\mathbb{B}'}}(A) := \frac{1}{\min\{\|\wp\|_2 : \wp \in \Theta_{\mathbb{B}'}, \det(I - A\wp) = 0\}}. \tag{1}$$

In above given Definition 2.1, the quantity $det(\cdot)$ denotes determinants of operator $(I - A\wp)$. The above Definition 2.1 for the case of pure complex perturbation takes the form:

$$\mu_{\Theta_{\mathbb{B}}}(A) = \frac{1}{\min\{\|\wp\|_2 : \wp \in \Theta_{\mathbb{B}},\ \rho(A\wp) = 1\}}. \tag{2}$$

In Equ. (2.2), the quantity $\rho(\cdot)$ is known as the spectral radius that is $\rho(A\wp) = max|\lambda_i|$ where λ_i is the spectrum of an operator $A\wp$.

Structured spectral value sets. Consider the given input arguments $A \in \mathbb{C}^{r\times r}$ and ϵ, the desired perturbation level. The structured spectral value set is the set containing all the eigenvalues of an operator $(\epsilon A\wp)$ and is defined as:

$$\Lambda_{\epsilon^*}^{\Theta_{\mathbb{B}'}}(A) = \{\lambda \in \Lambda(\epsilon^* A\wp) :\ \wp \in \Theta_{\mathbb{B}'}\}. \tag{3}$$

Here, $\Lambda(\cdot)$ is the set of all eigenvalues of an operator and $\|\wp\|_2 = 1$. If we consider both mixed real and complex perturbations, then structured spectral value set is of the form:

$$\Sigma_{\epsilon^*}^{\Theta_{\mathbb{B}'}}(A) = \{\eta = 1 - \lambda_1 : \lambda_1 \in \Lambda_{\epsilon^*}^{\Theta_{\mathbb{B}'}}(A)\}. \tag{4}$$

The above formulation in Equ. (2.4) helps us to write down the alternative definition of structured singular value as given in Equ. (1.2) as follows:

$$\mu_{\Theta_{\mathbb{B}'}}(A) = \frac{1}{\arg\,\min\{0 \in \Sigma_{\epsilon^*}^{\Theta_{\mathbb{B}'}}(A)\}}. \tag{5}$$

While for the case when we have only pure complex perturbations, then Equ. (2.3) allows us to alternatively express structured singular values as

$$\mu_{\Theta_{\mathbb{B}}}(A) = \frac{1}{\arg\,\min\{\max|\lambda_1| = 1\}}. \tag{6}$$

Problem under consideration. Our goal is to solve following optimization problem,

$$\xi(\epsilon^*) = \arg\min|\eta|. \tag{7}$$

In above Equ. (2.7), $\eta \in \Sigma_{\epsilon^*}^{\Theta_{\mathbb{B}'}}(A)$ for a fixed parameter $\epsilon > 0$. In order to solve the optimization problem addressed in Equ. (2.7), we suggests a two-level algorithm. In the inner algorithm, we give a solution of Equ. (2.7) by constructing and then solving a gradient system of ordinary differential equations. In the outer algorithm, with the help of an iterative method we first vary the perturbation level ϵ.

First we address the case of a purely complex perturbations when $\Theta_{\mathbb{B}}$ by taking the inner algorithm in order to compute a local extremizer for

$$\lambda(\epsilon) = \arg\max|\lambda_1|. \tag{8}$$

In above Equ. (2.8), $\lambda_1 \in \Lambda_{\epsilon}^{\Theta_{\mathbb{B}}}(M)$ which gives us a lower bound for structured singular values in case of pure complex uncertainties that is $\mu_{\Delta_{\mathbb{B}}}(M)$. First we consider the case of pure complex perturbations that is by taking into account the perturbations set $\Theta_{\mathbb{B}}$ instead of $\Theta_{\mathbb{B}'}$.

3. Pure Complex perturbations

In this section, we compute the solution to optimization problems as discussed in Equ. (2.8). For this we consider the estimation of $\mu_{\Theta_{\mathbb{B}}}(A)$ for the given operator $A \in \mathbb{C}^{r,r}$. Also, in this case we consider the pure complex perturbations that is

$$\Theta_{\mathbb{B}} = \{diag(\alpha_1 I_1, ..., \alpha_n I_n; \wp_1, ..., \wp_F) : \alpha_i \in \mathbb{C}, \wp_j \in \mathbb{C}^{m_j, m_j}\}. \tag{9}$$

The following Lemma 3.1 gives the behavior of the spectrum of a matrix valued function.

Lemma 3.1. Consider the matrix valued function $\Upsilon : \mathbb{R} \to \mathbb{C}^{n,n}$. Also consider the fact that $\lambda(t)$ is an eigenvalue of matrix valued function $\Upsilon(t)$ which approaches to simple eigenvalue that is λ^* of $\Upsilon_0 = \Upsilon(0)$ as $t \to 0$. Then $\lambda(t)$ is analytic near $t = 0$ with

$$\frac{d\lambda}{dt} = \frac{w_0^* \Upsilon_1 v_0}{w_0^* v_0},$$

where $\Upsilon_1 = \dot{\Upsilon}(0)$ and v_0, w_0 are right and left eigenvectors of Υ_0 associated to λ^*.

As now our goal is to deal with the an optimization problem as mentioned in Equ. (2.8). This needs the computation of an uncertainty $\wp_{local}$ so that $\rho(\epsilon A \wp_{local})$ achieves maximum growth along the perturbation $\wp \in \Theta_{\mathbb{B}'}$ with $\|\wp\|_2 \leq 1$. In below we consider that λ be the greatest eigenvalue when $|\lambda|$ equals to the spectral radius.

Definition 3.2. A matrix valued function $\wp \in \Theta_{\mathbb{B}}$ such that $\|\wp\|_2 = 1$ and $(\epsilon A \wp)$ possesses the maximum eigenvalue which increases the modulus for structured spectral vale set $\Lambda_\epsilon^{\Theta_{\mathbb{B}}}(A)$, known as a local maximizer. In theorem 3.3, we give the characterization of local maximizer of the gradient system of ordinary differential equations.

Theorem 3.3 [11]. Consider that

$$\wp_{local} = diag(\alpha_1 I_1, ..., \alpha_n I_n; \wp_1, ..., \wp_F).$$

Here, $\wp_{local}$ is such that $\|\wp_{local}\|_2 = 1$ and is a local maximizer for $\Lambda_\epsilon^{\Theta_{\mathbb{B}}}(A)$. Additionally, we consider that an operator $(\epsilon A \wp_{local})$ having a simple maximum eigenvalue which is $\lambda = |\lambda| e^{i\theta}$, having right and left eigenvectors v and w and are scaled so that $s = e^{i\theta} w^* v > 0$. Upon partitioning, we get

$$v = (v_1^{\mathrm{T}},\ \ldots,\ v_n^{\mathrm{T}},\ v_{n+1}^{\mathrm{T}},\ \ldots\ , v_{n+F}^{\mathrm{T}})^{\mathrm{T}}; \tag{10}$$

$$u = (u_1^{\mathrm{T}},\ \ldots\ , u_n^{\mathrm{T}},\ u_{n+1}^{\mathrm{T}},\ \ldots\ , u_{n+F}^{\mathrm{T}})^{\mathrm{T}}. \tag{11}$$

Where $u = A^* w$. Additionally we assume that

$$u_k^* v_k \neq 0 \quad \forall\ k = 1, \ldots, n, \tag{12}$$

$$\|u_{n+h}\|_2 \cdot \|v_{n+h}\|_2 \neq 0 \quad \forall\ h = 1, \ldots, F. \tag{13}$$

Then

$$|s_k| = 1 \quad \forall\ k = 1, \ldots, n \quad \text{and} \quad \|\Delta_h\|_2 = 1 \quad \forall h = 1, \ldots, F,$$

4. A system of ODEs to compute extremal points of $\Lambda_\epsilon^{\wp_{\mathbb{B}}}(A)$.

In order to compute the local extremizer to $|\lambda|$ such that $|\lambda| \in \Lambda_\epsilon^{\Theta_{\mathbb{B}}}(A)$. To do so first we compute matrix valued function $\wp(t)$ so that the maximum eigenvalue that is $\lambda(t)$ of an operator $(\epsilon A \wp(t))$ attains the maximum value. We then construct and give an optimal solution to a gradient system of ordinary differential equation's. This system of ordinary differential equations satisfies the choice of $\wp(t)$.

4.1. Local optimization problem

Let $\lambda = |\lambda_1|e^{i\theta}$ is the a simple eigenvalue of $(\epsilon A_{\wp}(t))$. Further consider that the corresponding eigenvectors v, w are normalized as

$$\|w\| = \|v\| = 1, \qquad w^*v = |w^*v|e^{-i\theta}. \tag{14}$$

By the help of Lemma 3.1, we get

$$\frac{d}{dt}|\lambda_1|^2 = 2|\lambda_1|Re\Big(\frac{u^*\dot{\wp}v}{e^{i\theta}w^*v}\Big) = \frac{2|\lambda_1|}{|w^*v|}Re(u^*\dot{\wp}v), \tag{15}$$

where $u = A^*w$.

Suppose that $\wp \in \Theta_{\mathbb{B}}$ and we search the direction $\dot{\wp} = \tau$ that given maximum local growth of the modulus of λ_1. This gives us

$$\tau = diag(\omega_1 I_{r_1}, \ldots, \omega_s I_{r_N}, \Omega_1, \ldots, \Omega_F). \tag{16}$$

This acts as a solution to the maximization problem

$$\begin{gathered}\tau^* = \arg\max\{Re(u^*\tau x)\} \\ \text{subject to } \; Re(\overline{\delta}_i\omega_i) = 0, \;\; i = 1:N, \\ \text{and} \quad Re\langle\wp_j, \Omega_j\rangle = 0, \; j = 1:F.\end{gathered} \tag{17}$$

In Lemma 4.1, we give the solution τ^* to the maximization problem as discussed in the Equ. (3.4).

Lemma 4.1 [11]. The solution τ^* with

$$\tau^* = diag(\omega_1 I_{r_1}, \ldots, \omega_N I_{r_N}, \Omega_1, \ldots, \Omega_F), \tag{18}$$

with

$$\omega_i = \nu_i \left(v_i^* u_i - Re\left(v_i^* u_i \overline{s}_i\right) s_i\right), \quad i = 1, \ldots, N \tag{19}$$

$$\Omega_j = \zeta_j \left(u_{N+j} v_{N+j}^* - Re\langle\wp_j, u_{N+j} v_{N+j}^*\rangle\wp_j\right), \quad j = 1, \ldots, F. \tag{20}$$

The coefficient $\nu_i > 0$ is the reciprocal of the absolute value of the expression that appears in the right-hand side in Equ. (4.6) when it's different from zero and the coefficient $\nu_i = 1$ else. While on the other hand the coefficient $\zeta_j > 0$ is the reciprocal of the Frobenius norm of an operator that appear in the right hand side of Equ. (4.7) if it's different from zero and the coefficient $\zeta_j = 1$ else.

Now we express the result as obtained in the previous Lemma 3.1 as:

$$\tau^* = S_1 P_{\Theta_{\mathbb{B}}}\left(u(t)v(t)^*\right) - S_2\wp. \tag{21}$$

In above Equ. (4.8), $P_{\Theta_{\mathbb{B}}}(\cdot)$ is the orthogonal projection while $S_1, S_2 \in \Theta_{\mathbb{B}}$ are diagonal operators.

4.2. Gradient system of ordinary differential equations

The result in the previous Lemma 4.1 allows us to have the following differential equation on the manifold $\Theta_{\mathbb{B}}$:

$$\dot{\wp}(t) = S_1 P_{\Theta_{\mathbb{B}}}(u(t)v(t)^*) - S_2\wp(t). \tag{22}$$

Here, $v(t)$ is eigenvector with $\|v(t)\|_2 = 1$ and is associated with the simple eigenvalue $\lambda(t)$ for an operator $(\epsilon A\wp(t))$ for fixed perturbation level that is $\epsilon > 0$. The differential equation (4.9) is a gradient system of ODE's because of the fact that it's right-hand side is nothing but projected gradient of $\tau \mapsto Re(u^*\tau v)$.

4.3. Choice of initial value matrix and ϵ

[11]. For the computation of the admissible perturbation level ϵ, we take the perturbation $\wp$ which is obtained for the previous value that is ϵ_1 as the initial value matrix. In order to produce the maximal growth for the eigenvalue $|\lambda(t)|$, we take the initial value matrix as:

$$\wp_0 = S\, P_{\wp_{\mathbb{B}}}(w(t)v(t)^*). \tag{23}$$

The operator S is chosen such that $\wp_0 \in \Theta_{\mathbb{B}}$. For a very natural choice of the initialization of the perturbation level, we consider ϵ as:

$$\epsilon = \frac{1}{\widehat{\mu}_{\Theta_{\mathbb{B}}}(A)}. \tag{24}$$

In the above equation, $\widehat{\mu}_{\Theta_{\mathbb{B}}}(A)$ is the upper bound of μ-value approximated by MATLAB function mussv, which approximates both upper and lower bounds of structured singular values.

5. Numerical Testing

In the very final section of this artile, we present numerical experimentations for both lower and upper bounds of structured singular values. The numerical results are computed by well-known MATLAB function mussv and the algorithm [11].

Example 1. Consider two dimensional real matrix A_1.

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The set of block diagonal matrices is taken as:

$$\Theta_{\mathbb{B}'} = \{diag(\delta_1 I_1, \delta_2 I_1, \wp_1) : \delta_1, \delta_2 \in \mathbb{R}, \wp_1 \in \mathbb{C}^{3,3}\}.$$

The admissible perturbation structure $\widehat{\wp}$ obtained by using MATLAB routine mussv is:

$$\widehat{\wp} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The $\|\widehat{\wp}\|_2 = 1$. The computed upper bound is $\mu_{PD}^{upper} = 2.3224$. The same lower bound is obtained, that is, $\mu_{PD}^{lower} = 2.3224$. By using algorithm [11], the perturbation structure $\epsilon^* \wp^*$ is obtained as:

$$\wp^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, $\epsilon^* = 1.0000$ and $\|\wp^*\|_2 = 1$. The obtained lower bound is as $\mu_{New}^{lower} = 2.3224$. In the following Figure 1, we give the comparison of lower bounds of structured singular values approximated by our New algorithm with the Lower and Upper bounds approximated by MATLAB function mussv for matrix valued function $B_1(w)$ for w=1:9, where $w \in \Omega$ and Ω represents the frequency range in $\mathbb{R}_+$. Frequency response w is measure of output of $(B_1 - \wp)$ system.

Example 2. Consider two dimensional real matrix A_2.

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 3 & 2 & -3 & -2 & -4 & 1 \\ -1 & 2 & -2 & -4 & -2 & 3 & 2 & 4 & -1 \\ 1 & -3 & 2 & 4 & 2 & -3 & -2 & -5 & 1 \\ 1 & -1 & 0 & 2 & 2 & -1 & 0 & -2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 3 & 2 & -3 & -2 & -4 & 0 \end{bmatrix}.$$

The set of block diagonal matrices is taken as:

$$\Theta_{\mathbb{B}'} = \{diag(\delta_1 I_1, \wp_1) : \delta_1 \in \mathbb{R}, \wp_1 \in \mathbb{C}^{8,8}\}.$$

The admissible perturbation structure $\widehat{\wp}$ obtained by using MATLAB routine mussv is:

$$\widehat{\wp} = \begin{bmatrix} -0.0641 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0000 & -0.0100 & 0.0105 & -0.0118 & -0.0047 & -0.0008 & 0.0011 & -0.0097 \\ 0 & 0.0000 & 0.0065 & -0.0069 & 0.0077 & 0.0030 & 0.0005 & -0.0007 & 0.0063 \\ 0 & 0.0000 & 0.0140 & -0.0147 & 0.0164 & 0.0065 & 0.0011 & -0.0015 & 0.0136 \\ 0 & 0.0000 & 0.0083 & -0.0087 & 0.0098 & 0.0039 & 0.0007 & -0.0009 & 0.0081 \\ 0 & -0.0000 & -0.0115 & 0.0121 & -0.0135 & -0.0054 & -0.0009 & 0.0012 & -0.0112 \\ 0 & -0.0000 & -0.0075 & 0.0079 & -0.0088 & -0.0035 & -0.0006 & 0.0008 & -0.0073 \\ 0 & -0.0000 & -0.0166 & 0.0175 & -0.0196 & -0.0078 & -0.0013 & 0.0018 & -0.0162 \\ 0 & 0.0000 & 0.0028 & -0.0030 & 0.0033 & 0.0013 & 0.0002 & -0.0003 & 0.0028 \end{bmatrix}.$$

The $\|\widehat{\wp}\|_2 = 0.1258$. The computed upper bound is $\mu_{PD}^{upper} = 15.6004$. The lower bound is obtained, that is, $\mu_{PD}^{lower} = 15.5921$. By using algorithm [11], the perturbation structure $\epsilon^* \wp^*$ is obtained as:

$$\wp^* = \begin{bmatrix} -1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0005 & -0.1559 & 0.1640 & -0.1834 & -0.0729 & -0.0123 & 0.0167 & -0.1518 \\ 0 & 0.0003 & 0.1016 & -0.1069 & 0.1195 & 0.0475 & 0.0080 & -0.0109 & 0.0989 \\ 0 & 0.0007 & 0.2178 & -0.2291 & 0.2563 & 0.1018 & 0.0171 & -0.0233 & 0.2120 \\ 0 & 0.0004 & 0.1296 & -0.1363 & 0.1525 & 0.0606 & 0.0102 & -0.0139 & 0.1262 \\ 0 & -0.0005 & -0.1794 & 0.1887 & -0.2110 & -0.0838 & -0.0141 & 0.0192 & -0.1746 \\ 0 & -0.0003 & -0.1164 & 0.1224 & -0.1369 & -0.0544 & -0.0092 & 0.0125 & -0.1133 \\ 0 & -0.0008 & -0.2596 & 0.2731 & -0.3054 & -0.1213 & -0.0204 & 0.0278 & -0.2527 \\ 0 & 0.0001 & 0.0443 & -0.0466 & 0.0521 & 0.0207 & 0.0035 & -0.0047 & 0.0431 \end{bmatrix}.$$

In this case, $\epsilon^* = 1.0000$ and $\|\wp^*\|_2 = 0.0641$. The obtained lower bound is as $\mu_{New}^{lower} = 15.5995$.

In the following Figure 2, we give the comparison of lower bounds of structured singular values approximated by our New algorithm with the Lower and Upper bounds approximated by MATLAB function mussv for matrix valued function $B_2(w)$ for w=1:9, where $w \in \Omega$ and Ω represents the frequency range in $\mathbb{R}_+$. Frequency response w is measure of output of $(B_2 - \wp)$ system.

Example 3. Consider two dimensional real matrix A_3.

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

The set of block diagonal matrices is taken as:

$$\Theta_{\mathbb{B}'} = \{diag(\delta_1 I_1, \delta_2 I_1, \wp_1) : \delta_1, \delta_2 \in \mathbb{R}, \wp_1 \in \mathbb{C}^{3,3}\}.$$

The admissible perturbation structure $\widehat{\wp}$ obtained by using MATLAB routine mussv is:

$$\widehat{\wp} = \begin{bmatrix} 0.4354 & 0 & 0 & 0 & 0 \\ 0 & 0.4354 & 0 & 0 & 0 \\ 0 & 0 & 0.0337 & 0.0337 & -0.2740 \\ 0 & 0 & 0.0337 & 0.0337 & -0.2740 \\ 0 & 0 & 0.0226 & 0.0226 & -0.1840 \end{bmatrix}.$$

The $\|\widehat{\wp}\|_2 = 0.4354$. The computed upper bound is $\mu_{PD}^{upper} = 2.2966$. The same lower bound is obtained, that is, $\mu_{PD}^{lower} = 2.2966$. By using algorithm [11], the perturbation structure $\epsilon^* \wp^*$ is obtained as:

$$\wp^* = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0.0773 & 0.0774 & -0.6293 \\ 0 & 0 & 0.0773 & 0.0774 & -0.6293 \\ 0 & 0 & 0.0519 & 0.0520 & -0.4227 \end{bmatrix}.$$

In this case, $\epsilon^* = 1.0000$ and $\|\wp^*\|_2 = 2.2966$. The obtained lower bound is as $\mu_{New}^{lower} = 0.4354$.

In the following Figure 3, we give the comparison of lower bounds of structured singular values approximated by our New algorithm with the Lower and Upper bounds approximated by MATLAB function mussv for matrix valued function $B_3(w)$ for w=1:6, where $w \in \Omega$ and Ω represents the frequency range in $\mathbb{R}_+$. Frequency response w is measure of output of $(B_3 - \wp)$ system.

Example 4. Consider two dimensional real matrix A_4.

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

The set of block diagonal matrices is taken as:

$$\Theta_{\mathbb{B}'} = \{diag(\delta_1 I_1, \delta_2 I_1, \wp_1) : \delta_1, \delta_2 \in \mathbb{R}, \wp_1 \in \mathbb{C}^{3,3}\}.$$

The admissible perturbation structure $\widehat{\wp}$ obtained by using MATLAB routine mussv is:

$$\widehat{\wp} = \begin{bmatrix} 0.4354 & 0 & 0 & 0 & 0 \\ 0 & 0.4354 & 0 & 0 & 0 \\ 0 & 0 & 0.0337 & 0.0337 & -0.2740 \\ 0 & 0 & 0.0337 & 0.0337 & -0.2740 \\ 0 & 0 & 0.0226 & 0.0226 & -0.1840 \end{bmatrix}.$$

The $\|\widehat{\wp}\|_2 = 0.4354$. The computed upper bound is $\mu_{PD}^{upper} = 2.2966$. The same lower bound is obtained, that is, $\mu_{PD}^{lower} = 2.2966$. By using algorithm [11], the perturbation structure $\epsilon^* \wp^*$ is obtained as:

$$\wp^* = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0.0773 & 0.0774 & -0.6293 \\ 0 & 0 & 0.0773 & 0.0774 & -0.6293 \\ 0 & 0 & 0.0519 & 0.0520 & -0.4227 \end{bmatrix}.$$

In this case, $\epsilon^* = 1.0000$ and $\|\wp^*\|_2 = 2.2966$. The obtained lower bound is as $\mu_{New}^{lower} = 0.4354$.

In the following Figure 4, we give the comparison of lower bounds of structured singular values approximated by our New algorithm with the Lower and Upper bounds approximated by MATLAB function mussv for matrix valued function $B_4(w)$ for w=1:2, where $w \in \Omega$ and Ω represents the frequency range in $\mathbb{R}_+$. Frequency response w is measure of output of $(B_4 - \wp)$ system.

In the following Figures [5-14], we give the comparison of lower bounds of structured singular values approximated by our New algorithm with the Lower and Upper bounds approximated by MATLAB function mussv for the various matrix valued functions.

6. Conclusion

In this article we have considered the numerical approximation of μ-values for the matrix representations of finite symmetric groups S_n over the filed of complex numbers by using well-known MATLAB function mussv and our algorithm [11]. The experimental results indicates the different behaviors of lower bounds of μ-values with once computed by mussv and our algorithm.

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Appendix

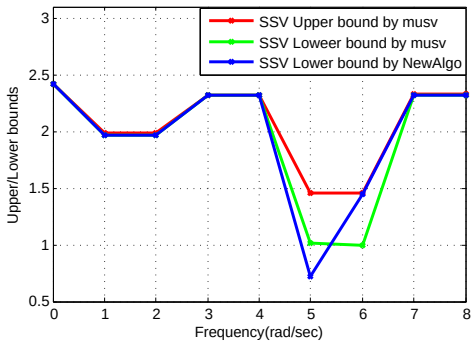


Figure 1: Comparison of Lower and Upper bounds of SSV

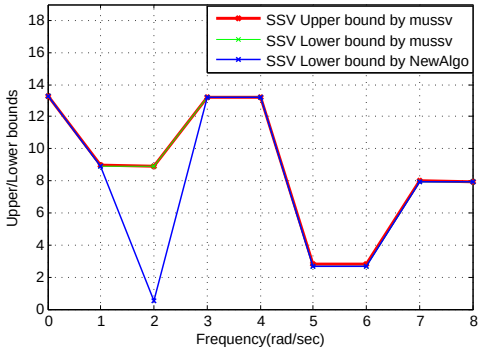
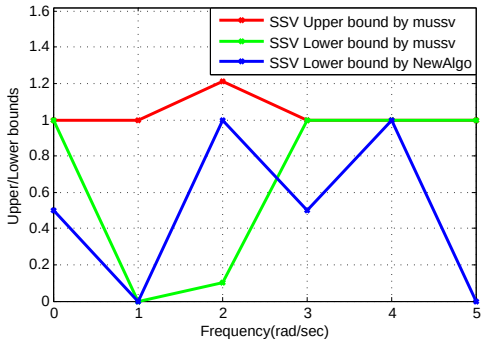


Figure 2: Comparison of Lower and Upper bounds of SSV



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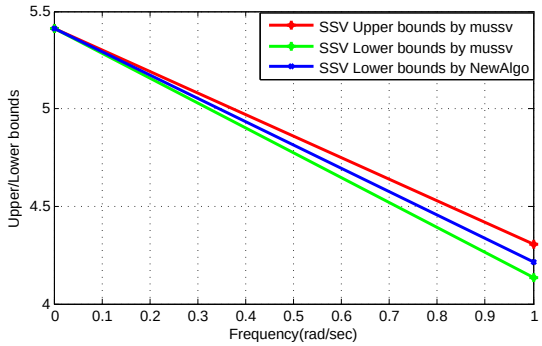
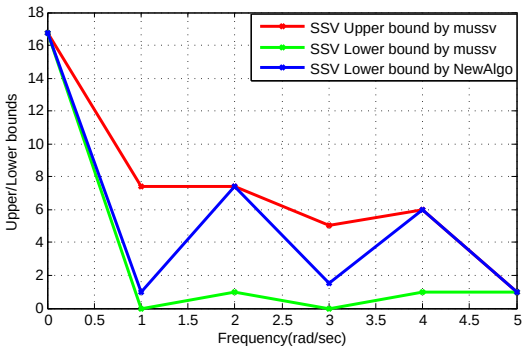


Figure 4: Comparison of Lower and Upper bounds of SSV



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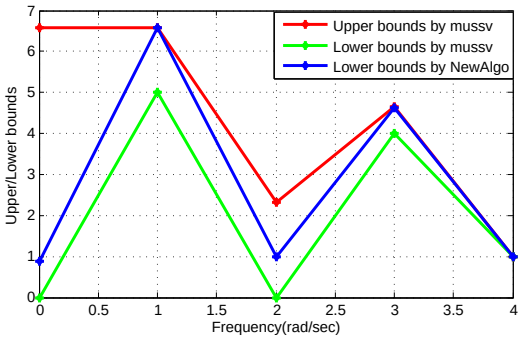

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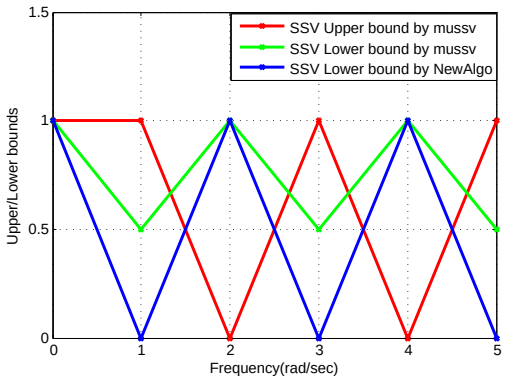

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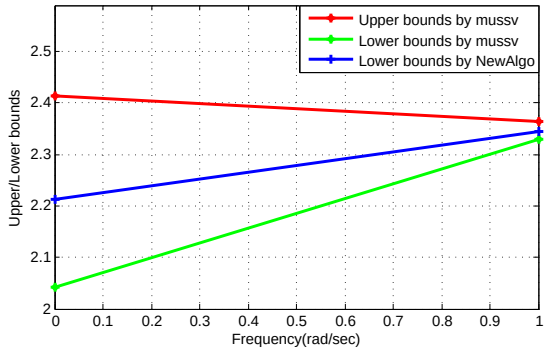

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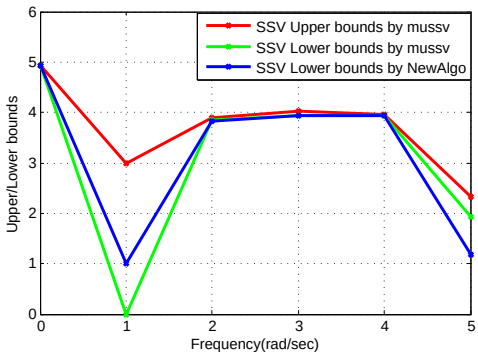

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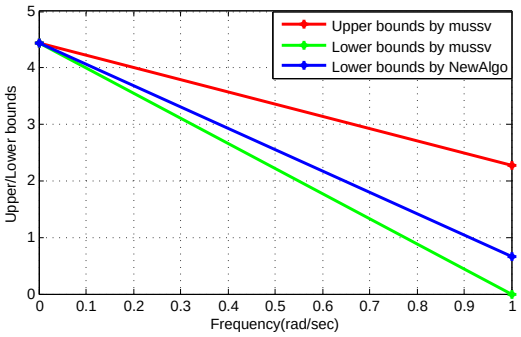
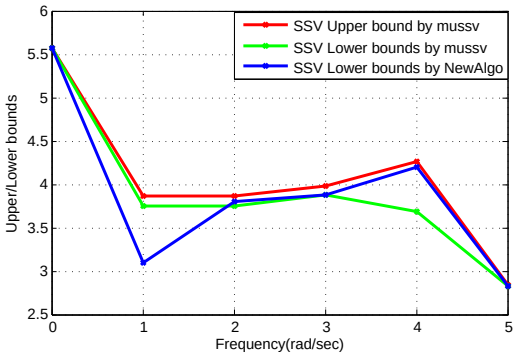


Figure 10: Comparison of Lower and Upper bounds of SSV



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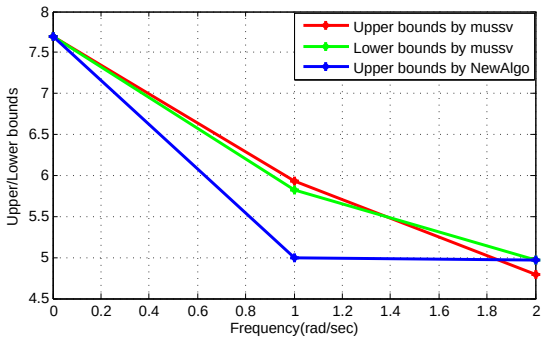

Figure 12: Comparison of Lower and Upper bounds of SSV

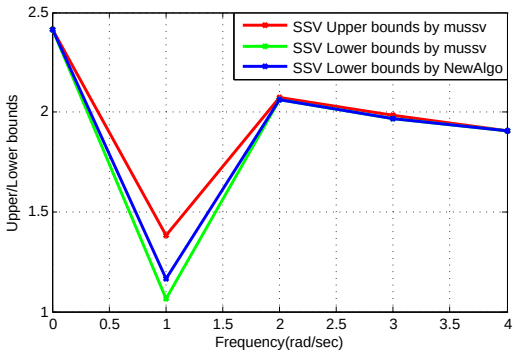

Figure 13: Comparison of Lower and Upper bounds of SSV

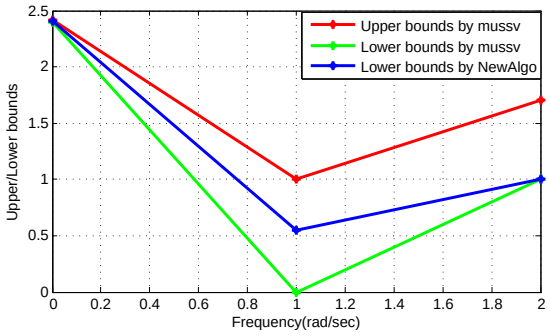

Figure 14: Comparison of Lower and Upper bounds of SSV